ON THE DYNAMICS OF THE CIRCULAR VORTEX WITH APPLICATIONS TO THE ATMOSPHERE AND ATMOSPHERIC VORTEX AND WAVE MOTIONS.

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The theoretical investigations which form the subject of this paper, have developed parallel to the empirical of which the most striking result arrived at has been the introduction of the »polar front» on the synoptic charts of the Norwegian Weather Service. Further information concerning these empirical results will be found in papers by meteorologists who have been attached to this service, namely: *J. Bjerknes*: »On the Structure of Moving Cyclones«, *J. Bjerknes* and *H. Solberg:* »Meteorological Conditions for the Formation of Rain«, and other papers expected soon to follow from the hands of these authors as well as from their previous colleague *T. Bergeron*, now at the Meteorological Office, Stockholm.[1])

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I. General Contributions to the Dynamics of Baroclinic Fluids.

1. *Baroclinic and barotropic fields of mass.* — In the following investigations on fluid motion special attention must be paid to the distribution of mass in its relation to the distribution of pressure.

The field of a single scalar in space is represented geometrically by a system of equiscalar surfaces, most conveniently drawn for unit intervals of the variable. The sur-

[1]) See Vol. 1 no. 2 and Vol. 2 no. 3 of Geofysiske Publikationer, Kristiania 1919 and 1921. — must emphasize here that the young meteorologists of the Norwegian Weather Service have been very much retarded in the publication of their scientific results due to the great burden of labour connected with the foundation and maintenance of this service under difficult circumstances during the first critical years. For this reason I cannot avoid in this paper referring to results which these forecasters have found long ago, but not yet had time to prepare for publication. — A summary of these results was communicated in a lecture which I gave before the Royal Meteorological Society Nov. 7, 1919, »On the Structure of the Atmosphere when Rain is Falling« (Quarterly Journal of the Roy. Met. Soc. April 1920); and in a Note »The Meteorology of the Temperate Zone and the General Atmospheric Circulation«, (Nature, June 24, 1920; Monthly Weather Review, January 1921).

faces then divide the space into a set of equiscalar unit sheets. If these sheets increase infinitely in thickness, the ultimate result will be that the scalar is reduced to a constant in all space. Excepting this case of degeneration, it will always be possible by suitable choice of units to get the unit sheets as thin as we like.

When two scalars are given in the same space, we get two sets of equiscalar surfaces as well as unit sheets. The surfaces will in general cut each other, giving a well-defined system of curves along which both scalars are constant, double equiscalar curves, — and a system of tubes with parallelogrammatic cross-sections, the double equiscalar unit tubes. By suitable choice of units we may get these tubes as thin as we like, exception being made for two cases of degeneration.

The first of these cases is the ultimate result of a convergence of the angle between the intersecting surfaces to zero. This gives coincidence of the two sets of surfaces, so that any curve drawn on one of these surfaces will be a double equiscalar curve. At the same time the unit tubes have swelled and lost their identity as tubes. One family of surfaces represents two scalars: the integer values of the one scalar is represented by one set of individual surfaces, and the integer values of the other scalar by another set. These two different selections of surfaces define two different sets of unit sheets. This case of degeneration appears whenever a relation exists between the two scalars, so that one of them can be expressed uniformly by the other, the constancy of the one involving the constancy of the other.

The other case of degeneration is presented when the unit sheets of the one scalar swell to infinite thickness. This scalar then becomes a constant, every surface in space an equiscalar surface of it and every curve in an equiscalar surface of the other scalar is a double equiscalar curve. The unit tubes have swelled and lost their character as tubes.

These general principles concerning two scalars in space must be remembered when we are to consider the mutual relations between the fields of mass and pressure.

For the description of a field of mass, two different variables are in use: the density, ϱ, defined as mass per unit volume, and its reciprocal value, the specific volume, α, which gives the volume of the unit mass. Both are represented by the same family of equiscalar surfaces. One set of individual surfaces corresponds to integer values of the density, while another set corresponds to integer values of the specific volume. Thus, accordingly, as we use the one or the other variable we get different divisions of the space into unit sheets. We shall call the surfaces of equal density *isopycnic*, those of equal specific volume *isosteric* surfaces. Occasionally it will be convenient to refer to the surfaces without specifying the variable used. In that case we shall use the expression *equisubstantial* surfaces. The degeneration of the field of mass, when the unit isosteric or the unit isopycnic sheets swell to infinite thickness, leads to the important case of complete homogeneity.

For the field of pressure we are concerned with only one scalar, pressure p, which is represented by the isobaric surfaces. The case of degeneration of this field, when the isobaric unit sheets swell to infinite thickness, has little importance for its appearance is quite exceptional.

Equisubstantial surfaces will in general be inclined to and cut the isobaric surfaces, and form a well defined system of curves of intersection and of unit tubes. According to the variable used to represent the field of mass we shall call them isobaric-isosteric or isobaric-isopycnic curves and unit tubes.

In all cases of well-defined curves of intersection and unit tubes, we shall call the field of mass *baroclinic*. In the case of degeneration we shall call it *barotropic*. The case of barotropy appears, first when the equisubstantial surfaces coincide with the isobaric, i. e., *when the specific volume, or the density, is a function of the pressure,*

(a) $$\alpha = f_1(p) \quad \text{or, equivalently,} \quad \varrho = f_2(p)$$

and secondly, *in the case of complete homogeneity,*

(b) $$\alpha = \text{constant, or, equivalently,} \quad \varrho = \text{constant.}$$

Equations (b) may be considered as formally contained in (a), which may therefore be called the *barotropic condition.* The case when the pressure is reduced to a constant may be considered as unimportant.

2. *Baroclinic and barotropic conditions from physical point of view.* — We have introduced equation 1 (a) merely to register a geometric fact, without seeking for the physical origin. Now the density of a body may be expressed as a function of a number of variables, of which pressure p is always one, and temperature ϑ another. In addition, a number of other variables may turn up, as the humidity of the atmospheric air, or the salinity of sea water, and so on. Thus

(a) $$\alpha = F(p, \vartheta, \ldots)$$

and a corresponding equation may be written for the density.

This equation is essentially physical as contrasted with 1 (a). It is valid for each moving individual of the fluid and may be different from individual to individual. Therefore in the most general case it must contain not merely physical parameters, but also coordinates of identification for the fluid particles. That is, we must use the so-called Lagrangean method of describing fluid motion. This is also what we should do in principle, though thanks to the simple character of the problem it will not be necessary to write the Lagrangean equations in their fully developed form.

It is seen that barotropy appears then and only then when equation (a) can be reduced to the form 1 (a), so that the second member takes the form of a function of p in which coordinates and time, if they enter, enter implicitly, because p is a function of x, y, z, t, but not explicitly. This reduction to barotropic form may take place for a moment during the motion (transient barotropy), or permanently through one of two reasons: on account of the special type of motion (permanent but accidental barotropy), or on account of the physical properties of the fluid (intrinsic barotropy).

Let us consider for instance a salt solution, which for simplicity may be supposed to be incompressible, so that the density of a fluid particle depends only upon its salinity. In the case of equilibrium both the surfaces of equal pressure and the surfaces of equal density, i. e., of equal salinity, are horizontal planes. But as soon as motion is introduced, the surfaces of equal salinity will in general no longer coincide with the surfaces of equal pressure, and the conditions will in general be baroclinic. But if the motion has the character for instance of regular standing waves, the isohalinic and the isobaric surfaces will periodically take the form of coinciding horizontal planes when the system passes the equilibrium position. Thus transient barotropy presents itself periodically. But then we may also define an infinity of different motions by which isohalinic surfaces coincide permanently with the isobaric, while the least disturbance would produce baroclinic conditions. This would be the case of permanent but accidental barotropy. A simple example is a permanent shearing motion by which horizontal planes slide upon each other. Other examples of this accidental barotropy will be given later in this paper. But intrinsic barotropy will occur only in one case, namely when the salt solution is homogeneous. Then and only then any motion whatever which the system may take will be barotropic.

Instead of an incompressible salt solution we may consider an ideal gas. No transfer of heat by conduction or radiation shall take place so that every process is adiabatic. Equilibrium will again be barotropic: isobaric, isosteric and isothermal surfaces being horizontal planes. In the case of motion surfaces of all three sets will in general separate. But by motions of the same special type as those considered in the previous example transient barotropy or permanent accidental barotropy may appear, and in one and only one case we shall have intrinsic barotropy, when in the initial state of equilibrium we have the adiabatic vertical temperature gradient.

The characteristic feature of the two cases which lead to intrinsic barotropy may be thus described: we have equations of one and the same form to describe both the geometry of the field of density during equilibrium and the physics of the process by which the density of a fluid individual is changed during the motion. In the homogeneous salt solution we had the condition $\varrho = \varrho_0$, giving the geometry of the density distribution during the equilibrium, and the same equation $\varrho = \varrho_0$, defining the incompressibility of a particle during its motion. And in the ideal gas we have in both cases to work with the equation of Poisson

$$\varrho = C p^{1/\varkappa}$$

$\varkappa$ being the ratio of the two specific heats of a gas. In the case of equilibrium, this equation defines the density of the gas from level to level as a function of the pressure p at the different levels. And during the motion this same equation defines the change of density of one and the same particle, as a function of the varying pressure.

It is seen that we arrive at the case of intrinsic barotropy only with the condition of a pre-established harmony between the initial state and the intrinsic properties of the fluid. In the »classical» hydrodynamics this harmony is always supposed to exist. It is realized in the simplest form when the fluid is supposed to be homogeneous and incompressible. And it is attained in the most general way when the general equation of condition (a) takes the form of one and the same relation between density and pressure both when it is used to define physically the change of density of one and the same particle in time, and when it is used to define geometrically the change of density from particle to particle in space. When this harmony exists we shall call *the fluid barotropic.*

The motion of barotropic fluids is only of a peculiarly restricted type. Circulations and vortices are conserved in accordance with the theorems of Helmholtz and Lord Kelvin. It is strange that authors on hydrodynamics have acquiesced for such a long time within the boundaries of this restricted theory. Formation and annihilation of circulations and vortices are seen to go on incessantly in the motion of actual fluids, and especially so in the atmosphere and sea. In addition, the theory is exceedingly easy to generalize. In the well known calculus which leads to the theorems of Helmholtz and Lord Kelvin the only thing to observe is *not* to drop the term which becomes zero in virtue of the barotropic condition, but to retain and discuss it. Although it was a long time before this step was taken, it is still more astonishing that after it was made the generalized theory with its numerous applications has had such difficulty in finding its way into treatises and textbooks. After more than twenty years only one treatise has entered upon the subject as far as I know.[1])

A problem which will especially interest us is that of determining the internal distribution of mass in the moving fluid. This is an insignificant question as long as the fluid is homogeneous and incompressible, and it is a simple supplementary problem which is solved by the barotropic relation when the density is a function of the pressure. For fluids of this type the problem of determining the distribution of mass is an independent one only in as much as it involves the determination of the external boundary surface. But even the determination of the internal distribution of mass is an independent problem as soon as the fluid is of the general baroclinic type, and subject to formation and annihilation of circulation and vortices.

The problem can conveniently be solved as one of generalized hydrostatics, when the state of equilibrium is supposed to be entertained by non-conservative instead of conservative forces. As an introduction we shall develop the principles of this hydrostatics.

3. *Generalized hydrostatics.* — The hydrostatical equations in their most general for L, referred to the rectangular coordinates x, y, z, may be written

$$\text{(a)} \qquad \frac{\partial p}{\partial x} = \varrho X', \quad \frac{\partial p}{\partial y} = \varrho Y', \quad \frac{\partial p}{\partial z} = \varrho Z'.$$

p denoting pressure, ϱ density, and X', Y', Z' the components of the exterior force referred to unit mass. Generally it is supposed that this force is conservative, and thus

[1]) *Appell*, Traité de Mécanique Rationelle, Troisième Edition, Tome III, Chapitre XXXVII, Fluides Baroclines, Paris, 1921.

L. Silberstein seems to be the first who has made the step leading to the general principle of the formation of vortices, but with a careful reservation as to the physical reality of it, and without any other application than the return to the theorem of Helmholtz. (Bulletin international de l'Académie des Sciences de Cracovie 1896–97). For my own part I was led to observe that formation and annihilation of vortices occur as a physical fact of fundamental importance in connection with hydrodynamic actions at a distance, and I developed the general vortex theory as a method of generalizing the theory of these actions. To serve this purpose it was given in terms, not of velocities but of momenta (hydrodynamic field-intensity). A natural subsequent step was to give it also in terms of velocities, and to apply it in that form to the discussion of air and sea motions. The following papers will be of special interest in connection with the subject treated below:

V. Bjerknes:	Über die Bildung von Cirkulationsbewegungen und Wirbeln in reibungslosen Flüssigkeiten, Videnskabsselskabets Skrifter, Kristiania 1898.
—»—	Über einen hydrodynamischen Fundamentalsatz und seine Anwendung besonders auf die Mechanik der Atmosphäre und des Weltmeeres. K. Svensk. Vet. Akad. Handl. 31, Stockholm 1898.
—»—	Das dynamische Princip der Cirkulationsbewegungen in der Atmosphäre. Meteorologische Zeitschrift 1900.
—»—	Cirkulation relativ zu der Erde, ibid., 1902.
J. W. Sandström:	Über die Beziehung zwischen Temperatur und Luftbewegung in der Atmosphäre unter stationären Verhältnissen. Öfv. Svensk. Vet. Ak. Förh. 58, Stockholm 1901.
—»—	Über die Beziehung zwischen Luftbewegung und Druck in der Atmosphäre unter stationären Verhältnissen, ibid. 59, 1902.
—»—	Über die Temperaturverteilung in den allerhöchsten Luftschichten. Arkiv för Mat., Astr. och Fysik 3, no. 25, Stockholm 1907.

depending upon a potential. But now it shall be subject to no other restrictive conditions than those following from the equations of equilibrium (a) themselves.

Multiplying the equations (a) by the projections dx, dy, dz of any line-element ds, and adding, we get,

$$dp = \varrho\,(X'dx + Y'dy + Z'dz) \quad \text{, or,} \tag{b}$$

$$X'dx + Y'dy + Z'dz = \alpha dp \tag{c}$$

The trinomic expression $X'dx + Y'dy + Z'dz$ has a simple dynamical interpretation. It represents *the elementary work performed upon a unit mass which is displaced the length ds in the field of the exterior force.* Thus equations (b) or (c) give the conditions of equilibrium in the simple form: the differential of pressure along a line-element is ϱ times the elementary work along the same element or, the work along a line-element is α times the differential of pressure along the same element.

To see the meaning of this statement in a more developed form, we first remark that the differential equation of an isobaric surface is

$$dp = 0. \tag{d}$$

Introducing this in any of the equations (b) or (c), we get

$$X'dx + Y'dy + Z'dz = 0. \tag{e}$$

As now (d) leads to (e) and, vice versa, (e) leads back to (d), we see that *every isobaric surface is a surface of zero work in the field of the exterior force,* and vice versa, *every surface of zero work in the field of the exterior force is an isobaric surface in the fluid.*

Now a surface of zero work means a surface which is normal to the exterior force. If it were hard and smooth, an investigator, who is under the action of the force, should be able to walk upon it without sliding, and he would call it *level.* As a vector of unlimited generality has no normal surfaces, a force of the most general type will consequently have no surface which might be called level. On the other hand, the existence of level surfaces is merely a necessary, but not a sufficient condition for making the force conservative. For this it must be demanded in addition that the transfer of a unit mass from one level surface to another requires an uniquely determined amount of work. Thus we conclude that forces may be imagined which have level surfaces, but still are non-conservative, as the work of transfer from one level surface to another depends upon the path of transfer.

When we are now to deal with work in the fields of such non-conservative forces we must specify the path of transfer. We shall agree to choose it as a straight line-element leading from one level surface to an infinitely near one. The work along this element is independent of its angle with the surfaces, and therefore equal to the work along the normal line-element. It is therefore called the work of transfer *across* the elementary level sheet. This elementary work of transfer across a level sheet varies from place to place along the sheet as long as the force is non-conservative. But it becomes a characteristic constant for the sheet, equal to the differential of potential between its boundary surfaces, when the force is conservative.

We may now choose a unit of pressure of suitable order of magnitude, so that we can put $dp = 1$ in equations (b) or (c). We see then that the work of transfer across an isobaric unit sheet is equal to the specific volume, or to the reciprocal of the density which the fluid has at the place. Summing up we may then state the conditions of equilibrium in the following form which will be found convenient when the problem is to determine the field of mass:

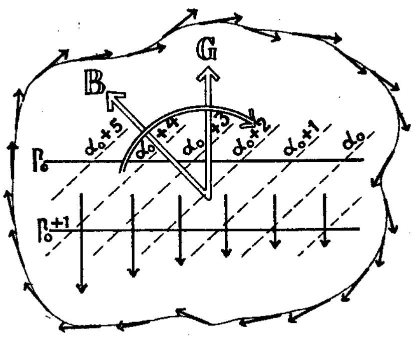

Fig. 1 A. Baroclinic field of mass; variable work of transfer; non-conservative force.

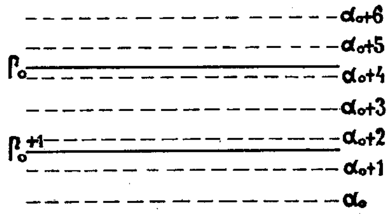

Fig. 1 B. Barotropic field of mass; constant work of transfer, conservative force.

(A) *The exterior force may be non-conservative, but must have level surfaces.*

(B) *The isobaric surfaces must coincide with these level surfaces.*

(C) *Specific volume represents the direct and density the reciprocal value of the work of transfer across an isobaric unit sheet.*

As long as the force is non-conservative the work of transfer varies from place to place along a sheet. The same will then be the case with density and specific volume and consequently the surfaces representing the field of mass must cut those representing the field of pressure: the distribution of mass is baroclinic. (Fig. 1 A). Vice versa, for baroclinic ditribution of mass, when the two kinds of surfaces cut each other, the elementary work must vary along the sheet, and the force which maintains the equilibrium is non-conservative.

But as soon as the work of transfer becomes constant, and thus the force conservative, density or specific volume will be invariable along the isobaric sheets. That is, the surfaces representing the field of mass will coincide with those representing the field of pressure: the distribution of mass is barotropic (Fig. 1 B). Vice versa, a barotropic field of mass, with its constancy of density or specific volume along the isobaric surfaces, involves a corresponding constancy of the work of transfer across the isobaric sheets. The force will be conservative, and the work of transfer from one isobaric or level surface to another will represent the difference of potential between these surfaces.

In this case of barotropy and conservative forces the numerical relation between the fields of pressure, potential and mass takes the simple form: the number expressing the specific volume represents the number of equipotential unit sheets contained in an isobaric unit sheet; or the number expressing the density represents the number of isobaric sheets contained in an equipotential unit sheet.[1])

4. *Energetics of the non-conservative force.* — As seen from the figure the baroclinic case is characterized by asymmetry, the barotropic by symmetry of the field of mass relative to the direction of the force. If in the figure 1 A, we had had the same conservative force as in 1 B evidently no equilibrium would have been possible. The lighter masses to the left would have moved upwards relatively to the heavier to the

[1]) *Bjerknes*: Dynamic Meteorology and Hydrography, T. I. Statics, page 47. Washington 1910.

right. A turning motion would have set in, round axes tangential to the isobaric-isosteric curves, in the direction from the vector **B**, the volume ascendant, to the vector **G**, the pressure gradient. This tendency to form a rotary motion round the isobaric-isosteric curves is very important. We take the opportunity to use it for defining a positive direction of these curves and the corresponding tubes:

(A) *Isobaric-isosteric curves and tubes will be counted positive in the direction which is positive relative to the direction of rotation from the volume ascendant* **B** *to the pressure gradient* **G**, *when the positive screw rule is used.*

We recall at the same time the following principle for the determination of signs: A given positive direction along a closed curve determines by the positive screw rule a positive direction of the normal to the surface which has this closed curve as its contour. We can therefore speak of curves and tubes cutting through this surface in the positive and negative direction, and count algebraically the number of tubes cutting through the surface or comprised by the curve.

Thé above mentioned production of turning motion is prevented through the non-conservative nature of the force. To get a measure for the required degree of non-conservatism we can move a unit mass along a closed contour in the field of the force and measure the amount of work to be performed for the accomplishment of the cycle. We can call this work the *energetic value of the cycle.* It is identically zero for every cycle in the field of a conservative force, but differs from zero when the force is non-conservative.

We shall first consider a special cycle which is accomplished by the following four steps:

1. A displacement across a unit isobaric sheet from the surface p_0 to the surface $p_0 + 1$ along the isosteric surface of specific volume α_0,
2. A displacement along the surface $p_0 + 1$ to a place where the specific volume is $\alpha_0 + n$,
3. A displacement along the isosteric surface $\alpha_0 + n$ back to the isobaric surface p_0,
4. A displacement along the isobaric surface p_0 back to the starting point.

The displacement (2) and (3) require no work, as the isobaric surfaces are level relative to the force. But by displacement (3) we have to yield the work $\alpha_0 + n$ against the force which entertains the fluid equilibrium while by the displacement (1) we gain he work α_0 from this force. For a complete cycle we have thus to supply the amount of work n, equal to the number of isobaric-sosteric solenoids which are contained in the isobaric sheet between the places where the transfer takes place, or contained in the contour along which the unit mass has been moved. If an isosteric surface cuts the isobaric sheet more than once the number n of solenoids must be counted algebraically in accordance with the rule just given.

The result is easily generalized by considerations according to a well known scheme: any cycle may be considered as equivalent to a succession of cycles of the special type considered; therefore to convey a unit mass along any closed curve an amount of work will be required equal to the sum of the amounts for each of the special cycles; and this leads immediately to the following results:

(B) *The energetic value of any cycle in the field of a non-conservative force which maintains baroclinic equilibrium is equal to the algebraical number of isobaric-isosteric solenoids embraced by the closed contour of the cycle.*

5. *The field of mass in a moving fluid.* — By the principle of d'Alembert we pass from the equations of equilibrium to those of motion in introducing the force of inertia. Then let u, v, w be the velocity of a fluid particle, $\dot{u}, \dot{v}, \dot{w}$ its acceleration, and $-\dot{u}$, $-\dot{v}$, $-\dot{w}$ consequently the corresponding force of inertia, referred to the unit mass of the particle. Adding it to the exterior force per unit mass X, Y, Z, we get the »apparent force «

$$X' = X - \dot{u} \;, \quad Y' = Y - \dot{v} \;, \quad Z' = Z - \dot{w}. \tag{a}$$

When we give this value to the force X', Y', Z' in equations 3 (a) of fluid equilibrium these become the required equations of motion.

The apparent force (a) has an obvious dynamical interpretation. Let us solidify the boundary surface of a fluid element, and remove the masses within it, while care is taken that it continues to move just as before. An observer who is placed within this cage will feel himself subject to the force (a). And it will be natural for him to refer to this force the phenomena which he observes in the fluid surrounding him.

But this application to hydrodynamics of the formal laws of hydrostatics is allowed only after the extension of these laws to the case of non-conservative forces. It is true that the exterior force X, Y, Z is always conservative. But the force of inertia $-\dot{u}, -\dot{v}, -\dot{w}$ which is present and disappears with the motion, has in general that rotational distribution in space, which would lead to contradiction with the principle of the conservation of energy in case of a permanently acting force X, Y, Z.

Remembering this we can discuss the field of mass in its relation to the fields of pressure and motion by means of the laws of generalized hydrostatics which we have just developed. Or we find explicitly for a moving fluid:

(A) *The apparent force, which is the vector sum of exterior force and force of inertia is in general non-conservative, but has level surfaces.*

(B) *The isobaric surfaces coincide with the level surfaces of this apparent force.*

(C) *Specific volume represents the direct, and density the reciprocal value of the work of transfer across an isobaric unit sheet.*

6. *The general vortex theory.* — Without entering upon the detailed demonstration we shall mention the fundamental theorems of the general vortex theory. The fundamental quantities of this theory are a scalar, the *circulation*, and a vector, the *vorticity*. The circulation is defined as the line integral of the velocity, i. e., of the tangential component calculated along a closed curve.

$$C = \int u dx + v dy + w dz \tag{a}$$

The vorticity is defined as the curl of the velocity,

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{b}$$

and has the kinematical significance of the double angular velocity of the moving medium at the considered place. The vortex lines thus run tangential to the instantaneous axes of rotation. As the vorticity is a solenoidal vector it may be represented completely by the unit vortex tubes: it is everywhere directed tangential to the axis of the tube, and is numerically given by the reciprocal value of its cross-section.

It is important to emphasize two different ways of considering circulations and vorticities. We may consider them locally: The vorticity *at* a resting geometrical point, and the circulation *along* a resting geometrical curve in space. Or we may consider them as belonging to the individual elements of fluid: the circulation *of* a moving physical curve, or the vorticity *of* a moving particle. These distinctions are of fundamental importance when we have to consider time-derivatives of the quantities, while they are irrelevant as long as we consider geometrical properties at a given moment of time.

Thus irrespectively of the one or the other interpretation, circulation and vorticity are analytically connected with each other by the identity called *Stokes's Theorem*, which we may express thus:

(A) *The circulation — along a geometrical curve or of a physical curve — is equal to the number of vortex tubes comprised by the curve.*

On the other hand the laws giving the dynamics of circulation or of vortex motion refer to moving curves or moving particles. We have first to observe a kinematical identity due to Lord Kelvin. For every moving closed curve we have

$$\text{(c)} \qquad \frac{d}{dt}\int u dx + v dy + w dz = \int \dot{u} dx + \dot{v} dy + \dot{w} dz$$

i. e., the rate of increase of the circulation of the curve is identical with the line-integral of the acceleration along it. But for this line-integral we may find another expression by the hydrodynamical equations. And in reality we have already found it: we have only to bring the theorem 4 (B) into application to the apparent non-conservative force 5 (a) in the moving fluid, considering the field of the force *at a definite time t*. Then the conservative part X, Y, Z of this force gives no contribution to the energetic value of a cycle, all comes from the non-conservative part, the force of inertia $-\dot{u}, -\dot{v}, -\dot{w}$. But the work performed against this force is under the defined conditions simply identical with the line-integral (c) of the acceleration. This line integral is on the one side identical with the rate of increase of the circulation, and on the other side equal to the energetic value at the time t of the corresponding cycle. Using theorem 4(B) and representing by N the number of solenoids we find

$$\text{(d)} \qquad \frac{dC}{dt} = N$$

or in words:

(B) *The rate of increase of the circulation of any moving curve is equal to the number of isobaric-isosteric solenoids comprised by the curve.*

This law is illustrated by the Fig. 1 (A), which we have already used to illustrate the asymmetry of the field of mass, and the non-conservatism of the force which entertained the baroclinic equilibrium. By Stokes's Theorem we may derive the corresponding law for the generation of vorticity.

When the number N of isobaric-isosteric solenoids is zero, we come back to the barotropic case. Then equation (d) makes C constant, and the theorem of the formation of circulations is reduced to the theorem of Lord Kelvin: *the circulation of a moving curve in a barotropic fluid is conserved.* From this we deduce, by use of the theorem of Stokes, Helmholtz's principle of the conservation of vortices: a material line which is once a vortex line will always remain a vortex line; and the fluid mass which once forms a vortex tube will always form a vortex tube, of invariable »strength«, or content of unit vortex tubes.

7. *General characteristics of baroclinic and barotropic conditions.* — Summing up we see in the following four points the contrasts presented between the general baroclinic and the special barotropic conditions:

Baroclinic field of mass:	*Barotropic field of mass:*
1. Variable work of transfer across one and the same isobaric sheet.	1. Constant work of transfer across an isobaric sheet.
2. Non-conservative apparent force.	2. Conservative apparent force depending upon a potential.
3. Rotational distribution of acceleration.	3. Irrotational distribution of acceleration.
4. Formation and annihilation of circulations and vortices.	4. Conservations of circulations and vortices.

Thus in classical hydrodynamics we have, in consequence of the intrinsic barotropy, permanent coincidence of three sets of surfaces, those representing the field of mass, those representing the field of pressure, and those representing the potential of the apparent force. Using this latter potential as one of the fundamental variables, we find the formal laws of ordinary hydrostatics fulfilled in hydrodynamics: thus the specific volume represents the number of equipotential unit sheets contained in an isobaric unit sheet, or the density represents the number of isobaric unit sheets contained in an equipotential unit sheet. By this rule we can derive the field of mass from the fields of pressure and of motion.

To solve the same problem in the general case of a baroclinic fluid we must go back to the determination of the work of transfer across the isobaric sheets, 5 (C).

8. *Pressure and isobaric surfaces.* — Introducing the value 5 (a) of the apparent force into equation 3 (b), we get for the differential of pressure during the motion

(a) $$dp = \varrho\,[(X - \dot{u})\,dx + (Y - \dot{v})\,dy + (Z - \dot{w})\,dz]\,,$$

Consequently the differential equation of the level surfaces of the apparent force and of the isobaric surfaces will be simultaneously

(b) $$(X - \dot{u})\,dx + (Y - \dot{v})\,dy + (Z - \dot{w})\,dz = 0\,,$$

while the differential equation for the level surfaces of the true exterior force, or for the isobaric surfaces in the case of equilibrium is simply

(c) $$X dx + Y dy + Z dz = 0.$$

When the motion is known, the field of pressure can always be found by integration of equation (a). If it has succeeded as an introductory problem by integration of (b) to find the equation of the isobaric surfaces

(d) $$\Psi(x\,,y\,,z) = \text{const.}$$

we know that the integral of (a) will be of the form

$$p = F(\Psi). \tag{e}$$

The integration will always involve implicitly the determination of the corresponding field of mass. And we can make it explicit by use of equation (a), or by the general theorems 5 (A)—(C).

In the barotropic case all integrations may be reduced upon quadratures. First we know that the exterior force X, Y, Z always depends upon the potential

$$\Phi_e = -\int Xdx + Ydy + Zdz \tag{f}$$

Then in the barotropic case even the force of inertia $-\dot{u}, -\dot{v}, -\dot{w}$, depends upon a potential, viz.

$$\Phi_i = \int \dot{u}dx + \dot{v}dy + \dot{w}dz \tag{g}$$

The potential of the apparent force will then be

$$\Phi' = \Phi_e + \Phi_i, \tag{h}$$

and $\Phi' =$ constant represents any level surface of this force or any isobaric surface in the fluid.

It is useful to remark that the equation (h) gives a very convenient graphical method of determining the isobaric surfaces. As soon as the two partial potentials (f) and (g) have been determined, we may represent each of them graphically by their equiscalar surfaces, and then find the corresponding equiscalar surfaces representing the potential (h) by graphical addition of the two partial fields. We shall make extensive use of this method below.

After the potential Φ' has been found, we may write (a) in the form $dp = -\varrho d\Phi'$ and find the pressure for given values of the potential by the integral

$$p = -\int \varrho d\Phi' \text{ which gives } p = F(\Phi'). \tag{i}$$

Or we may work with (a) in the form $d\Phi' = -\alpha dp$ and find the potential for given values of the pressure by the integral

$$\Phi' = -\int \alpha dp \text{ which gives } \Phi' = f(p). \tag{j}$$

The last integral (j) is performed by use of the barotropic relation in the form $\alpha = f_1(p)$. And the integral (i) must be performed by use of the corresponding relation existing between density and potential.

The integrations (i) or (j) are the same as those by which the fundamental hydrostatical problems are solved. As examples we may write down the following solutions, which are wellknown from hydrostatics: In case of a homogeneous and incompressible liquid $\varrho = \varrho_0$, or $\alpha = \alpha_0$, we get

$$p = p_0 - \varrho_0 \Phi' \tag{k}$$

When the fluid is a perfect gas, and a linear relation exists between temperature and potential

(l) $$\vartheta = \vartheta_0 - \gamma \Phi'$$

we get for the distribution of pressure and of mass[1])

(m) $$p = p_0\left(1-\frac{\gamma}{\vartheta_0}\Phi'\right)^{\frac{1}{R\gamma}}, \quad \varrho = \varrho_0\left(\frac{p}{p_0}\right)^{1-R\gamma} = \varrho_0\left(1-\frac{\gamma}{\vartheta_0}\Phi'\right)^{\frac{1}{R\gamma}-1},$$

R denoting the gas constant. When γ converges to zero we get the isothermal case

(n) $$\vartheta = \vartheta_0\ , \ p = p_0 e^{-\frac{\Phi'}{R\vartheta_0}}, \quad \varrho = \varrho_0 \frac{p}{p_0} = \varrho_0\, e^{-\frac{\Phi'}{R\vartheta_0}}$$

Introducing from (h) the value of Φ', we find the same distributions (m) or (n) as functions of coördinates and time x, y, z, t. These solutions, which always represent barotropic distributions of mass, may be used to exemplify an important difference between accidental and intrinsic barotropy.

If we make two fluid masses, which are situated under different pressure, interchange their position, they will then spontaneously interchange even their density only when the exponent $R\gamma$ has the special value $R\gamma = \frac{\varkappa - 1}{\varkappa}$, $\varkappa$ being the ratio of the two specific heats of the gas. And only in this case, i. e., of indifferent equilibrium, we have the intrinsically barotropic field of mass which retains its barotropy even by motion.

9. *Discontinuity in the field of mass.* — The case may appear that one of the fundamental quantities changes its value very rapidly in a thin layer. When the thickness of the layer decreases it may finally be impossible to decide if we have still con-

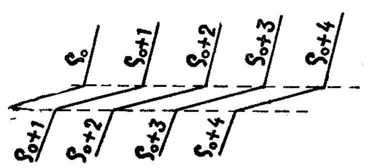

Fig. 2 A. Density varying rapidly in a layer of finite thickness.

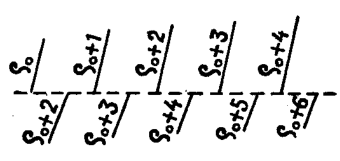

Fig. 2 B. Abrupt change of density at a surface of discontinuity.

tinuous variation in a layer of finite thickness or an abrupt change at a surface. According as it may be convenient we consider ourselves in such cases entitled to substitute a layer of continuous rapid variation to a surface of discontinuity, or *vice versa*, a surface of discontinuity to a layer of continuous rapid variation.

Let first a layer be given in which the density changes continuously but much more rapidly than elsewhere (Fig. 2 A). The equisubstantial surfaces must then occur in greater number inside than outside this layer; as they enter it on the one side they are

[1]) *Bjerknes*, Dynamic Meteorology and Hydrography, T. I., Chap. V.

refracted to approximate parallelism with it and then refracted back again as they go out on the other side.

Now let the thickness of this layer decrease infinitely (Fig. 2 B). Inside it the different equisubstantial surfaces then come to coincidence over finite lengths. The layer is changed into a surface of discontinuity which everywhere contains a number of coinciding equisubstantial surfaces. Every such surface which immerges into the surface of discontinuity from the one side will follow it over a finite length before it emerges on the other. Wherever we pass through the surface we find an abrupt change of density equal to the number of absorbed isopycnic surfaces or an abrupt change of specific volume equal to the number of absorbed isosteric surfaces. In the simplest case that the fluid layers on the two sides are homogeneous, the surface of discontinuity consists of a group of equisubstantial surfaces which have come to coincide in their entire length.

While thus any discontinuity may be presented in the field of mass, the field of our other fundamental scalar, pressure, is necessarily continuous. For in virtue of the principle of equal action and reaction, the pressure must have the same value on the two sides of any surface.

10. Discontinuities in the field of motion. — Let the fundamental vector in the field of motion, velocity, be represented by its lines and tubes of flow, and let these

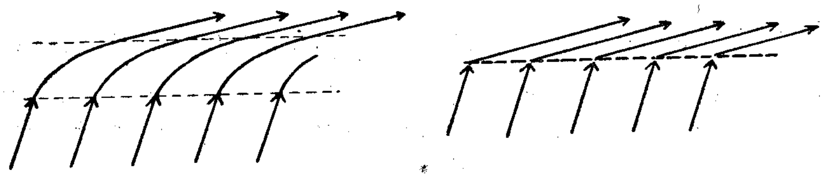

Fig. 3 A. Curvature of streamlines in a layer of finite thickness. Fig. 3 B. Refraction of stream-lines at a surface of discontinuity.

pass a layer in which they change direction with any degree of rapidity (Fig. 3 A). If then the thickness of this layer converges to zero, we get a sudden refraction of the lines and tubes at a surface of discontinuity (Fig. 3 B).

In a case of this kind the normal velocity component must have the same value on both sides of the surface. For otherwise there would be an outflow from or an inflow to it, involving the impossible idea of a creation or annihilation of mass at the surface. If u, v, w be the velocity on the one side and u', v', w' on the other, this condition of the continuity of the normal component may be expressed by the equation

(a) $$(u - u') \cos n, x + (v - v') \cos n, y + (w - w') \cos n, z = 0$$

n being the normal to the surface. If we apply vector notations and represent by $\mathbf{v}$ the velocity as a vector, by $\mathbf{N}$ the unit normal to the surface, and use the dot notation of *Gibbs* for the scalar product, we may write the same equation in the form

(b) $$\mathbf{N} \cdot (\mathbf{v} - \mathbf{v}') = 0.$$

The equation says directly that the discontinuity consists in a sliding motion tangential to the surface of discontinuity represented by the two-dimensional vector

(c) $$\mathbf{S} = \mathbf{v} - \mathbf{v}'$$

which with its vector lines is contained in the surface. **S** may be found by the parallelogram construction performed upon the vectors **v** and **v**′ directly: their vector difference **S** falls always in the surface of discontinuity, and their plane cuts this surface along the tangent to the vector lines of **S**. We may also first project the vectors **v** and **v**′ on the surface, and then perform the construction in the surface with these two-dimensional vectors.

This sliding or slipping motion, which represents the discontinuity in the field of velocity, may be considered as the ultimate case of a shearing motion in a layer of finite

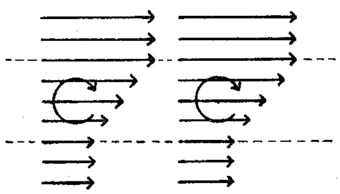

Fig. 4 A. Shearing motion in a layer of finite thickness.

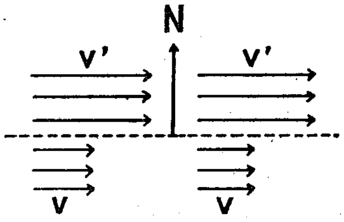

Fig. 4 B. Sliding motion at a surface of discontinuity.

thickness (fig. 4). Let us consider a curve which consists of four parts: a part I of finite length situated in the one boundary surface of the layer where the velocity is **v**; an infinitely short part II going through the layer; a finite part III in the other boundary surface of the layer, where the velocity is **v**′, everywhere opposite to part I; an infinitely short part IV which goes across the layer and closes the curve. It is seen that the circulation of this curve is obtained simply by integrating the velocity difference **v** — **v**′ along the part I (or along the part III) of the curve. This circulation retains its value when the thickness of the layer converges to zero. As long as the layer has a finite thickness Stokes's Theorem gives a distribution of finite vorticity which characterizes the shearing motion.

If the motion is simply parallel to the plane of the figure, the vortex lines and tubes will be normal to this plane, and, in conformity with the positive screw rule, be directed away from the reader. If the motion is irrotational on both sides of the layer, the vortex lines and tubes will be confined to the layer. But if we have the rotational motion on two sides, the vortex lines and tubes will enter the layer from the one side, and emerge from it on the other. At the boundary surfaces of the layer they will undergo a refraction, so that in the layer they hold a course nearly parallel to it. This is illustrated in figure 5 (A) which is drawn in a plane normal to that of figure 4 (A).

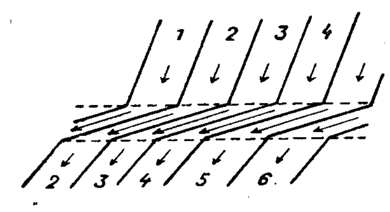

Fig. 5 A. Vortex tubes passing a layer of shearing motion.

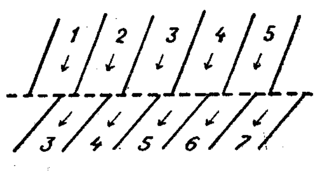

Fig. 5 B. Vortex tubes apparently interrupted by a surface of discontinuity.

When the layer becomes infinitely thin, the vortex tubes in it will become infinitely flat bands, which cannot be seen in the figure 5 (B). But they may still be imagined to establish a uniform correspondence between the tubes on the two sides: as seen from

Stokes's Theorem the correspondence will be complete if we introduce the idea of *a surface vortex* which has the numerical value of the velocity of slip, and is normal to it. In vector symbols it may be represented by the vector product of the unit normal to the surface in the velocity of slip

$$\text{(d)} \qquad \mathbf{N} \times (\mathbf{v} - \mathbf{v}')$$

This surface vortex must always be taken into account when full generality shall be given to the theorems concerning the geometrical and dynamical properties of vortices. Therefore wherever we have a sliding motion of surface upon surface the corresponding surface vortices must always be remembered.

Finally we remark that discontinuous velocity will in general lead to discontinuous acceleration. The discontinuity in this vector will appear, not merely in the tangential but also in the normal component. A discontinuous tangential velocity component gives a discontinuous normal component of acceleration which depends upon the curvature of the surface, and can only disappear under exceptional conditions. Vice versa, discontinuity of acceleration involves discontinuity of velocity. If there is a discontinuity in the tangential component of the acceleration this shows that the production of a discontinuity in the velocity is going on, sliding motion and surface vortices being formed. If the discontinuity is limited to the normal component of the acceleration, no change is taking place in the correlated discontinuity of velocity: the velocity of slip and the surface vortices are conserved.

On account of this close relation between the different kinds of kinematical discontinuity, we may generally speak of a *surface of discontinuity in the field of motion.* This expression will in general involve a discontinuity of the acceleration which may extend to both components, and a discontinuity of the velocity limited to the tangential component. Finally it involves the existence or the formation of surface vortices.

11. Determination of a surface of discontinuity. — Now let us represent pressure, density, velocity, and acceleration on the one side of a surface of discontinuity by p, ϱ, u, v, w, $\dot{u}$, $\dot{v}$, $\dot{w}$, and on the other side by p', ϱ', u', v', w', $\dot{u}'$, $\dot{v}'$, $\dot{w}'$. Then in the first place the kinematical surface condition 10 (a) must be fulfilled. In the second place the dynamical condition of equal action and reaction leads to the same value of the pressure on the two sides. For all points which at the time t form the surface we must have

$$\text{(a)} \qquad p(x, y, z, t) - p'(x, y, z, t) = 0$$

Thus when the two fields of pressure $p(x, y, z, t)$ and $p'(x, y, z, t)$ are known, their difference equated to zero represents the equation of the surface of discontinuity in finite form.

This principle is especially useful for the graphical construction of the surface of discontinuity when we know the graphical representations of the pressure in the fluid on both sides of it. We have simply to perform the graphical subtraction of the two fields, the surface for difference zero will then be the required surface. The construction is of special importance for the investigation of such surfaces in atmosphere and sea. For while observations from the surface of discontinuity itself may be rare, we may often have sufficient observations for determining the field of pressure in the surrounding sea or atmosphere. Then the course of the surface separating the two currents is found simply by graphical subtraction of the two fields of pressure.

If the pressure is unknown we may eliminate it. Differentiating (a) we get

(b) $$dp - dp' = 0$$

Here the differentials of pressure have the values

(c) $$dp = \varrho\,[(X - \dot{u})\,dx + (Y - \dot{v})\,dy + (Z - \dot{w})\,dz]$$

(d) $$dp' = \varrho'\,[(X - \dot{u}')\,dx + (Y - \dot{v}')\,dy + (Z - \dot{w}')\,dz]$$

Equating separately dp and dp' to zero, we get the differential equations of the isobaric surfaces on the two sides

(e) $$(X - \dot{u})\,dx + (Y - \dot{v})\,dy + (Z - \dot{w})\,dz = 0$$

(f) $$(X - \dot{u}')\,dx + (Y - \dot{v}')\,dy + (Z - \dot{w}')\,dz = 0$$

Introducing the expressions (c) and (d) into (b), and dividing by the factor $\varrho - \varrho'$, we get the differential equation of the surface of discontinuity in the form

(g) $$(X - \dot{u}^*)\,dx + (Y - \dot{v}^*)\,dy + (Z - \dot{w}^*)\,dz = 0$$

where

(h) $$\dot{u}^* = \frac{\varrho\,\dot{u} - \varrho'\,\dot{u}'}{\varrho - \varrho'} \quad , \quad \dot{v}^* = \frac{\varrho\,\dot{v} - \varrho'\,\dot{v}'}{\varrho - \varrho'} \quad , \quad \dot{w}^* = \frac{\varrho\,\dot{w} - \varrho'\,\dot{w}'}{\varrho - \varrho'}$$

We have thus represented the surface of discontinuity by an equation of the same form as that for the isobaric surfaces (e) or (f). The latter surfaces are normal to the apparent forces $X - \dot{u}$, $Y - \dot{v}$, $Z - \dot{w}$ and $X - \dot{u}'$, $Y - \dot{v}'$, $Z - \dot{w}'$ respectively, and the surface of discontinuity normal to the complex apparent force $X - \dot{u}^*$, $Y - \dot{v}^*$, $Z - \dot{w}^*$.

Two extreme cases are important. When ϱ' converges to ϱ so that the discontinuity of density disappears, it is easily seen that equation (g) is reduced to the form

(i) $$(\dot{u} - \dot{u}')\,dx + (\dot{v} - \dot{v}')\,dy + (\dot{w} - \dot{w}')\,dz = 0$$

Thus in this case the course of the surface of discontinuity is independent of the exterior force X, Y, Z. It runs normal to the vector $\dot{u} - \dot{u}'$, $\dot{v} - \dot{v}'$, $\dot{w} - \dot{w}'$ which represents the discontinuity of acceleration. When on the other hand the discontinuity of acceleration drops out, $\dot{u}'$ converging to $\dot{u}$, $\dot{v}'$ to $\dot{v}$, $\dot{w}'$ to $\dot{w}$ it is seen that we get

(j) $$\dot{u}^* = \dot{u}' = \dot{u} \;,\; \dot{v}^* = \dot{v}' = \dot{v} \;,\; \dot{w}^* = \dot{w}' = \dot{w}$$

which shows that there is a full identity of the three equations (e), (f), (g). That is, the surface of discontinuity must coincide with an isobaric surface.

Both cases, (i), and (j), agree in respect to the fact that there is no discontinuity in the tangential component of the acceleration, i. e., the acceleration fulfills the irrotational surface condition. This involves the fact that a tangential discontinuity of velocity, if there is any, suffers no change, so that no formation of surface vortices is going on. Thus all the features characterizing the barotropic state are present in so far as conditions on the surface are concerned. If, vice versa, we suppose that there is no formation of surface vortices, this involves continuity of the tangential component of acceleration.

This condition may be fulfilled in two ways: the surface of discontinuity may be normal to the discontinuity of acceleration, and thus fulfill equation (i); but this can be identical with (g) only by the condition that $\varrho' = \varrho$, which excludes discontinuity in the field of mass. Or the acceleration may be entirely continuous, which leads back to equation (j), involving the coincidence of the surface of discontinuity with an isobaric surface.

To appreciate the content of these results we must remember that a surface of discontinuity in the field of mass contains a complex of equisubstantial surfaces, so that its intersection with the isobaric surfaces is a baroclinic phenomenon. and loses this character only in the two cases: when it either coincides with an isobaric surface, or when the supposed discontinuity in the mass distribution drops out. We may then concisely sum up the general result thus:

(A) *The general surface of discontinuity is a phenomenon of baroclinic character but with two cases of degeneration to barotropic discontinuity:*

1°. *when the discontinuity is merely substantial, i. e. appears only in the field of mass: then the surface of discontinuity coincides with an isobaric surface;*

2°. *when the discontinuity is merely kinematical, i. e. appears only in the field of motion: then the surface of discontinuity runs normal to the discontinuity of acceleration.*

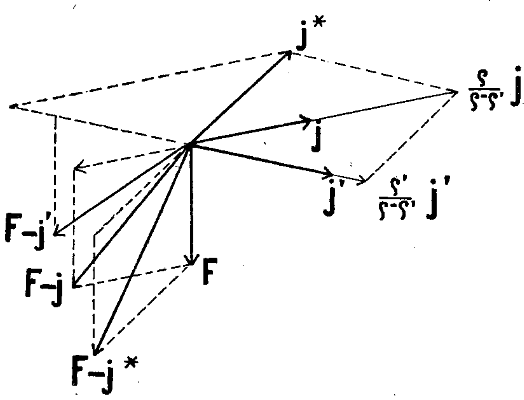

Fig. 6. Construction of the normal $\mathbf{F} - \mathbf{j}^*$ to a surface of discontinuity.

The course of the surface of discontinuity relatively to the isobaric surfaces may easily be constructed. Let the accelerations $\dot{u}, \dot{v}, \dot{w}$ and $\dot{u}', \dot{v}', \dot{w}'$ and the complex acceleration $\dot{u}^*, \dot{v}^*, \dot{w}^*$ be represented by the vector symbols $\mathbf{j}$, $\mathbf{j}'$ and $\mathbf{j}^*$ respectively. Then according to the formula (h) the vector $\mathbf{j}^*$ is given by the vector equation

$$\text{(k)} \qquad \mathbf{j}^* = \frac{\varrho}{\varrho - \varrho'}\,\mathbf{j} - \frac{\varrho'}{\varrho - \varrho'}\,\mathbf{j}'$$

and can be constructed as shown in fig. 6.

When $\mathbf{F}$ represents the exterior force X, Y, Z as a vector we can proceed further as shown in the figure to form the vector differences $\mathbf{F} - \mathbf{j}$, $\mathbf{F} - \mathbf{j}'$, and $\mathbf{F} - \mathbf{j}^*$ which are the normals respectively to the isobaric surfaces and the surface of discontinuity. The construction gives directly the coincidence of all three surfaces when $\varrho = \varrho'$. In the other exceptional case we simply construct the vector $\mathbf{j} - \mathbf{j}'$ which is then normal to the surface of discontinuity.

12. *Special choice of coordinates.* — When we deal with the angles which the different surfaces form with each other or with a given plane in space, it will be convenient to make a special choice of coordinates. The true exterior force X, Y, Z is normal to the equipotential surfaces, which, in case of complete equilibrium, are at the same time the isobaric surfaces 8 (c). The apparent force $X - \dot{u}$, $Y - \dot{v}$, $Z - \dot{w}$ is normal to its level surfaces, or the isobaric surfaces during the motion 8 (b). The complex force $X - \dot{u}^*$, $Y - \dot{v}^*$, $Z - \dot{w}^*$ is normal to the surface of discontinuity 11 (g). If we then turn the system of coordinates so that dx becomes tangential to the surface in

question at the considered point, we must introduce $X = 0$ in equation 8 (c); $X - \dot{u} = 0$ in equation 8 (b), and $X - \dot{u}^* = 0$ in equation 11 (g). The three] differential equations may then be written

$$\text{(a)} \qquad \frac{dz}{dy} = -\frac{Y}{Z} \quad , \quad \frac{dz}{dy} = -\frac{Y - \dot{v}}{Z - \dot{w}} \quad , \quad \frac{dz}{dy} = -\frac{Y - \dot{v}^*}{Z - \dot{w}^*} \; .$$

These equations give directly the angles of inclination of the surfaces in question relatively to the xy-plane; the first for the level surfaces of the exterior force, the second for the isobaric surfaces, and the third for a surface of discontinuity where we are to remember the values 11 (h) of $\dot{v}^*$ and $\dot{w}^*$.

II. Various Examples to the Preceding Principles.

13. Vertical acceleration and horizontal acceleration. — Before we take up our main problem it will be useful to illustrate the use of the developed principles and formulae by a few obvious examples. Suppose the force to be constant gravity, $Y = 0$, $Z = -g$, so that formulae 12 (a) for the three kinds of surfaces reduce to

$$\text{(a)} \qquad \frac{dz}{dy} = 0 \quad , \quad \frac{dz}{dy} = -\frac{\dot{v}}{g + \dot{w}} \quad , \quad \frac{dz}{dy} = -\frac{\dot{v}^*}{g + \dot{w}^*} \; .$$

When the horizontal accelerations are zero, $\dot{v} = 0$, $\dot{v}' = 0$, and thus even $\dot{v}^* = 0$, we see that all these equations give the inclination zero. Thus in the case of vertical acceleration both the isobaric surfaces and the surface of discontinuity remain horizontal and coincident with the level surfaces as by equilibrium. The only difference from statical conditions will be a change of distance between the isobaric planes, in inverse proportion to the forces $g/(g + \dot{w})$, and in case of compressibility a changed density corresponding to the new distribution of pressure. The isobaric unit sheets become thinner than under statical conditions by a positive $\dot{w}$, i. e., by acceleration upwards, but thicker by a negative $\dot{w}$ smaller than g, i. e., by an acceleration downwards, smaller than that of free fall. In the case of free fall itself, $\dot{w} = -g$, the sheets become infinitely thick, corresponding to the disappearance of every difference of pressure in the fluid. With still greater acceleration downwards we again get finite thickness of the isobaric sheets but now with pressure inversion in addition. A reversed bottle which is moved downwards with an acceleration so great that the water does not run out furnishes a small scale example of these conditions. In great scale we meet with the same phenomenon by the formation of Saturn Rings considered below. In the great atmospheric motions the vertical accelerations are too small to have an effect on the pressure which can be observed by common meterological instruments. And pressure inversions only appear quite locally and transiently, in sound waves or waves of explosion propagating vertically.

When $\dot{w} = 0$, and thus the acceleration is horizontal, the tangent to the angle of inclination of the isobaric surfaces is $-\dot{v}/g$. Thus it is in the second quadrant, and the slope of the surface has the direction of the acceleration. As the vertical component of the apparent force remains the same as in equilibrium, the *vertical* distance between the isobaric surfaces will remain unchanged, but by virtue of their inclination their *normal* distances are reduced. If a bottle which is completely filled with water is accelerated horizontally, the isobaric surfaces will remain plane, and take the indicated slope in the direction of the acceleration; and in taking this inclination they will retain their vertical distance, but have their normal distance reduced.

Now let the same bottle be filled with two layers of homogeneous and incompressible not-mixing fluids of different densities. A sudden acceleration of the bottle gives again a slope of the isobaric surfaces in the direction of the acceleration. (Fig. 7 A). But this slope is greater in the lighter layer which relatively rushes ahead, and fills the foremost part of the bottle, and smaller in the heavier layer which relatively remains behind, and fills the rear part. The result is the wellknown slope of the boundary surface of the two layers, in the same direction as that of the isobaric surfaces, but smaller. The tangent to the angle of inclination, calculated in the first approximation merely from the horizontal acceleration is $-\dot{v}^*/g$, and in the expression 11 (h) of $\dot{v}^*$ we have $\varrho \dot{v}$ greater than $\varrho' \dot{v}'$ although $\dot{v}'$ is greater than $\dot{v}$. That by the impulsive motion the surface of discontinuity gets a smaller slope than the isobaric surfaces is easily foreseen: the isobaric surfaces take their new position instantaneously while the surface of discontinuity can take it only as a result of the motion. The construction of Fig. 6 A (p. 18) correponds to this case of an impulsively accelerated bottle, when we take into consideration also that the acceleration has a component upwards in the heavy and a component downwards in the lighter fluid.[1])

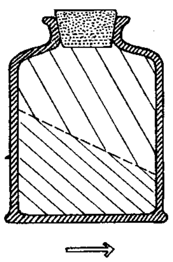

Fig. 7 A. Accelerated bottle.

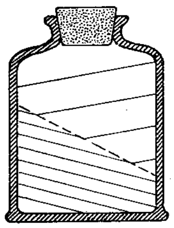

Fig. 7 B. Oscillations in a resting bottle.

Conditions may change completely during the later phases of this motion. When the bottle is left at rest, oscillations will follow. Then the horizontal accelerations of the two fluid layers will always have opposite directions. That is, the isobaric surfaces have opposite slope in the two fluid layers, and the boundary surface gets a slope of the same direction as the isobaric surfaces in the lower layer, but numerically greater (Fig. 7 B). The result is easily seen from a construction as that of Fig. 6, or from the expression 11 (h) of $\dot{v}^*$.

This case of oscillating motions and corresponding wave motions is, however, so important both for the general illustration of the difference in the character of the motion in baroclinic and barotropic fluids, and for the special problems of this paper, that we must consider it in detail.

14. Waves due to gravity. — Waves can in general be seen only on the surface of liquids. But if we were to colour any thin horizontal layer inside the fluid, it should be seen to perform a similar undulating motion as the surface. Such undulating motions may exist invisibly in the interior of a fluid system quite independent of the presence of any visible free surface, and in order to recognize them their character must be known.

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[1]) The refraction of the isobaric surfaces seen in Fig. 7 A is connected with a refraction of the lines of flow which is analogous to the refraction of electric or magnetic lines of force passing from one medium to another. Cf. *V. Bjerknes*: Die Kraftfelder, p. 56. Braunschweig 1909.

Let u_z be the vertical and u_x the horizontal displacement of a particle belonging to an undulating layer. The simplest type of motion of any particle belonging to it will then be represented by the equations:

$$\text{(a)} \qquad u_z = a \cos m (x - ct) , \quad u_x = b \sin m (x - ct) ,$$

a being the vertical, b the horizontal amplitude, $2\pi/m$ the wave length, and c the velocity of propagation. If t is constant and x variable, the first equation gives the profile curve of the waves at the time t as a cosine curve. If then we let t vary, we see the waves of this curve propagate with the velocity c in the direction of the positive x. When, on the other hand, x has definite values, and t is variable, the equations (a) show how this undulating motion of the entire layer results from a revolution of every definite particle in an elliptic orbit, with the vertical axis a and the horizontal axis b, the phase of the motion changing from particle to particle as we proceed in the direction of x. These differences of phase underlie the propagation.

If in the equation (a) we write $x + ct$ instead of $x - ct$, we get a train of waves precisely similar which propagate in the opposite direction. By interference the two motions form standing waves

$$\text{(b)} \qquad u_z = 2\, a \cos mx \cos mct , \quad u_x = 2\, b \sin mx \cos mct .$$

Here all differences of phase from particle to particle drop out, with the effect that we no longer have any propagation. The orbits of the particles are reduced to straight line-elements, which are vertical in crests and troughs, where u_z has its greatest values, horizontal in the nodes where u_x has its greatest values, and have intermediate inclinations in the intermediate points.

In the simplest fluid wave motions, all fluid layers will have a motion of the type (a) or (b) only with amplitudes which vary from level to level

$$\text{(c)} \qquad a = f_1 (z) \quad , \quad b = f_2 (z) .$$

These equations together with (a) or (b) then determine the motion in the entire fluid. To get a picture of this motion we may use the instantaneous stream-lines. They are the same for both cases, standing and propagating waves, with the difference that in the one case they are stationary in space, while in the other they follow the propagation. To find their equations we must know explicitly the functions (c). But certain general features of the lines are seen independently of details.

First, wherever the horizontal velocity is zero, i. e.

$$\text{(d)} \qquad \frac{du_x}{dt} = 0 ,$$

we have a straight vertical stream-line. This is seen in equation (a) or (b) to take place for every half wave-length. On the other hand, in every level where the vertical amplitude is zero, i. e., for all values of z fulfilling the equation

$$\text{(e)} \qquad f_1 (z) = 0,$$

we have a straight horizontal stream-line. Thus, in general, we have two sets of straight stream-lines, one vertical and equidistant, the other horizontal and in general not equidistant, which divide the space into a set of rectangular cells. All other stream-lines are thus bound to be closed curves in these cells.

Fig. 8 gives an example of such lines. It may represent standing oscillations in a heterogeneous salt-solution within a rectangular box. Among the stream-lines, of which the equation will be given below, the innermost is reduced to a point, which defines a nodal line, normal to the plane of the paper. The motion in each cell has the character of a pendulum motion round this nodal line, the heavy masses of the lower strata being lifted alternately on the one side and then the other. The rotative oscillations go oppositely in adjacent cells. The particles which by equilibrium are situated in the horizontal plane through the nodal lines are marked by points in the one, and by small circles in the other extreme position which is reached during the oscillations. They are seen to form a surface in standing oscillations. All other originally horizontal surfaces perform similar oscillations, only with smaller amplitudes.

By a slight modification the same diagram would represent waves which propagate between two rigid horizontal planes. It is sufficient to imagine the waving surfaces dis-

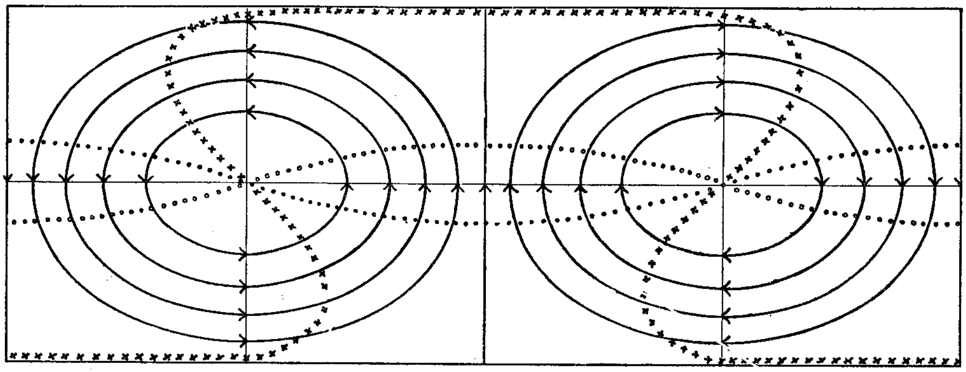


Fig. 8. Standing waves in a fluid with continuous density distribution.

placed $1/4$ wave length relatively to the stream-lines, and the nodal axes in the successive cells to be centered alternately a little higher and a little lower, so that they come at the crests and troughs of the undulating surface, and not at the nodes.

Now we may modify the distribution of mass in the fluid system of Fig. 8. Let it consist of two homogeneous layers, separated from each other by a transitional thin sheet in which the density varies continuously but rapidly. We should in that case get standing or propagating waves of essentially the same character. The stream-lines would remain closed curves within the rectangular cells, only with a slightly varied shape in as much as they show reduced curvature within the homogeneous layers, but increased curvature as they pass through the transitional sheet. Fig. 9 represents the extreme case that this sheet is infinitely thin, and thus we have an abrupt change of density from the one layer to the other and consequently a sudden refraction of the lines of flow. Therefore, analytically these lines are represented by different equations on the two sides of the surface of discontinuity. On the lower side they may resemble common catenaries, on the upper side reversed curves of the same description. If they be continued they would go asymptotically to the vertical stream-lines. That is, the rectangular cell to which each set of these stream-lines belong is open either upwards or downwards. This is the case when the equation (e) has only one root. Further the figure 9 is drawn with the supposition that this root is infinitely distant, namely $z = -\infty$ for the lower and $z = \infty$ for the upper layer. That is, both layers are supposed to be infinitely thick.

The waves of the surface of discontinuity in Fig. 9 have been drawn with relatively great amplitudes, while the known solutions fulfill the equations only in case of infinitely small amplitudes. But just as in the case of the pendulum, we are entitled practically to apply to finite oscillations a solution which is exact only in the case of infinitely small amplitudes. Such strong extrapolation as in Fig. 9 leads, however, to some uncertainty in the details in the diagram; the curves have had to be slightly adjusted to avoid contradiction with the kinematical surface condition.

The relative situation of the stream-lines and the undulating surfaces in Fig. 9 is that which corresponds to propagating, not as in Fig. 8, to standing waves. While therefore, the paths of the particles were simple elements of the stream-lines in Fig. 8, they

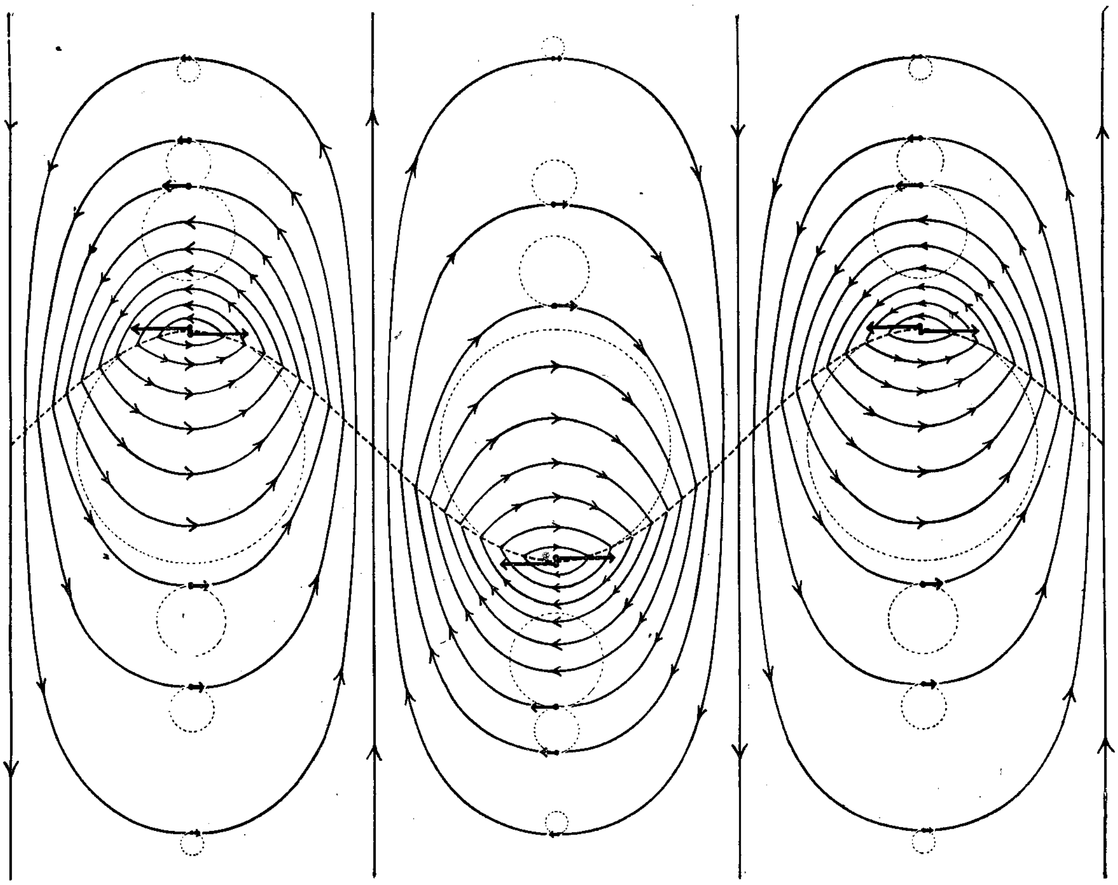


Fig. 9. Propagating waves at an internal surface of discontinuity.

are now closed orbits which on account of the infinite depth of the fluid to both sides have circular form. A number of such orbits are drawn in the figure and are seen to have rapidly decreasing radii as the distance from the surface of discontinuity decreases.

In the case of Fig. 9 if we let the density of the fluid in the upper layer converge to zero, in the ultimate case we are only concerned with one fluid layer with a free waving surface.

15. Complete formulae for simple mass distribution. — It will be useful to write down the complete formulae for a simple case of waves. The general principles by which they may be deduced are given in my paper »Ueber Wellenbewegung in kompressiblen schweren Flüssigkeiten, Erste Mitteilung«.[1]

[1] Abh. d. math. phys. Kl. d. k. sächs. Ges. d. Wiss. Vol. XXXV, No. II, Leipzig 1916.

Let p° and ϱ° be pressure and density at equilibrium, p and ϱ the same quantities during motion. Then the parameter

$$\text{(a)} \qquad \delta^\circ = \frac{d\varrho^\circ}{dp^\circ}$$

characterizes the geometric relation between density and pressure at equilibrium, and

$$\text{(b)} \qquad \delta = \frac{d\varrho}{dp}$$

defines the physical behaviour of a fluid particle during motion. The geometrical equation (a) and the physical equation (b) are to be considered as two special cases of one and the same general physical equation of condition of the form 2 (a). We shall always suppose that

$$\text{(c)} \qquad \delta^\circ \geqq \delta \ .$$

$\delta^\circ < \delta$ may be disregarded as it leads to unstable state of equilibrium. $\delta^\circ = \delta$ gives the case of barotropy and leads to indifferent equilibrium. $1/\sqrt{\delta}$ is the well-known velocity of propagation of sound waves in the medium. $\delta = 0$ gives incompressibility and makes this velocity of propagation infinite. $\delta^\circ = \delta = 0$ gives homogeneity and incompressibility.

For simplicity we shall limit ourselves to the case that these two parameters δ° and δ are constants, i. e., independent of the pressure and thereby of coordinates and time. Then the constancy of δ° in formula (a) leads to the well known exponential decrease of density upwards as function of the gravity potential

$$\text{(d)} \qquad \varrho^\circ = \varrho_0{}^\circ \, e^{-\delta^\circ \Phi} = \varrho_0{}^\circ \, e^{-\Phi/H^\circ}$$

Here $H^\circ = \frac{1}{\delta^\circ}$ is the difference of level, measured in dynamic decimeters, — or »leo-decimeters« as Mr. Whipple has proposed to call them[1]) — which gives the reduction of ϱ° to its e-th part. And this same H° is the height of a homogeneous fluid layer of density $\varrho_0{}^\circ$, which at the level $\Phi = 0$ exerts the same pressure as the unlimited layer with exponentially decreasing density. The height of the »homogeneous atmosphere« which exerts the pressure of 100 cbar at the ground is $H^\circ = 78380$ dynamic decimeters or 7838 dynamic meters.

The parameter δ is approximately independent of p for liquids. For ideal gases we must in general reckon with the adiabatic law, which makes δ dependent upon p and thereby upon coordinates and time. An exception will appear only in isothermal conditions of motion. But in waves of the type which interest us here, we shall have precisely these conditions.

The potential energy of the most general atmospheric waves will be partly due to the elasticity, partly to the gravity of the moving masses. The ultimate case on the one side appears as pure sound waves, in which gravitational energy is insignificant, and the elastic energy is due to adiabatic compression and expansion. The ultimate case on the other side is given by the pure gravitational waves, in which the potential energy is entirely due to the vertical displacement of the moving masses, while expansions and contractions come in only as adjustments following these displacements. In the case of such waves we may suppose δ to be practically constant. It can be demonstrated that the highest velocity of propagation which may be attained by waves of this

[1]) Quarterly Journal of the Royal Meteorological Society, April 1914.

type is the Newtonian, and not the Laplacean velocity of sound. This fact that our waves are propagated with smaller velocity than the Newtonian, and thus *a fortiori* smaller than the Laplacean velocity of sound is important to remember, and is expressed by the *inequality*

(e) $$1 - c^2 \delta > 0$$

In this case, when the potential energy of the waves is of gravitational origin, we shall get the simplest formula by using the gravity potential Φ instead of the height z as coordinate in the vertical direction. Thus, using the *MTS* system of units, we apply horizontally the coordinate x measured in meters, and vertically the coordinate Φ measured in dynamic decimeters (leodecimeters). Then u_Φ shall be the vertical displacement of a particle expressed in dynamic decimeters, and u_x the horizontal displacement expressed in meters. Further A is a parameter of the dimensions of a gravity potential, b a parameter of the dimensions of a length. Instead of the formulae (a) and (b) of the preceding section we thus have, for propagating waves

(f) $$u_\Phi = A \cos m(x - ct) \quad , \quad u_x = b \sin m(x - ct)$$

and for standing waves

(g) $$u_\Phi = 2\,A \cos mx \cos mct \quad , \quad u_x = 2\,b \sin mx \cos mct.$$

We can at any time return to geometric heights by the relations

(h) $$\Phi = gz \quad , \quad u_\Phi = g\,u_z \quad , \quad A = ga.$$

A and b are functions of Φ, which for the case of constant δ° and δ are given as follows. It can be shown that, according as

(i) $$\frac{\delta^\circ - \delta}{c^2} \gtreqless \tfrac{1}{4}\,\delta^{\circ 2} + (1 - c^2\,\delta)\,\frac{m^2}{g^2} \quad ,$$

we have the following three expressions for A, $\mathfrak{A}$ being a constant

(j) $$A = \mathfrak{A} \cdot e^{\frac{1}{2}\delta^\circ \Phi} \sin \mu\Phi \quad , \quad \text{where } \mu^2 = \frac{\delta^\circ - \delta}{c^2} - \tfrac{1}{4}\,\delta^{\circ 2} - (1 - c^2\,\delta)\,\frac{m^2}{g^2}$$

(k) $$A = \mathfrak{A} \cdot e^{\frac{1}{2}\delta^\circ \Phi}\, \Phi$$

(l) $$A = \mathfrak{A} \cdot e^{\frac{1}{2}\delta^\circ \Phi} \sinh \eta\Phi \quad , \quad \text{where } \eta^2 = \tfrac{1}{4}\,\delta^{\circ 2} + (1 - c^2\,\delta)\,\frac{m^2}{g^2} - \frac{\delta^\circ - \delta}{c^2} \quad ,$$

and b is in each case given as a function of A by the formula

(m) $$b = -\frac{1}{m} \cdot \frac{A' - A\,\delta}{1 - c^2\,\delta} \quad ,$$

A' denoting the derivative of A with respect to Φ. In these formulae the horizontal stream-line $\Phi = 0$ serves as horizontal coordinate axis. When as vertical coordinate axis we choose one of the straight vertical stream-lines, i. e., a stationary line for standing and a moving line for propagating waves, we may write the equation of the stream-lines

(n) $$A \cdot e^{\Phi\,\delta}\,(\sin mx)^{1 - c^2\,\delta} = \text{const.}$$

This equation gives an infinite number of straight horizontal stream lines which are equidistant as long as the vertical amplitude A is given by the expression (j). Then the complete system of stream lines is a pattern of equal rectangles, within which we have closed curves. But we only get a single straight horizontal stream line, the horizontal coordinate axis $\Phi = 0$, in the cases (k) and (l). In this case the pattern is changed in as much as the rectangles are open upwards or downwards, and these open rectangles contain curves of the type of hanging or reversed catenaries.

Taking the case (j) with the infinite number of straight horizontal stream lines, and introducing the condition of incompressibility, $\delta = 0$, we get the equation (n) explicitly in the form

$$\text{(o)} \qquad e^{\frac{1}{2}\delta^{\circ}\Phi} \sin \mu\Phi \sin mx = \text{const., where } \mu^2 = \frac{\delta^{\circ}}{c^2} - \tfrac{1}{4}\delta^{\circ 2} - \frac{m^2}{g^2}$$

Fig. 8 is drawn from this equation. In the case (l) we get the corresponding equation

$$\text{(p)} \qquad e^{\frac{1}{2}\delta^{\circ}\Phi} \sinh \eta\Phi \sin mx = \text{const., where } \eta^2 = \tfrac{1}{4}\delta^{\circ 2} + \frac{m^2}{g^2} - \frac{\delta^{\circ}}{c^2} .$$

If now in addition we should introduce the condition of homogeneity it is seen that $\delta^{\circ} = 0$ is incompatible with (o), while from (p) in connection with $\Phi/g = z$ we get

$$\text{(q)} \qquad \sinh (\pm mz) \sin mx = \text{const.}$$

These formulae for the case of a heterogeneous incompressible fluid are due to *Love*[1]).

In all these formulae the horizontal stream line $\Phi = 0$, or $z = 0$, serves as horizontal axis of coordinates. But when this stream line is infinitely distant, we must change the coordinates and (q) then takes the well known simple form

$$\text{(r)} \qquad e^{\pm mz} \sin mx = \text{Const.}$$

which represents the streamlines in infinitely deep layers of a homogeneous and incompressible fluid. The drawing of Fig. 9 is directed by this equation, with the value of $+m$ for the curves above and $-m$ for the curves below the waving surface of discontinuity.

When we have standing or propagating waves in a fluid layer which is contained between two given plane horizontal rigid boundary surfaces, the equation (j) for μ gives directly the velocity of propagation c, respectively the period $2\pi/mc$ corresponding to a given wave length $2\pi/m$. But the problems have in general the form that the planes containing the straight horizontal stream lines have to be found in connection with the determination of the velocity of propagation, the necessary data being given by the kinematical and dynamical conditions which must be fulfilled at a free boundary surface or at any internal surface of discontinuity. This leads to formulae showing a more or less complicated dependency of the velocity of propagation upon different parameters, among which the wave length is one. In one case we have the great simplification that the wave length drops out of the formulae. That is when the waves are long when compared to the depth of the fluid. In this important case of so-called long waves we have a series of simplifications. The stream lines are flattened down to practically straight horizontal lines, and the formulae for the propagation take remarkably simple forms.

[1]) *A. E. H. Love*: Wave Motion in a Heterogeneous Heavy Liquid. Proceedings of the London Math. Soc. 1891 p. 307.

We shall give these formulae for the case when the fluid system consists of two layers each in internal indifferent equilibrium, $\delta^0 = \delta$. The bottom of the lower layer shall be a horizontal plane. The pressure shall be p_2 at the bottom, p_1 at the internal boundary surface, and p_0 at the free surface, and the specific volume shall be α_2 at the bottom, α_1 just below and α_1' just above the internal boundary surface, and α_0 at the free surface. The difference $\alpha_1' - \alpha_1$ is considered small compared to α_1 and α_1'.

Two types of long waves may propagate in this system. The first may be called external waves, because we have maximal undulations at the external and only reduced copies of them at the internal boundary surface. The second type may be called internal waves because we have the maximal undulations at the internal boundary surface, and at the external boundary only a reduced negative copy of them, while between these oppositely undulating surfaces there exists a horizontal plane containing straight horizontal stream lines. We then find for the velocity of propagation of the external waves

$$c = \sqrt{\alpha_2 (p_2 - p_0)} \tag{s}$$

and of the internal waves

$$c = \sqrt{\frac{(\alpha_1' - \alpha_1)(p_2 - p_1)(p_1 - p_0)}{p_2 - p_0}} \tag{t}$$

In the special case when the fluid strata are both homogeneous and incompressible, we may express the differences of pressure by corresponding differences of potential and the formulae are reduced to well known forms. Thus (s) becomes $c = \sqrt{\Phi} = \sqrt{gh}$, Φ being the dynamic and h the geometric height of the free surface above the bottom. This is just the classical rule for the propagation of water waves: it is equal to the velocity obtained by a free particle falling the height equal to the half depth of the water.

We may apply the formulae to an idealized atmosphere consisting of two layers each in internal adiabatic equilibrium. The atmosphere will then be limited to a height of about 27300 meters. At the boundary we have $p = 0$ and formula (s) reduces to

$$c = \sqrt{\alpha_2 p_2} \tag{u}$$

which is the Newtonian sound velocity at sea level. By the gas equation we get $c = \sqrt{R\vartheta_2}$. For the value of the gas constant in the *MTS* system, $R = 287.042$, and for the temperature $\vartheta_2 = 273$ we find $c = 280$ m/sec. This is then the greatest velocity of propagation which long atmospheric waves may get. But short waves will attain the full Laplacean sound velocity when the potential energy due to adiabatic compression is great compared to that due to the displacement of the masses in the gravitational field.

In the formula for internal waves introduce $p_0 = 0$, and express the specific volumes by the temperatures. This gives

$$c = \sqrt{R\frac{(\vartheta' - \vartheta)(p_2 - p_1)}{p_2}} \tag{v}$$

for the velocity of propagation along an internal boundary surface in the atmosphere, when the pressure is p_2 at the ground and p_1 at the boundary surface, while $\vartheta' - \vartheta$ is the increase of temperature when we pass through the surface from below. Equating the pressure at the ground to 100 cbar, and introducing the value of R, we get

(w)
$$c = 1.69 \sqrt{(\vartheta' - \vartheta)\ (100 - p_1)}$$

The annexed table gives the velocities of propagation according to this formula, as function of the pressure p_1 and the drop of temperature $\vartheta' - \vartheta$ at the surface of discontinuity.

Table I. — *Velocities of propagation of internal atmospheric waves.* (m/sec.)

Pressure at the surface of discontinuity, cbar.	Corresp. height in isotherm. atmosph. of 0°C. Dyn. meters	Drop of temperature at the surface of discontinuity. (°C)												
		0.2	0.4	0.6	0.8	1	2	4	6	8	10	15	20	30
0		8	11	13	15	17	24	34	42	48	54	66	76	93
10	18044	7	10	13	15	16	23	32	40	46	51	62	72	88
20	12612	7	10	12	14	15	21	30	37	43	48	59	68	83
30	9435	6	9	11	13	14	20	28	35	40	45	55	64	78
40	7180	6	8	10	12	13	18	26	32	37	42	51	59	72
50	5432	5	8	9	11	12	17	24	29	34	38	47	54	66
60	4003	5	7	8	9	11	15	21	26	30	34	42	48	59
70	2795	4	6	7	8	9	13	18	23	26	29	36	42	51
80	1749	3	5	6	7	8	11	15	18	21	24	29	34	42
90	826	3	3	4	5	5	8	11	13	15	17	21	24	29
100	0	0	0	0	0	0	0	0	0	0	0	0	0	0

When we apply the values in the table to the propagation of waves in the highest known atmospheric surface of discontinuity, the »tropopause«, at the height of about 10 km., the wave length should be at least 60 km. Modifications of the results by still longer waves will be discussed later.

The table is extended to higher values for the drop of temperature than are found generally at an atmospheric surface of discontinuity. But as the atmospheric layers in general are in stable stratification the waves will propagate with greater velocity than that given by the table when we use the drop of temperature really observed, and we may have to resort to an artificial increase.

When the entire fluid system has a motion of translation with velocity u, we get a resultant propagation $u + c$, or $u - c$, according as the proper propagation goes with, or against the translation. But more complicated conditions originate when the two layers have different translations u and u'. Then the sliding motion $u' - u$ at the boundary surface interferes with the dynamics of propagation, so that in addition to the convective effect we get a modified velocity of propagation in the proper sense of this word.[1])

The convective effect is represented by an additive term, in the simplest cases a linear function of the velocities of the two currents, which may be said to represent a certain average current. Thus when both fluid layers are infinitely deep, the mean velocity which gives the convective term is simply

(x)
$$\frac{\varrho u + \varrho' u'}{\varrho + \varrho'}$$

For finite depths h and h' of the two fluid layers, even these parameters enter in the expression of the convective term (x).

[1]) For the simplest case of this kind cf. Lamb's Hydrodynamics p. 364 (Fourth Edition, London 1916).

Relatively to this mean velocity of the two currents the waves have a proper velocity of propagation, c, which depends, both upon the gravitational energy of the waves and upon the sliding velocity $u' - u$ at the boundary surface, increase of this sliding motion giving decrease of propagation. When this sliding velocity reaches a certain critical value, the proper propagation ceases, while the convective term persists. Then for still greater values of the sliding velocity the propagation becomes imaginary, and the solution changes character. The time t, which previously entered only as argument of the sine or cosine, comes in the expression of the amplitude A, and gives an increase of this amplitude with the time, initially according to an exponential law. But this exponential solution soon loses its validity. It only indicates the beginning of the process that the two layers whirl into each other. Then instead of waves we get convectively propagated vortices.

We shall consider this important process more closely in the next section. Here we shall only remark by the way that this final transformation of wave to vortex by too strong sliding motion is a corollary to the inital production of waves by more moderate values of the sliding motion. When the sliding velocity has reached a certain value, a plane boundary surface between the two layers becomes unstable, and takes undulating form. This is the well known principle for the formation of wind waves on water. But by too strong sliding motion even the wave motion loses its stability, and the two fluid layers whirl into each other.

16. Wave and vortex. — A physical pendulum with two weights, one below and one above the axis of rotation, will begin oscillations with a small disturbance. The nearer the center of gravity be brought to the axis by displacements of the weights, the longer will be the period, and the greater the elongations reached by one and the same impulse. Finally it will not return but continue to rotate always in the same direction, though with a periodically varying angular velocity as long as the center of gravity does not coincide exactly with the axis of rotation. As soon as this coincidence occurs, we get a permanent rotation with invariable angular velocity. If we perform the displacement suddenly during the motion, we can instantaneously have the oscillations changed into a permanent rotation, with direction and intensity depending upon the phase of the oscillation at the moment of displacement of the masses: equilibrium will follow if the displacement is performed while the pendulum has its maximum deviation, maximum velocity in the one or the other direction if the change is made at the moment when the pendulum passes its position of equilibrium. This is not in contradiction with the principle of the conservation of energy. For the displacement of the masses requires more work in the latter case than in the former.

Similar phenomena may occur in fluid oscillations. Standing waves, such as those represented by Fig. 8, give within each rectangular cell a pendulating motion of the entire fluid mass round the nodal lines as axis with a period depending upon the vertical asymmetry of the mass-distribution.

The fluid system which performs these oscillations may for the sake of argument be an incompressible heterogeneous salt solution or an ideal gas with a vertical temperature gradient that gives stability. Let the differences in density in the fluid be reduced, be it by a change of the saline concentration or by a thermal process. The period of the oscillations will then increase, and the same will be the case with the amplitudes. If the fluid be suddenly made completely homogeneous, the motion existing at that particular moment will continue as a permanent motion. The lines of flow must remain sensibly the same as before, only the motion along them changes from a periodically to a permanently circulating one.

Now let us consider this phenomenon from the point of view of the general vortex theory. The vorticity of the standing wave motion may easily be derived from the formulae of the preceding section. The vortex lines and tubes are normal to the plane of the paper. The vorticity is most strongly concentrated round the nodal lines, has decreasing intensity outwards, passing through the zero value and taking the opposite sign when we enter the next cell. Thus the fluid in every separate rectangular cell forms a separate vortex, and the successive cells contain vortices of the opposite sign. And within each cell the vortex motion changes periodically in time. Thus we have no conservation, but an incessant formation and annihilation of vortices in accordance with the general law 6 (B). But in the same moment as the fluid is made homogeneous, the periodical vortex is made permanent, and the vortex lines and vortex tubes are conserved materially in accordance with the theorems of Helmholtz. During this permanent vortex motion the originally horizontal plane through the nodal lines is within each cell progressively folded up in a double spiral. The approximate form of this spiral a short time after the passage to homogeneity is indicated in the diagram by the line of small crosses.

Similar phenomena would be observed in a discontinuous system such as that to which Fig. 9 refers: When here we have a standing wave motion, and then reduce the difference of density between the two strata, we get the same phenomena as in the preceding case: longer period, greater amplitudes, and finally a permanent motion which continues the phase of the oscillations at the moment when the difference of density disappears.

But from the point of view of the vortex theory an interesting difference should be observed. If each of the two strata were barotropic, the motion would be and always remain irrotational in both of them. The periodic vortices of the standing waves will exist merely in the transitional layer between them, in the ultimate case as surface vortices at the undulating surface of discontinuity. They change to permanent vortices at the moment when the difference of density disappears, and persist in the transitional layer, while this folds itself up as a double spiral in the manner illustrated in Fig. 8. But the fluid motion outside this spiral layer remains irrotational. The longer the process continues, the finer will the mixture of rotationally and irrotationally moving masses be. Even if we disregard the influence of turbulence and friction, the macroscopic result will soon be that of a single permanent vortex within each rectangular cell.

Corresponding considerations may, with slight modifications, be carried through for the case of propagating waves.

In standing waves we have periodically varying vortices, which are stationary in space. In propagating waves the vortices follow the waves by the propagation through the fluid. The departure from the Helmholtz laws of the conservation of vortices consists then no longer in periodic variation of vorticity of one and the same fluid individual, but in a transfer of vorticity from individual to individual, always in full accordance with the general law 6 (B).

In standing waves the direction and the intensity of the formed permanent vortices will depend upon the phase of the oscillations at the time when the fluid system is made barotropic. In propagating waves the time for the transition will be irrelevant. The transition may therefore go on quite gradually. The result will be that the velocity of propagation gradually decreases while the amplitudes increases. Finally, the velocity of propagation becomes zero, and the system of propagating waves is changed into a system of permanent stationary vortices.

If the fluid had a continually varying density from the beginning, the final result would be a continuous distribution of the permanent vortices. But if the fluid system consisted initially of two strata separated by a surface simultaneously of substantial and kinematic discontinuity, the resulting vortex will have the discontinuous structure with a surface of kinematical discontinuity folding itself up as a double spiral.

Finally, we must extend our considerations to the case alluded to at the end of the last section, viz. that the two fluid layers have different translational velocities. A sliding motion exists then at the boundary surface independently of the waves. When the value of this sliding motion exceeds a certain limit, the propagation relative to the defined mean velocity of the two currents ceases, while the amplitudes of the waves increase, initially according to an exponential law: this is the mathematical symptom that the transition from propagating waves to convectively moved vortices takes place. And now it takes place as soon as the difference of density between the two layers has diminished below a certain *finite* limit depending upon the value of the sliding velocity. This gives a violent whirling of the two layers into each other, against the statical forces which tend to keep the two layers separated.

This violent transformation of wave to vortex is, as emphazized already, the ultimate result of the same tendency which for more moderate values of the sliding velocity leads to the formation of surface waves. The tendency is to produce a mixture of the two fluid strata, the formation of the waves is a first attempt to attain this result, the transformation of wave to vortex the concluding step.

III. Dynamics of the Circular Vortex.

17. The steady circular vortex. — In a circular vortex the particles runin circular orbits round a common axis, all particles lying on one and the same circle having one and the same constant velocity. The steadiness of the motion involves the fact that the circles round the axis are at the same time the paths of the particles and the lines of flow representing the field of velocity.

Any plane normal to the *axis* shall be called the *plane* of the vortex. For axis we shall choose the axis of z and for plane the plane of xy. On account of the complete symmetry it will be sufficient to consider the conditions in a single meridian plane, that of yz.

To avoid circumlocution we shall confine ourselves to the case of an *attracting* exterior force. That is, the force components Y and Z shall be negative in the quadrant where y and z are positive, which is the only quadrant that needs to be considered. Further, we shall agree to call *upwards* the direction parallel to the axis of z, against the direction of the attracting force component Z. And we shall call *outwards* the direction away from, *inwards* the direction to the axis.

The force of inertia in this circular vortex will be simply the centrifugal force. It is directed outwards parallel to the axis of y. We have as expression of this force, referred to unit mass, when u is the linear and ω the angular velocity of the particle.

(a) $$-\dot{v} = \frac{u^2}{y} = \omega^2 y$$

In the most general case u or ω may be functions of y and z. The centrifugal force will then be a function of the same two coordinates. But a vector in the plane of yz which is parallel to the axis of y, will have a shearing or rotational field as long as it depends upon the variable z (cf. Fig. 4 A), while it will be irrotational then, and only then, when it is independent of z. According as $-\dot{v}$ is dependent upon or independent of z, the same will be the case with u and ω. As now the rotational or irrotational nature of the force of inertia decides the barotropic or the baroclinic nature of the field of mass, we find

(A) *The circular vortex has barotropic field of mass when the velocity is a function merely of the distance from the axis, but baroclinic field of mass when it varies also in the direction parallel to the axis.*

When the velocity varies only with the distance from the axis, all particles which are situated at the same distance from this axis will move as if they belonged to a rigid cylindrical surface. In the barotropic vortex the motion will therefore not in the least be modified by a solidification of coaxial cylindrical surfaces. The motion may be described as due to the rotation round a common axis of an infinite number of rigid cylindrical surfaces, which slide the one within the other. But if the condition of barotropy be given up, the velocity may also vary from circle to circle on each of these cylinders.

The steady circular vortex gives good illustrations of the general vortex theory. On account of the steadiness of the motion the circulation *along* any invariable curve in space remains invariable, and the vorticity *at* any given locality in space remains invariable. But the circulation *of* a physical curve or the vorticity *of* a moving volume element will be conserved in the sense of the theorems of Helmholtz only in the case of the barotropic vortex. It is seen by symmetry that the vortex lines are contained in the meridian planes. Further they are straight lines parallel to the axis in the barotropic case, but curves approaching to the axis in regions of intensive vortex motion and diverging from it in regions of less intensive vortex motion in the baroclinic case. In the barotropic vortex a material line which is once straight and parallel to the axis, will always remain straight and parallel to the axis, and therefore remain a vortex line which is materially conserved in the sense of the theorem of Helmholtz. But in the baroclinic vortex it is immediately seen that a material line which is once straight and parallel to the axis will in the next moment be twisted out of the meridian plane, and thus no longer be a vortex line.

It should be noticed when we apply the general theorem of circulation 6 (B) that the isobaric-isosteric solenoids run as circles round the axis of the vortex. Therefore every material curve which embraces a number of these solenoids will have variable circulation: a curve of this description has an upper and a lower part, and in the baroclinic vortex these two parts travel with different velocity, with the effect that the curve will embrace an always increasing or always decreasing number of vortex tubes. Then as by the theorem of Stokes the circulation of the curve is always equal to the number of vortex tubes which it embraces, we get corresponding increase or decrease of this circulation. But in the barotropic vortex any two points of the closed curve which are situated on one and the same cylindrical surface round the axis have the same motion round this axis. The curve will then during its motion embrace an invariable number of vortex tubes, and consequently have invariable circulations.

18. Barotropic vortices under constant gravity. — Preliminary to the study of the general baroclinic vortex we shall consider some types of barotropic vortices. When we know the exterior force and the law of variation of the velocity with the distance from the axis, we can construct the isobaric surfaces of the vortex. Then supplementary conditions will give the value of the pressure on each isobaric surface, and the corresponding barotropic field of mass.

To construct the isobaric surfaces of a barotropic vortex, we can use the method of sect. 8, viz., to add graphically the potentials of the exterior force and the centrifugal force. The latter potential is found by performing the integral

$$\text{(a)} \qquad \Phi_c = -\int \frac{u^2}{y}\, dy ,$$

which can always be evaluated when we know the law for the variation of the velocity with the distance from the axis, $u\ (y)$. The value of this potential as function of the variable is given below in a number of simple cases. At the same time we have written the value of y expressed as function of Φ_c. The values of y represent the radii of the cylindrical surfaces $\Phi_c = 1, 2, 3, \ldots,$ which give the graphical representation of the potential Φ_c.

When u has the same constant value u_o at all distances from the axis, we get the logarithmic potential

$$\text{(b)} \qquad u = u_o\ ,\ \Phi_c = -u_o^2 \log y\ ,\ y = e^{-\Phi_c/u_0^2}$$

When u has the value u_o in the distance b from the axis, and otherwise varies as the n-th power of this distance we have

$$\text{(c)} \qquad u = u_0 \left(\frac{y}{b}\right)^n ,\ \Phi_c = -\frac{u_o^2}{2n} \cdot \left(\frac{y}{b}\right)^{2n} ,\ y = b \cdot \sqrt[2n]{-\frac{2n}{u_o^2} \cdot \Phi_c}$$

For positive n the velocity converges to zero at the axis, and to infinity at infinite distance, *vice versa,* for negative n to infinity at the axis and to zero at infinite distance.

To avoid these infinities, if desirable, we may represent the velocity by a binomic expression. Adjusting its constants so that we get a maximum velocity $u = u_o$ at the distance $y = b$ from the axis, and introducing an abbreviation

$$\text{(d)} \qquad h = \left(\frac{2n}{2n+1}\right)^n ,\ k = -(2n+1).$$

we may write

$$\text{(e)} \quad u = \frac{u_o}{h} \cdot \frac{y}{b}\left(1 + \frac{1}{k} \cdot \frac{y^2}{b^2}\right)^n ,\ \Phi_c = \frac{u_o^2}{2h^2}\left(1 + \frac{1}{k} \cdot \frac{y^2}{b^2}\right)^{2n+1} ,\ y = b\left[k\left(\frac{2h^2}{u_o^2}\,\Phi_c\right)^{\frac{1}{2n+1}} - k\right]^{\frac{1}{2}}$$

Then we always have volocity zero at the axis, and velocity zero also at infinite distance when $n < -\frac{1}{2}$. In the critical case $n = -\frac{1}{2}$, formulae (e) become illusoric, and are then to be replaced by

$$\text{(f)} \quad u = \sqrt{2}\, u_o \frac{y}{b}\left(1 + \frac{y^2}{b^2}\right)^{-\frac{1}{2}} ,\ \Phi_c = -u_o^2 \log\left(1 + \frac{y^2}{b^2}\right) ,\ y = b\left(e^{-\Phi_c/u_0^2} - 1\right)^{\frac{1}{2}}$$

In this case the velocity has the value zero for $y = 0$; the value u_o, which is no longer a maximum, for $y = b$; and the value $\sqrt{2}\, u_o$ for $y = \infty$.

The exterior force controlling the vortices defined by these distributions of velocity shall first be constant gravity. This force has the potential

$$\Phi = gz. \tag{g}$$

This is represented graphically by horizontal planes which when we use the *MTS* system follow each other with a difference of level of one dynamic decimeter (leodecimeter).

Adding the potential (g) to any of the potentials (b), (c), (e) or (f) we get the potential Φ' of the apparent force in the vortex. $\Phi' =$ const. then represents the level surfaces of this force, or the isobaric surfaces of the vortex. On account of the barotropy they are at the same time equisubstantial surfaces of the fluid, and any surface of discontinuity in the vortex will be a definite surface $\Phi =$ const. The free boundary surface of the fluid, if there is any, will also be a definite surface $\Phi' = 0$. This strikingly visible surface will then show the configuration of the invisible isobaric surfaces in the interior of the fluid.

The entire system of these surfaces may be found graphically by drawing the diagonal curves in the network of lines formed by the non-equidistant straight lines $\Phi_c = 1, 2, 3,$ which are parallel to the axis of z, and the equidistant straight lines $\Phi = 1, 2, 3, \ldots$, which are parallel to the axis of y. An example of the complete construction is given in Fig. 11 below.

When Φ_c is of the form (c) we get

$$u = u_0 \left(\frac{y}{b}\right)^n, \quad \Phi' = gz - \frac{u_0^2}{2n}\left(\frac{y}{b}\right)^{2n} \tag{h}$$

with the special formulae

$$u = u_0, \quad \Phi' = gz - u_0^2 \log y \tag{i}$$

in the case $n = 0$. For positive values of n, and thus outwards increasing velocity, $\Phi' =$ const. represents surfaces of paraboloidal form. The circle $y = b$, at which the velocity is u_0, is situated

(j). $H = u_0^2/2n$ dynamic decimeters (leodecimeters), or $h = u_0^2/2ng$ meters

above the apex of the paraboloid. For negative values of n the surfaces are of hyperboloidal form, and the circle of velocity u_0 comes at the height (j) below the horizontal asymptotic plane of the hyperboloid. For $n = 0$ both apex and asymptotic plane are infinitely distant.

Fig. 10 shows the profile curve for the different surfaces for values of n from $n = 1$ to $n = -1$. All correspond to the case of one and the same velocity u_0 at the circle $y = b$. In the point of the diagram corresponding to this circle, all curves touch each other. It is seen that decreasing n gives increasing dip of the apex below the circle of velocity u_0. For $n = 0$ this dip becomes infinite, a vortex funnel being

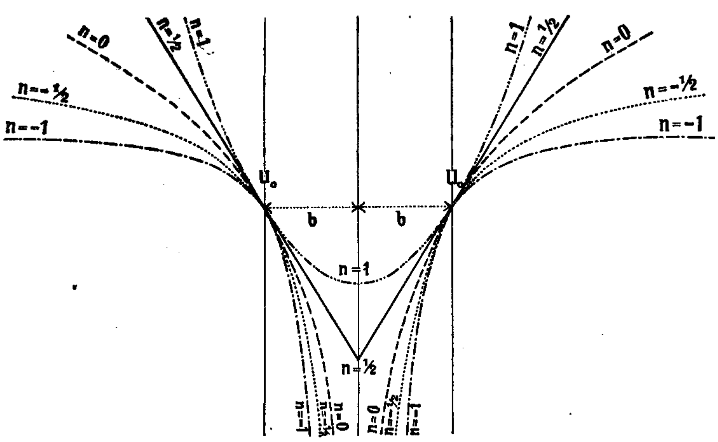


Fig. 10. Profile curves of isobaric surfaces.

formed. For an n still decreasing the funnel becomes always steeper, while at the same time the surfaces become flatter outwards. We now have an asymptotic plane which for decreasing n comes always nearer to the circle of velocity u_0.

Otherwise, $n = 1$ gives the simple and important case when the fluid rotates like a rigid body, and the isobaric surfaces and the exterior boundary surface of the fluid take the form of common paraboloids. For $n = \frac{1}{2}$, we get cones $z = Cy$, for $n = 0$, a logarithmic surface $z = C \log y$, and for $n = -\frac{1}{2}$, hyperboloids $zy = \text{const.}$ Finally, when $n = -1$, we have the important case where the velocity decreases in inverse ratio to the distance from the axis which gives the interesting irrotational circulation around the axis. This motion has a tendency to occur when the internal differences of velocity in the fluid are smoothed out by friction as far as compatible with the surface conditions. If the vorticity is formed in accordance with the formula 6 (b) it is seen to be zero everywhere in the fluid, except at the axis where it is infinite. The axis may be said to form an infinitely thin vortex tube of infinite strength. All closed curves surrounding the axis have one and the same circulation equal to the strength of this central tube, the isobaric surfaces and the free surface of a vortex of this type are represented by asymmetric hyperboloidal surfaces of the third degree, $zy^2 = \text{const.}$

From the simple vortices we may pass to »combined« vortices. Inside the cylinder $y = b$ we may use a solution with a positive and outside it a solution with a negative n. The kinematical surface condition, 10 (a) will then be fulfilled identically, and the dynamical surface condition 11 (a) will be satisfied if the fluid on both sides has the same density. For different values u_0 and u_0' of the velocity on both sides we then have only kinematical but no substantial discontinuity, i. e., a discontinuity of barotropic character according to the theorem 11 (A). Then the entire vortex remains barotropic. At the cylinder $y = b$ we have a sudden change of inclination of the isobaric surfaces. By continuity of the velocity, $u_0 = u_0'$, discontinuities will persist merely in the higher derivatives.

The most harmonic combinations are obtained by using the same numerical value of n outside and inside the junction. Then the circle of junction, where we have the maximum velocity $u = u_0$, is situated at precisely the half height between the asymptotic plane and the apex of the combined surface. Thus the total dip of a surface becomes $2h$ in length-measure and $2H$ in potential-measure, h and H being given by formulae (j).

Hyperboloids of the second degree in the exterior space which continue as cones in the interior are given by $n = \pm \frac{1}{2}$. For $n = \pm 1$ we get hyperboloids of the third degree in the exterior space and paraboloids of second degree in the interior. This is Rankine's »combined vortex«.[1])

The velocity distributions which are given by the binomic expressions (e) lead to vortices of similar features as the combined vortices, but with discontinuities excluded, even in the higher derivatives. For the case (f) of $n = -\frac{1}{2}$, the vortex will be represented by the formula

(k) $$u = \sqrt{2}\, u_0 \frac{y}{b}\left(1 + \frac{y^2}{b^2}\right)^{-\frac{1}{2}}, \quad \Phi' = gz - u_0^2 \log\left(1 + \frac{y^2}{b^2}\right).$$

The isobaric surfaces have finite dip at the axis, but mount outwards to infinite height according to the slow logarithmic law. The case $n = -1$ will be represented by the simple formula

[1]) Lamb's Hydrodynamics p. 27 (Fourth edition 1916).

(l) $$u = u_\circ \frac{2y/b}{1 + y^2/b^2}\,, \quad \Phi' = gz + \frac{2u_\circ^2}{1 + y^2/b^2}$$

and by the figure 11. It approaches to that of Rankine both infinitely near and infinitely far from the axis. But the circle $y = b$ of maximum velocity $u = u_\circ$ comes at the height of

(m) $H = u_\circ^2$ dynamic decimeters, or $h = u_\circ^2/g$ meters

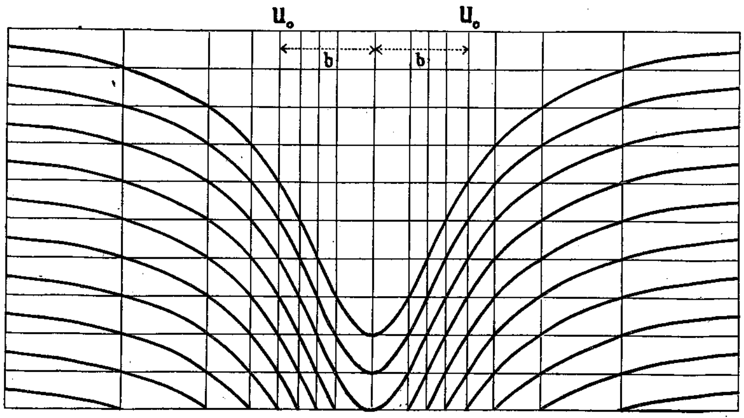


Fig. 11. Vortex of third degree.

above the lowest point of the surface, and the asymptotic plane at the same height above this circle just as by the conical-hyperboloidal vortex.

We may try to apply these formulae to the isobaric surfaces in atmospheric vortices of such small extent that the influence of the earth's rotation need not be considered, such as waterspouts and tornadoes. We should then find the following relation between the maximum velocity and the total dip from the asymptotic plane to the lowest point of the isobaric surface

Maximum velocity	1	10	100 m/sec.
Rankine's vortex ($n = \pm 1$)	0,102	10,2	1020 m.
Conical vortex ($n = \pm \frac{1}{2}$) / Continuous vortex of third degree	0,204	20,4	2040 m.

To the different systems of isobaric surfaces which we have thus considered may corespond any distribution of pressure from surface to surface, $p = F(\Phi')$, with the correlated distributions of density $\varrho = f(\Phi')$ which may be found as indicated in sect. 8.

As examples of possible pressure distributions with the correlated distributions of density and of temperature we may introduce the above values of Φ' into the expressions 8 (k), (m) and (n). In view of later applications we write the expression

(n) $$p = p_\circ - \varrho g z - \frac{2\varrho u_\circ^2}{1 + y^2/b^2}\,,$$

which corresponds to the case of a continuous vortex of the third degree in a homogeneous and imcompressible fluid.

19. Barotropic vortices under the action of an attracting centre. — In the same manner we may construct the isobaric surfaces when the vortices which are defined by formulae 18 (b), (c), (e), and (f), are under the action not of constant gravity, but of an attracting centre.

First, let the force be proportional to the distance from the centre, as the case would be within a homogeneous fluid sphere under mutual attraction of its elements. The force may then be written $-g_p r/r_p$, g_p denoting the acceleration of gravity at the pole, and r_p the polar radius. The potential of this force is $\frac{1}{2}\, g_p r^2/r_p + const.$ If we choose the constant so that the potential becomes zero for $r = r_p$, we arrive at the following form for the potential, and for the radii of the equipotential spheres, expressed as function of the potential

$$\text{(a)} \qquad \Phi_c = \tfrac{1}{2}\frac{g_p}{r_p}(r^2 - r_p^2)\,,\; r = \sqrt{r_p^2 + \frac{2r_p}{g_p}\Phi_c}$$

If we give the parameter Φ_c the values $1, 2, 3, \ldots$ in the last formula we find the radii of the spheres which give the graphical representation of the field of the attracting force.

To this field of attraction we may now add any of the fields of the centrifugal force 18 (b), (c), (e) or (f). We shall consider the case 18 (c) for $n = 1$, i. e., the case of constant angular velocity

$$\text{(b)} \qquad u = \Omega y\;,\; \Phi_c = -\tfrac{1}{2}\Omega^2 y^2\;,\; y = \sqrt{-2\Phi_c/\Omega}$$

The potential Φ' of the apparent force then becomes

$$\text{(c)} \qquad \Phi' = \tfrac{1}{2}\frac{g_p}{r_p}(r^2 - r_p^2) - \tfrac{1}{2}\Omega^2 y^2$$

The surface $\Phi' = \text{const.}$ is an ellipsoid of revolution. The solution may be adapted so as to give in the first approximation a representation of the isobaric surfaces in the fluid interior of the earth. $\Phi' = 0$ should then represent the surface of the earth itself. Neglecting the square of the small quantity $\Omega^2 r_p/g_p$ we get its equation in the form

$$\text{(d)} \qquad \frac{y^2}{r_p^2\left(1 + \frac{1}{2}\frac{\Omega^2 r_p}{g_p}\right)^2} + \frac{z^2}{r_p^2} = 1$$

This makes the equatorial radius of the earth $\frac{1}{2}\frac{\Omega^2 r_p^2}{g_p}$ or 21.9 km. longer than the polar, while the result determined by geodesy is 21.2 km. This shows that the assumption of a central force proportional to the distance from the earth's centre gives merely a first approximation.

Outside the solid earth we may suppose that the attracting force decreases in inverse ratio of the square of the distance from the earth's centre. The expression of the force may then be written $-g_p r_p^2/r^2$, and its potential $-g_p r_p^2/r + const.$ When we again determine the constant so that the potential becomes zero for $r = r_p$, we get for the potential, respectively for the radius of the equipotential spheres expressed as a function of the potential

$$\text{(e)} \qquad \Phi_c = -g_p r_p^2\left(\frac{1}{r} - \frac{1}{r_p}\right)\;,\; r = \frac{r_p}{1 - \frac{\Phi_c}{g_p r_p}}$$

Adding the potential to the centrifugal force we get

(f) $$\Phi' = -g_p r_p^2 \left(\frac{1}{r} - \frac{1}{r_p}\right) - \tfrac{1}{2}\,\Omega^2 y^2$$

Fig. 12 gives the graphical construction of the field of the potential Φ'.

The surfaces $\Phi' = \text{const.}$ which at the same time represent the level surface of the apparent force, the isobaric and the equisubstantial surfaces of the rotating fluid mass, as well as every surface of dicontinuity or external boundary surface, are seen to have

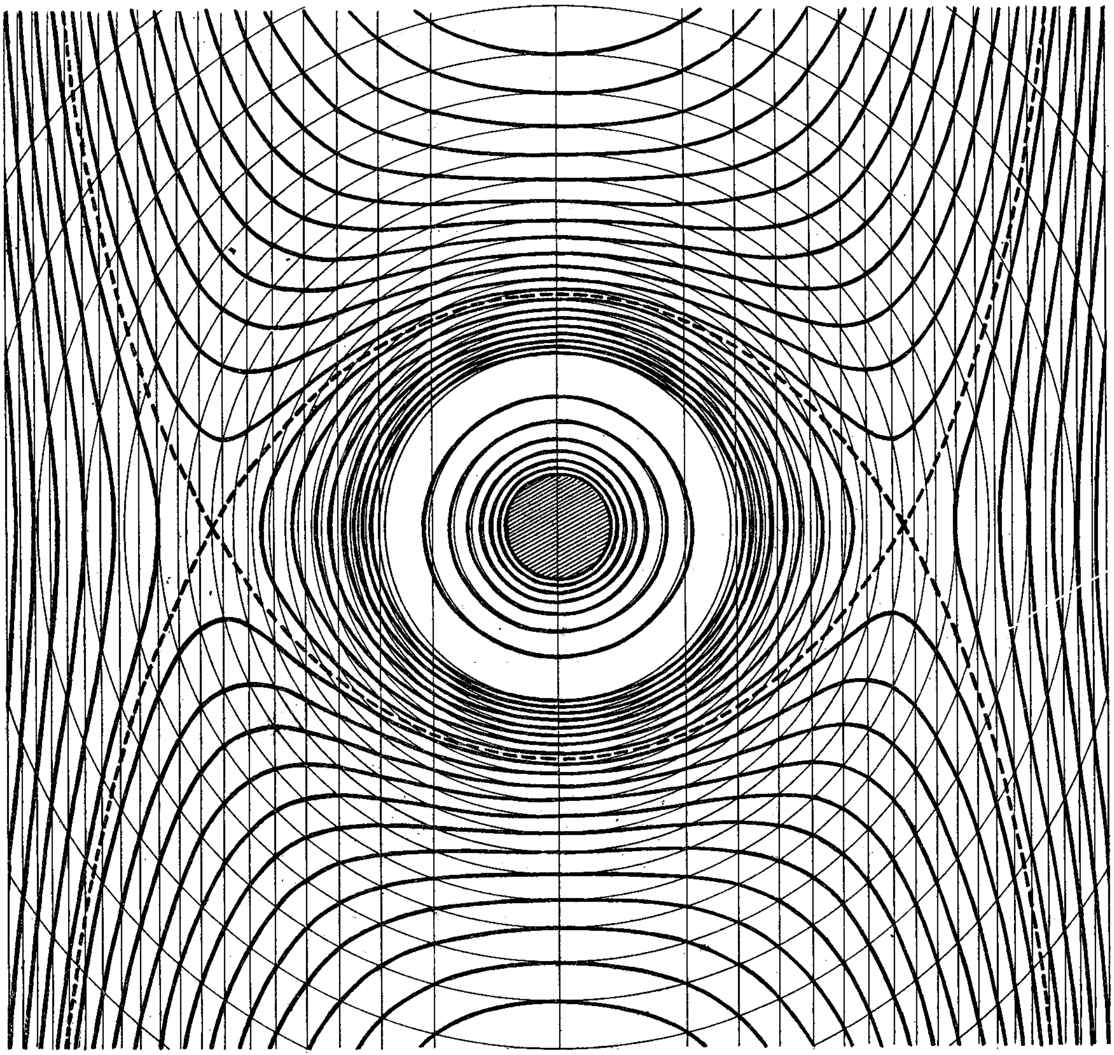


Fig. 12. Isobaric surface in an atmosphere following the earth's rotation.

the following shape: near the attracting centre they are nearly spherical, then, by and by, oblate. Among these, $\Phi' = 0$ should represent the surface of the earth. It is very nearly, but not exactly, an ellipsoid, and gives the equatorial radius 22.0 km. longer than the polar. Thus even this assumption gives merely a first approximation. Then, as we continue outwards the surfaces deviate always more from the ellipsoidal form, leading finally to a singular surface which cuts itself, and outside the circle of intersection continues to infinity, converging asymptotically to a cylindrical surface. Further out all surfaces continue in the same way to infinity, each asymptotically to a certain cylinder.

At the circle where the singular surface cuts itself, gravity and centrifugal force exactly balance. The gravitational attraction is not able to keep together rotating fluid masses outside the lenticular space which is limited by this singular surface. The surfaces outside this space would have no physical significance except in the case that the entire fluid system was inclosed within rigid boundaries. If, therefore, the atmosphere is limited, and follows completely the rotation of the solid earth, it cannot extend beyond this lenticular space. Its boundary must either be formed by this lenticular surface itself, or one of the surfaces inside it. The equatorial and the polar diameter of this lens are in the ratio 3 to 2, and the equatorial diameter amounts to 6.7, the polar 4.45 diameters of the earth.[1])

In the ideal case of a homogeneous and incompressible atmosphere the pressure would be given by

$$\text{(g)}\qquad p = p_\circ - \varrho_\circ \Phi' = p_\circ - \varrho_\circ \left[g_p r_p^2 \left(\frac{1}{r_p} - \frac{1}{r} \right) - \tfrac{1}{2}\, \Omega^2 y^2 \right].$$

By the linear relation, $\vartheta = \vartheta_\circ - \gamma \Phi'$ between temperature and potential, it would be

$$\text{(h)}\qquad p = p_\circ \left(1 - \frac{\gamma}{\vartheta_\circ} \Phi' \right)^{\frac{1}{R\gamma}} = p_\circ \left\{ 1 - \frac{\gamma}{\vartheta_\circ} \left[g_p r_p^2 \left(\frac{1}{r_p} - \frac{1}{r} \right) - \tfrac{1}{2} \Omega^2 y^2 \right] \right\}^{\frac{1}{R\gamma}}$$

and then the atmosphere would be limited at the surface

$$\text{(i)}\qquad \Phi'_* = \frac{\vartheta_\circ}{\gamma}$$

In the case of an isothermal atmosphere, finally, the pressure would be

$$\text{(j)}\qquad p = p_\circ e^{-\frac{1}{R\vartheta_\circ} \left[g_p r_p^2 \left(\frac{1}{r_p} - \frac{1}{r} \right) - \frac{1}{2} \Omega^2 y^2 \right]} + C,$$

where the constant has to be adjusted so that the pressure becomes zero at the lenticular boundary surface.

Finally, we may consider the vortex defined by 18 (e) for $n = -1$, when it is influenced by an attracting centre, of which the potential is given by (e). The potential of the apparent force in this vortex will then be

$$\text{(k)}\qquad \Phi' = -g_p r_p^2 \left(\frac{1}{r} - \frac{1}{r_p} \right) + \frac{2 u_\circ^2}{1 + y^2/b^2}$$

By sufficiently small values of the maximum velocity u the attraction will everywhere be in excess, and the field will be represented by a system of more or less oblate, but always closed surfaces. For sufficiently great value of this velocity even not closed surfaces will appear, and then we may have the case that in an intermediate zone the centrifugal force is in excess of the attraction, while the attraction is in excess both for all smaller and greater distances. We have then in the field a singular surface which cuts itself along a line in the equatorial plane, and which divides the space into three distinct regions: an inner closed space, where all surfaces are closed; an outer space where also

[1]) *Tisserand*, Traité de Mécanique céleste. T. IV, p. 236. Paris, 1896.

all surfaces are closed and simply connected; and finally, an annular space in which all surfaces are of annular form, ending with a circle in the middle of the space. Between this circle and the circle along which the singular surface cuts itself, the centrifugal force is in excess, and we have here pressure inversion, with a gradient directed towards the attracting centre. (Fig. 13).

This diagram gives an idea of the possible pressure distributions within a Laplacean nebula. As each of the isobaric surfaces may be a surface of pressure zero, which

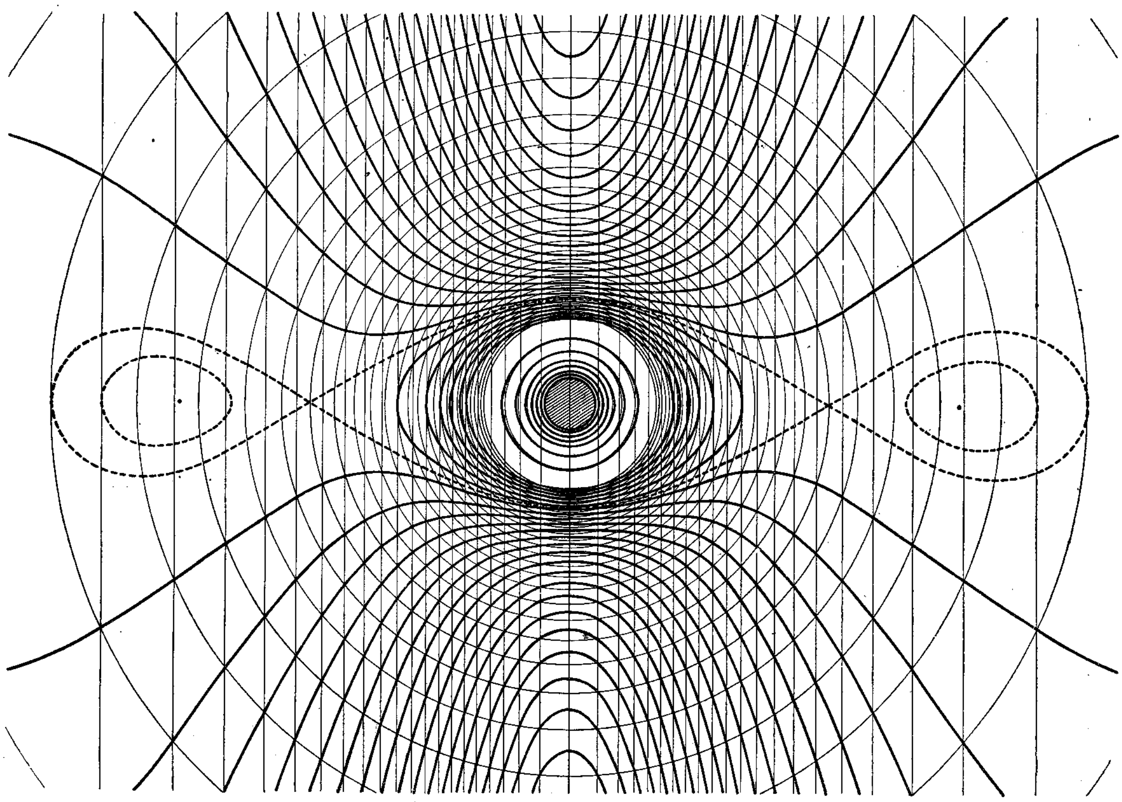


Fig. 13. Saturn-Ring-shaped isobaric surfaces.

limits the fluid mass against the empty space, we see that the fluid mass may have many different exterior shapes — and one of them is that of the central planet, surrounded by a Saturn Ring. It should be remembered, that our formulae only show the formal possibility of these different forms without information concerning their stability or instability.

20. Isobaric surfaces and surfaces of dicontinuity in the general circular vortex. — Now to pass from the special case of the barotropic to the general case of the baroclinic vortex, we may use equation 12 (a) for discussing the inclination of different surfaces. As in the circular vortex we have no acceleration parallel to the axis of the vortex, we get $\dot{w} = 0$ and $\dot{w}^* = 0$, so that the equations take the form

(a) $$\frac{dz}{dy} = -\frac{Y}{Z}, \quad \frac{dz}{dy} = -\frac{Y - \dot{v}}{Z}, \quad \frac{dz}{dy} = -\frac{Y - \dot{v}^*}{Z}$$

where the first gives the inclination of the level surfaces of the exterior force, the second that of the isobaric surfaces, and the third that of a surface of discontinuity, relatively to the plane of the vortex. In the last two equations the expressions of the centrifugal force per unit mass, $-\dot{v}$, respectively $-\dot{v}'$, and of the complex quantity, $-\dot{v}^*$ should be remembered. We then have

$$\text{(b)} \qquad -\dot{v} = \frac{u^2}{y}\ ,\ -\dot{v}' = \frac{u'^2}{y}\ ,\ -\dot{v}^* = \frac{\varrho u^2 - \varrho' u'^2}{\varrho - \varrho'} \cdot \frac{1}{y}$$

The equations (a) give the tangents to the angles in question. We can see that, $(-Z)$ being always positive, we pass from the first tangent to the second by adding the always positive quantity $(-\dot{v})/(-Z)$, and to the third by adding the quantity $(-\dot{v}^*)/(-Z)$, which is seen to be positive, zero or negative according as $\varrho u^2 \gtreqless \varrho' u'^2$, i. e., according as the kinetic energy *per unit volume* is in excess in the denser or less dense fluid.

For comparing $-\dot{v}^*$ with $-\dot{v}$ and $-\dot{v}'$, it is easily seen that we have $(-\dot{v}^*)$ greater than, equal to, or smaller than $(-\dot{v})$ and $(-\dot{v}')$ according as $u^2 \gtreqless u'^2$, i. e., according as the kinetic energy *per unit mass* (or scalar velocity) is in excess in the denser or less dense fluid. When we remember that algebraic increase of the tangent always gives algebraic increase of the corresponding angle, the following result is apparent when we compare the angles which the level surfaces, the isobaric surfaces and the surface of discontinuity form with the plane of the vortex:

(A). *The angle of the isobaric surfaces is greater than that of the level surfaces, and is contained between this angle and the next greater angle which has infinite tangent;*

(B). *A surface of discontinuity coincides with a level surface of the exterior force by equality of the kinetic energy per unit volume and with an isobaric surface by equality of the velocity on both sides of the surface;*

the angle is greater or smaller than that of the level surfaces according as the kinetic energy per unit volume is in excess in the denser or less dense fluid;

greater or smaller than that of the isobaric surfaces according as the scalar value of the velocity is in excess in the denser or less dense fluid;

and it will be limited by the next greater or the next smaller angle which has infinite tangent.

The result for the isobaric surfaces is illustrated already by the diagrams which have been drawn for the barotropic vortices. In the case of constant gravity the level surfaces of the exterior force are horizontal planes, forming the angle zero with the plane of the vortex. The angle which the isobaric surfaces form with this plane will thus be contained between 0 and $\pi/2$, i. e., the surfaces are always concave as seen in the diagram of Fig. 10 and 11. The level surfaces surrounding an attracting centre at the origin are spheres and thus form angles in the second quadrant with the plane of the vortex.

The angle of the isobaric surfaces will then be greater and consequently contained in this or the next quadrant. As long as the angle remains in the second quadrant, the result will be such oblate surfaces as are seen in the central spaces of Fig. 12 and 13. But as soon as the angle enters the third quadrant we get the more complex exterior surfaces of these figures.

The result for the surfaces of discontinuity is illustrated by Fig. 14, A—E, for the simple case of constant gravity $Y = 0\ ,\ Z = -g$. The first equation (a) is then reduced to $dz/dy = 0$ and defines the level surfaces as horizontal planes while the other equations (a) and (b) reduce to

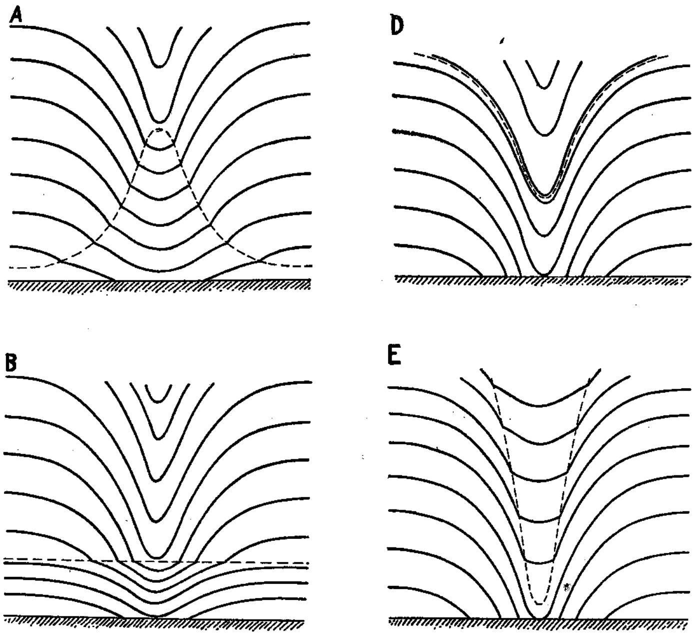
$$\text{(c)} \qquad \frac{dz}{dy} = \frac{1}{g} \cdot \frac{u^2}{y}, \quad \frac{dz}{dy} = \frac{1}{g} \cdot \frac{u^{**}}{y}, \quad u^{**} = \frac{\varrho u^2 - \varrho' u'^2}{\varrho - \varrho'}$$

The diagrams Fig. 15 A—E, give the corresponding illustrations for the case of an attracting centre.

Diagram A of both figures represents the case when the fluid of smaller density has an excess of kinetic energy, and thus, a fortiori, also of velocity. We then have the smallest value of the angle: In the case of constant gravity, a negative angle between 0 and $-\pi/2$, and in the case of an attracting centre an angle in the second quadrant, smaller than that of the spherical level surfaces of the attracting force, and greater than $\pi/2$. In these cases the lighter fluid attains the greater distance from the axis, while the heavy fluid is concentrated round the axis. Thereby heavy fluid is lifted against gravity and stretched along the axis of the vortex: we recognize the action of the centrifugal pump.

Diagram B of both figures gives the case of no discontinuity of the kinetic energy, and consequently coincidence of the surface of discontinuity with a level surface, namely a horizontal plane in the case of constant gravity, and a sphere in the case of the attracting centre.

Diagram C of both figures then gives the case where the heavier fluid has gained the greater kinetic energy per unit volume while the lighter has still the greater volocity, with the effect that the surface of discontinuity has an angle contained between that of the level and that of the isobaric surfaces. In the case of constant gravity, the surface of discontinuity has become concave, but with smaller concavity than the isobaric surface. In the case of the attracting centre it has become oblate, but less oblate than the isobaric surface.



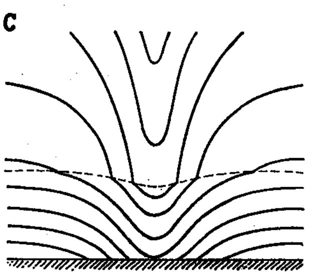


Fig. 14. Isobaric surfaces and surface of discontinuity in a circular vortex under the action of constant gravity:
A. Excess of kinetic energy in the upper fluid mass.
B. Continuity of kinetic energy.
C. Excess of velocity in the upper, of kinetic energy in the lower fluid mass.
D. Continuity of velocity. Barotropic case.
E. Excess of velocity in the lower fluid mass.

Diagram D of both figures gives the barotropic case, when there is no discontinuity of velocity, and the surface of discontinuity thus coincides with an isobaric surface. In the case of the attracting centre even a case as that of Fig. 13 may occur and any of the isobaric surfaces of this figure may be a surface of discontinuity. I e., we may also have a surface of discontinuity forming the boundary of a central core and a ring, both of greater density than the surrounding medium.

Finally, diagram E of both figures gives the case when the heavy fluid has the greater velocity. Then the angle of the surface of discontinuity exceeds that of the isobaric surfaces. In the case of constant gravity this surface then is more concave than the isobaric surfaces, in the case of the attracting centre it is more oblate. The surface of discontinuity may then take an annular form even in cases where no annular isobaric surfaces exist. Striking centrifugal effects are again apparent: the heavy fluid gains the greatest distance from the axis, the lighter fluid is sucked down to lower levels in the regions of the axis, and even annular openings may be produced. The heavier masses which are contained inside the annular surface of discontinuity are carried partly by the centrifugal force, and partly by the surrounding lighter masses in accordance with the Archimedian principle.

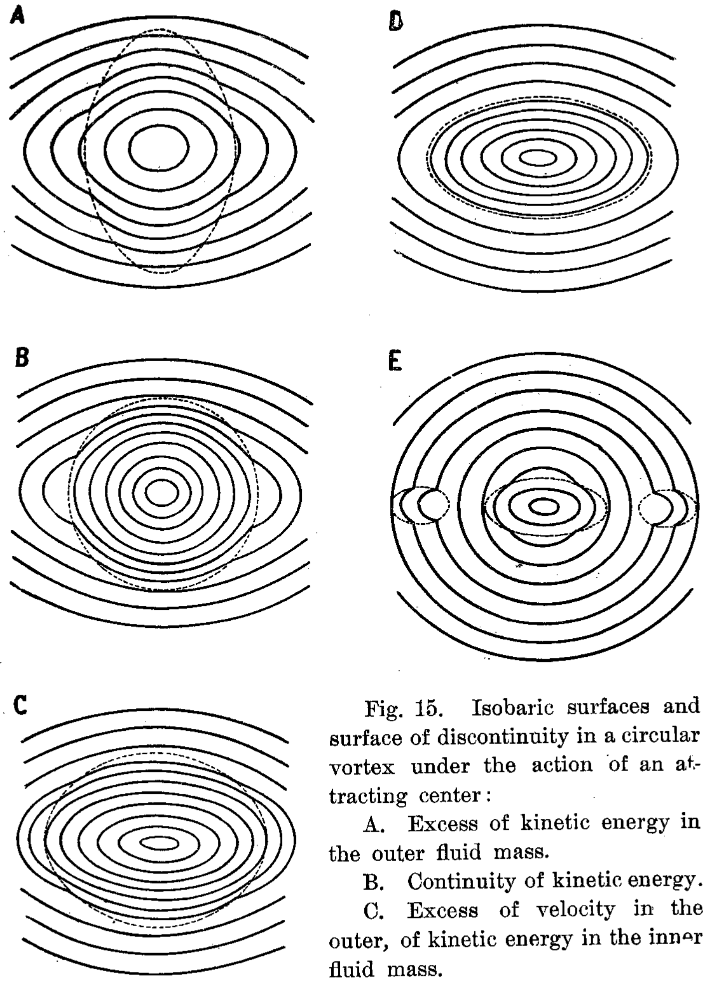


Fig. 15. Isobaric surfaces and surface of discontinuity in a circular vortex under the action of an attracting center:
A. Excess of kinetic energy in the outer fluid mass.
B. Continuity of kinetic energy.
C. Excess of velocity in the outer, of kinetic energy in the inner fluid mass.
D. Continuity of velocity. Barotropic case.
E. Excess of velocity in the inner fluid mass.

The interest of this case lies in the density inversions which show that at least super-adiabatic temperature gradients should not be considered as excluded in the atmosphere; at the surface of separation between an overlying strong and an underlying weaker west wind there may be an increase of density as we pass through the surface from below, and still more so if the underlying wind is an east wind. But it is not clear whether this arrangement may have sufficient stability to last for a longer time.

It should be noticed that the equation of the surface of discontinuity can be written in finite form when we know the pressure in the fluid masses on both sides of it. Thus, if formula 18 (n) represents the pressure in the two layers when we write it once with the parameters ϱ_0 and u_0, and once with the parameters ϱ_0' and u_0', we find the equation of the surface of discontinuity in the form

$$\text{(d)} \qquad gz + \frac{2u_0^*}{1+y^2/b^2} = 0 \;, \quad \text{where } u_0^* = \frac{\varrho_0 u_0^2 - \varrho_0' u_0'^2}{\varrho_0 - \varrho_0'}$$

This surface is of the same type as the isobaric surfaces 18 (l), the only difference being that the parameter u_0^* may take negative values with the effect that the surface is elevated and not depressed. The equation may be applied in representing the boundary surface of the internal core, for instance of a waterspout, when this lifts heavier air or water masses from below, or sucks down lighter masses from the clouds. The diagrams of Fig. 14 are drawn for equation 18 (l) of the isobaric surfaces, and the equation (d) for the surface of discontinuity. A second approximation for the representation of the phenomena of tornadoes or waterspouts may be obtained when we take into account the compressibility and the temperature distribution of the air, using the formula 8 (k), (m), or (n) in connection with the equation 18 (l) for the isobaric surfaces.

In the same manner, if the pressure in the two fluid masses surrounding an attracting centre is given by the expression 19 (g), written with the parameters ϱ_0 and Ω for the heavier, and ϱ_0' and Ω' for the lighter masses, we arrive at the equation

$$g_p r_p^2 \left[\frac{1}{r} - \frac{1}{r_p}\right] + \tfrac{1}{2}\,\Omega^* y^2 = 0\,,\ \text{where}\ \Omega^* = \frac{\varrho_0 \Omega^2 - \varrho_0' \Omega'^2}{\varrho_0 - \varrho_0'}\,, \tag{e}$$

for the surface of discontinuity separating the two masses of different density. Again we have the same type of equation as for the isobaric surfaces, only with a parameter Ω^* which may change sign, and thereby give rise to surfaces of the elongated as well as the oblate form. The equation may be used in the first approximation for representing the great internal surfaces of discontinuity in the atmosphere, which will be discussed below. And higher approximations may be obtained if we introduce pressures of the type 8 (k), (m), or (n), in connection with the equation 19 (f) for the isobaric surface.

21. General mass distribution in the circular vortex. — A surface of discontinuity reveals certain striking features of the distribution of mass, but we must also reckon with the continuous variation in the mass distribution. For this we shall apply the general results of section 5 to the circular vortex.

Considering the cylindrical surfaces round the axis of the vortex, we find in the barotropic case that they move as if they were solidified while in the baroclinic case the velocity differs from circle to circle upwards on one and the same cylinder. This gives different laws for the inclination of the isobaric surfaces. The tangent to the angle which these surfaces form with the plane of the vortex is according to 20 (a).

$$-\frac{(-Y)}{(-Z)} + \frac{(-\dot{v})}{(-Z)}\,, \tag{a}$$

where the quantities within the parentheses are positive.

Now let the velocity, and with it the centrifugal force $-\dot{v}$, in one case be constant and in another increased upwards: if a certain isobaric surface then forms the same angle with the plane of the vortex in both cases, the next higher surface will form greater angle in the case when the velocity increases upwards. But the greater angle gives a more rapid increase outwards in the thickness of the isobaric sheet. The work of transfer against the same force Z must then also increase outwards instead of being constant as in the barotropic case, and the same is true with the specific volume. The opposite result will follow when the velocity decreases with increasing z. We then arrive at the general result for a circular vortex, governed by attracting forces:

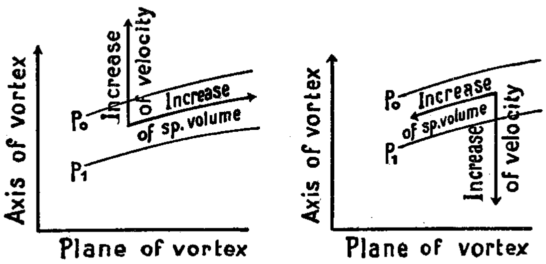
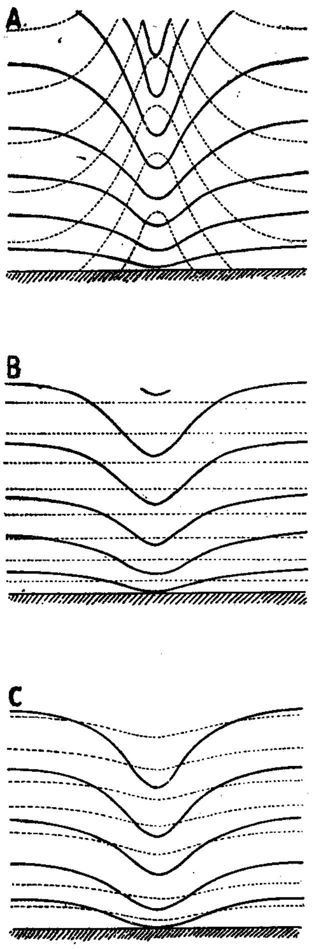


Fig. 16. Correlated distributions of velocity and mass in a circular vortex.

(A). *Increase of velocity upwards, parallel to the axis, gives increase of specific volume outwards from the axis when we follow an isobaric sheet; and increase of velocity downwards parallel to the axis gives increase of specific volume inwards to the axis when we follow an isobaric sheet* (Fig. 16).



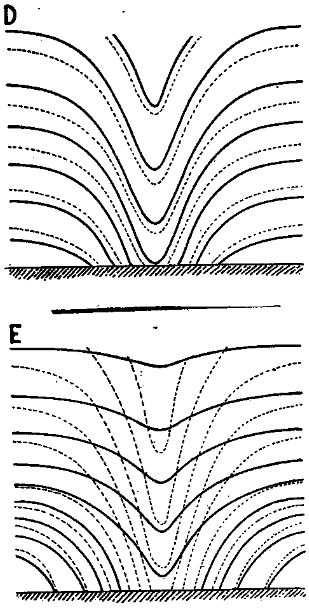


Fig. 17. Isobaric and equisubstantial surfaces in a circular vortex under the action of constant gravity:

A. Increase of velocity upwards parallel to the axis.

B. The same, smaller rate.

C. The same, still smaller rate.

D. No variation of velocity parallel to the axis. Barotropic case.

E. Decrease of velocity upwards parallel to the axis.

Finally, in the direction upwards, against the force Z, we have statical conditions, involving an increase of specific volume upwards along every line parallel to the axis. Combining this decrease upwards with the decrease or increase along the isobaric sheets we arrive at this result concerning the equiscalar surfaces of the field of mass.

(B.) *The equisubstantial surfaces will form a smaller angle with the plane of the vortex than the isobaric surfaces when the velocity increases, will coincide with them when the velocity remains invariable, and will form a greater angle when the velocity decreases in the direction upwards parallel to the axis.*

It is seen that the general rule is the same as for the surface of discontinuity. The diagrams of Fig. 17 and 18 correspond to those of Fig. 14 and 15, and will be understood without further explanation.

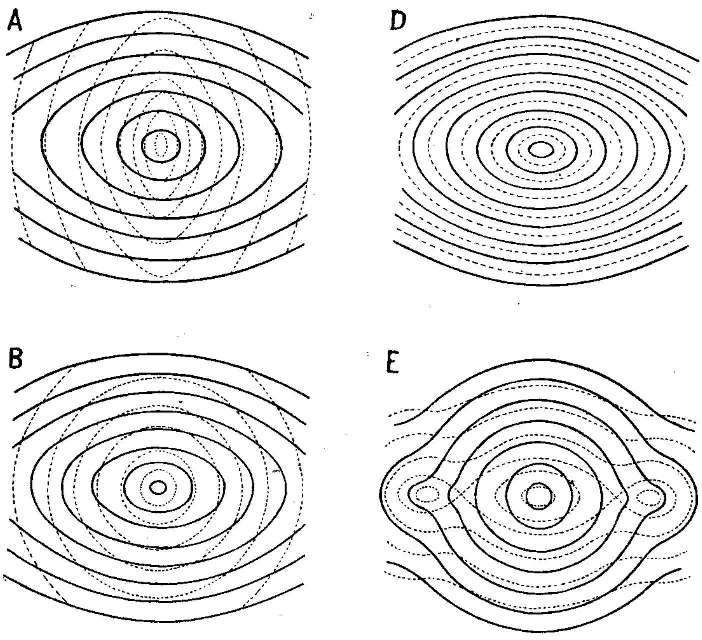
22. *Corresponding distribution of temperature.* — In all cases where the density depends merely upon the two variables, pressure and temperature, the knowledge of the fields of pressure and mass involves that of temperature. When density increases along an isobaric sheet, temperature will decrease, and vice versa. The fall of temperature along the isobaric sheet is defined by the component of the temperature gradient tangential to it. To avoid circumlocution, we shall call it the *isobaric temperature gradient.* On account of the nearly horizontal course of the isobaric surfaces in the atmosphere the isobaric temperature gradient will in most cases be practically identical with the horizontal. But still the distinction between them is important, as cases where they are oppositely directed may occur. Then, in terms of temperature, the theorem 21 (A) takes the following form:

(A). *Increase of velocity upwards parallel to the axis gives an isobaric temperature gradient directed inwards to the axis and increase of velocity downwards parallel to the axis gives an isobaric temperature gradient directed outwards from the axis.*

The course of the isothermal surfaces may also be found. While temperature and specific volume vary in the same way parallel to the isobaric surfaces, conditions may be different in the direction normal to them. In general, decreasing pressure gives decrease of temperature but increase of volume. When temperature and volume have the same variation parallel to, but opposite variation normal to the isobaric surfaces, the isothermal and the equisubstantial surfaces will have opposite inclinations relatively to them. But temperature inversion gives increase both of temperature and volume for decreasing pressure. Then the isothermal and the equisubstantial surfaces must have the same inclination relatively to the isobaric surfaces. Thus we get the useful rule:

(B). *The isothermal and the equisubstantial surfaces are under normal conditions inclined oppositely relatively to the isobaric surfaces, but in the same direction under the conditions of temperature inversion.*

23. Analytical examples of baroclinic vortices. — Examples of baroclinic vortices are easily given: we may choose any distribution of pressure, subject only to the condition that the isobaric surfaces form a greater angle with the plane of the vortex than the level surfaces of the exterior force, and derive from it the corresponding fields of motion and of mass. The resulting vortex will in general be baroclinic, and only in exceptional cases barotropic.



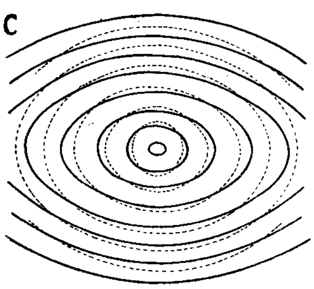


Fig. 18. Isobaric and equisubstantial surfaces in a circular vortex under the action of an attracting centre:

A. Increase of velocity upwards parallel to the axis.

B. The same, smaller rate.

C. The same, still smaller rate.

D. No variation of velocity parallel to the axis. Barotropic case.

E. Decrease of velocity upwards parallel to the axis.

But in order to find vortices of simple structure we shall start with the barotropic vortices which we already know, and pass to corresponding baroclinic vortices by the method of varying the constants. In order to retain as much as possible of the properties of the simple vortex, the best method will be to substitute for the constant not an arbitrary explicit function of y and z, but a function of that parameter Φ' of which constant values define the isobaric surfaces. Then the isobaric surfaces in the new vortex will be of the same type as in the old, and on each of them we shall have the distribution of velocity which we know already. The vortex, however, is made up of a new combination of these surfaces.

In the formulae of sections 18 and 19, we may thus consider any of the constants u_0^2 or b, or both simultaneously, as functions of Φ'. As an example we may retain the value of b and consider u_0^2 as a linear function of Φ'

(a) $$u_0^2 = u_{00}^2 \left[1 - \Phi'/\Phi_1'\right],$$

where u_{00}^2 and Φ_1' are constants. When we introduce this expression in the formulae of sect. 18, we get systems of isobaric surfaces as those previously described, but with a dip which varies from surface to surface. And simultaneously with the varying dip a corresponding variation of the distribution of velocity will follow from surface to surface. Then introducing (a) into the formulae 18 (l) for the continuous vortex of the third degree, we get

(b) $$u = u_{00} \cdot \frac{y}{b} \cdot \frac{2\sqrt{1 - \Phi'/\Phi_1'}}{1 + y^2/b^2}, \quad \Phi' = gz + \frac{2u_{00}^2 [1 - \Phi'/\Phi_1']}{1 + y^2/b^2}.$$

The second equation gives the isobaric surfaces, the first the distribution of velocity, expressed by the coordinates y and Φ'. If we wish to express the velocity as a function of y and z we must solve the second equation with respect to Φ' (equation (c) below) and introduce the value of Φ' in the first equation (b).

It is seen that $\Phi' = \Phi_1'$ gives the velocity zero, and the corresponding horizontal plane as isobaric surface, $\Phi' = 0$ leads back to the formulae 18 (l), i. e., it gives a surface with a dip $2u_{00}^2/g$. In the space between the plane $\Phi' = \Phi_1'$, and the surface $\Phi' = 0$, we have all intermediate forms. Choosing $\Phi_1' < 0$ we have the plane below and upwards increasing dip, with corresponding increasing velocity. Choosing, on the other hand, $\Phi_1' > 0$, we get the reverse case of upward decreasing dip, with upward decreasing velocity.

When we solve the equation of the isobaric surfaces with respect to the parameter Φ', we get

(c) $$\Phi' = \Phi_1' \cdot \frac{gz(1 + y^2/b^2) + 2u_{00}^2}{\Phi_1'(1 + y^2/b^2) + 2u_{00}^2}$$

The parameter Φ' has now, in this case of the baroclinic vortex, lost completely the significance of a potential of the apparent force. For being non-conservative, this force has no potential, and Φ' has retained merely its purely mathematical significance as a parameter of which a constant value defines a surface normal to the apparent force, or an isobaric surface which always coincides with the level surfaces.

If now the distribution of pressure from surface to surface is known, $p = F(\Phi')$, we derive the corresponding density by considering the elementary work of transfer across an isobaric sheet. Along an element dz parallel to the axis, this work will be $-gdz$, and we find

(d) $$\varrho = -\frac{dp}{gdz} = -\frac{1}{g} \cdot \frac{dp}{d\Phi'} \cdot \frac{d\Phi'}{dz},$$

which is simply the last equation 3 (a) for the force $Z' = -g$. Thus, if we take the law of pressure

(e) $$p = p_0 \cdot \left[1 - \frac{\gamma}{\vartheta_0} \cdot \Phi'\right]^{\frac{1}{R\gamma}},$$

we find the corresponding distribution of density expressed as a function of the coordinates y and Φ'

(f) $$\varrho = \varrho_0 \cdot \frac{\Phi_1'[1 + y^2/b^2] \cdot \left[1 - \frac{\gamma}{\vartheta_0}\Phi'\right]^{\frac{1}{R\gamma} - 1}}{\Phi_1'[1 + y^2/b^2] + 2u_{00}^2}$$

From the density we pass to the temperature by the gas equation, $\vartheta = p/R\varrho$. This gives

(g) $$\vartheta = \vartheta_0 \cdot \frac{\left[1 - \frac{\gamma}{\vartheta_0}\Phi'\right] \cdot [\Phi_1'(1 + y^2/b^2) + 2u_{00}^2]}{\Phi_1'(1 + y^2/b^2)}$$

If we here introduce the value of Φ' from (c), we find

$$\text{(h)} \qquad \frac{\vartheta}{\gamma} - \frac{\vartheta_0}{\gamma} = -gz - \frac{2u_{00}^2\,[1-\vartheta_0/\gamma\Phi_1']}{1+y^2/b^2}\,,$$

which represents the field of temperature as a function of y and z.

Infinite values of Φ_1' lead back to the barotropic case. Density (f) and temperature (g) then present themselves merely as functions of Φ'

$$\text{(i)} \qquad \varrho = \varrho_0 \left[1 - \frac{\gamma}{\vartheta_0}\Phi'\right]^{\frac{1}{R\gamma}-1}, \quad \vartheta = \vartheta_0\left[1-\frac{\gamma}{\vartheta_0}\Phi'\right],$$

so that equisubstantial surfaces and isothermal surfaces coincide with the isobaric. (h) is seen in this case to take explicitly the form of the equation 18 (l) of the isobaric surfaces with the constant dip $2H = 2u_{00}^2$. When then Φ_1' becomes finite, we get another value of the dip, but still constant from surface to surface, so that it can no longer coincide with the isobaric surfaces, which have a varying dip. We may consider especially the normal case of a positive γ, i. e., of decrease of temperature upward. When then $\Phi_1' < 0$, so that we have upward increasing velocity and upward increasing dip of the isobaric surfaces, the isothermal surfaces will have a dip greater than $2u_0^2$, and thus exceeding that of all isobaric surfaces in the field: this is the case when the strong circulation in the upper layers lifts air from below, so that they by adiabatic cooling form a cold and heavy core of the vortex. In the opposite case $\Phi_1' > 0$, when we have decrease of velocity upwards and decrease of dip upwards in the isobaric surfaces, the isothermal surfaces get a smaller dip than the isobaric. This dip will be changed into an elevation when γ is negative. This is the case when the intensive circulation in the lower strata sucks down from the higher levels air masses which are heated adiabatically, and thereby form a warm core of the vortex.

IV. Relative Motion.

24. General formulae in the case of relative motion. — We have hitherto taken the point of view of »absolute motion«, to which we must always return in order to get a clear insight into the dynamics of the phenomena. But in view of practical applications it will be useful to adapt some of our formulae and results to the case when the observer is supposed to belong to a rigid rotating system.

Let x, y, z be rectangular coordinates which are fixed in this rotating system. Relatively to the fundamental system let them have the angular velocity Ω with the components $\Omega_x, \Omega_y, \Omega_z$ along the axes x, y, z. Further, let u, v, w be velocities, and $\dot{u}, \dot{v}, \dot{w}$ accelerations, relatively to these rotating axes, and let X, Y, Z be the force which determines equilibrium relatively to them. To pass from the equations of equilibrium to those of motion relatively to the moving system we must introduce into the equations of equilibrium 3 (a) an »apparent force« X', Y', Z' which is more general than that of 5 (a), in as much as besides the force X, Y, Z and the force of inertia $-\dot{u}, -\dot{v}, -\dot{w}$ it contains also an additional force, the force of *Coriolis*, which is the vector product of the relative velocity u, v, w into the double angular velocity of the rotating system. For the »apparent force« we shall then write more completely.

$$\text{(a)} \qquad \begin{aligned} X' &= X + 2(v\Omega_z - w\Omega_y) - \dot{u} \\ Y' &= Y + 2(w\Omega_x - u\Omega_z) - \dot{v} \\ Z' &= Z + 2(u\Omega_y - v\Omega_x) - \dot{w} \end{aligned}$$

Introducing these expressions of X', Y', Z' into formula 3 (b) we get for the differential of pressure

$$\text{(b)} \quad dp = \varrho\Big([X + 2(v\Omega_z - w\Omega_y) - \dot{u}]\,dx + [Y + 2(w\Omega_x - u\Omega_z) - \dot{v}]\,dy + [Z + 2(u\Omega_y - v\Omega_x) - \dot{w}]\,dz\Big).$$

In another part of the fluid where velocity is u', v', w', density ϱ', and pressure p', the same differential will be

$$\text{(c)} \quad dp' = \varrho'\Big([X + 2(v'\Omega_z - w'\Omega_y) - \dot{u}')\,dx + [Y + 2(w'\Omega_x - u'\Omega_z) - \dot{v}']\,dy + [Z + 2(u'\Omega_y - v'\Omega_x) - \dot{w}']\,dz\Big).$$

Thus, the differential equations of the isobaric surfaces in the two parts of the fluid will be

$$\text{(d)} \quad [X + 2(v\Omega_z - w\Omega_y) - \dot{u})]\,dx + [Y + 2(w\Omega_x - u\Omega_z) - \dot{v}]\,dy + [Z + 2(u\Omega_y - v\Omega_x) - \dot{w}]\,dz = 0,$$

$$\text{(e)} \quad [X + 2(v'\Omega_z - w'\Omega_y) - \dot{u}']\,dx + [Y + 2(w'\Omega_x - u'\Omega_z) - \dot{v}']\,dy + [Z + 2(u'\Omega_y - v'\Omega_x) - \dot{w}']\,dz = 0.$$

Equating the difference $dp - dp'$ to zero, and dividing for convenience by the difference of density $\varrho - \varrho'$, we get the differential equation for a surface of discontinuity at which we have a sudden change of density ϱ to ϱ', of velocity from u, v, w to u', v', w', and of acceleration from $\dot{u}$, $\dot{v}$, $\dot{w}$ to $\dot{u}'$, $\dot{v}'$, $\dot{w}'$:

$$\text{(f)} \quad [X + 2(v^*\Omega_z - w^*\Omega_y) - \dot{u}^*]\,dx + [Y + 2(w^*\Omega_x - u^*\Omega_z) - \dot{v}^*]\,dy + [Z + 2[u^*\Omega_y - v^*\Omega_x) - \dot{w}^*]\,dz = 0.$$

Here we have for abbreviation

$$\text{(g)} \qquad u^* = \frac{\varrho u - \varrho' u'}{\varrho - \varrho'}\,, \quad v^* = \frac{\varrho v - \varrho' v'}{\varrho - \varrho'}\,, \quad w^* = \frac{\varrho w - \varrho' w'}{\varrho - \varrho'}\,,$$

$$\text{(h)} \qquad \dot{u}^* = \frac{\varrho \dot{u} - \varrho' \dot{u}'}{\varrho - \varrho'}\,, \quad \dot{v}^* = \frac{\varrho \dot{v} - \varrho' \dot{v}'}{\varrho - \varrho'}\,, \quad \dot{w}^* = \frac{\varrho \dot{w} - \varrho' \dot{w}'}{\varrho - \varrho'}\,.$$

Now we may make a special choice of coordinates as in section 12. That is, we turn the axes so that the line element dx becomes tangential to the examined surface element. The angle of elevation θ of this element relatively to the plane of xy will then be in the case of an isobaric surface

(i) $$\operatorname{tg}\theta = \frac{dz}{dy} = -\frac{Y + 2(w\Omega_x - u\Omega_z) - \dot{v}}{Z + 2(u\Omega_y - v\Omega_x) - \dot{w}}$$

and in the case of a surface of discontinuity

(j) $$\operatorname{tg}\theta = \frac{dz}{dy} = -\frac{Y + 2(w^*\Omega_x - u^*\Omega_z) - \dot{v}^*}{Z + 2(u^*\Omega_y - v^*\Omega_x) - \dot{w}^*}$$

where the expressions (g) and (h) for u^*, v^*, w^*, $\dot{v}^*$, $\dot{w}^*$ are to be remembered.

For most applications it will be useful to give an orientation to the system of coordinates which is natural for an observer who belongs to the rotating system. The basis for his orientation in space is formed by the surfaces which he calls level, and which are surfaces of revolution around the axis of rotation. He may then choose the axis of z normal to these surfaces, and directed upwards against the force which he feels as gravity. The xy plane then becomes a tangential plane to one of the level surfaces with the origin of coordinates as point of taction. Under these circumstances we have in the vicinity of this point

(k) $$Y = 0\ ,\ Z = -g\ ,$$

g being the scalar value of the force which the inhabitants of the rotating system call gravity.

The axis z forms with the plane of rotation an angle φ, the *angle of latitude*, which determines the parallel circles on the level surfaces. This angle of latitude is in the first quadrant for the ellipsoidal level surfaces of the rotating earth, but in the second for the paraboloidal level surfaces which are produced by the experiment with a rotating vessel. Let further ψ be the angle which the vertical plane xz forms with the meridian plane. The projections of the angular velocity Ω on the three axes x, y, z will then be

(l) $$\Omega_x = \Omega\cos\varphi\cos\psi\ ,\quad \Omega_y = \Omega\cos\varphi\sin\psi\ ,\quad \Omega_z = \Omega\sin\varphi$$

When we introduce (k) and (l), formula (i) for the inclination of an isobaric surface takes the form

(m) $$\operatorname{tg}\theta = \frac{dz}{dy} = \frac{2\Omega(w\cos\varphi\cos\psi - u\sin\varphi) - \dot{v}}{g - 2\Omega(u\cos\varphi\sin\psi - v\cos\varphi\cos\psi) + \dot{w}}\ ,$$

and formula (j) for the inclination of a surface of dicontinuity,

(n) $$\operatorname{tg}\theta = \frac{dz}{dy} = \frac{2\,\Omega(w^*\cos\varphi\cos\psi - u^*\sin\varphi) - \dot{v}^*}{g - 2\Omega(u^*\cos\varphi\sin\psi - v^*\cos\varphi\cos\psi) + \dot{w}^*}$$

The formulae show that in order to determine exactly the inclination of an isobaric surface, we must know all three components u, v, w of the velocity and two components of the acceleration, $\dot{v}$ and $\dot{w}$, while the third one $\dot{u}$, which is tangential to the surface, does not enter. To find the exact inclination of a surface of discontinuity we must know the same quantities on the two sides of the surface. In common practice wind observations do not give all this information but still give, in general, sufficient data if we disregard small quantities.

25. *Curved horizontal motion. Circular vortex in the relative motion.* — We may now restrict the generality of the motion by conditions which are always fulfilled in the atmosphere with a certain degree of approximation.

First, the motion shall be what the inhabitants of the rotating system call horizontal. That is, the vertical velocity, which is always small, is neglected completely, $w = 0$.

Then, the velocity shall always be tangential to the examined surface, so that $v = 0$. For isobaric surfaces in the free atmosphere this is known always to be very nearly the case. When we apply the same condition to a surface of discontinuity, $v = 0, v' = 0$, it means that this surface does not move in a direction normal to itself, i. e., we examine a surface of this kind which is in equilibrium, not in motion.

At the considered point, the origin, we are consequently concerned only with the horizontal velocity u, which is tangential to the surface. We shall suppose this velocity has an invariable scalar value, so that $\dot{u} = 0$.

Under these conditions no other acceleration appears than that which is due to the curvature of the path. As the axis of x is tangent to the path, the osculating circle of the path is contained in a plane which cuts the xy-plane along the axis of x, and forms a certain angle χ with it. If then R is the radius of the circle, the centripetal acceleration will be u^2/R. The sign is positive because the centre is situated to the positive side of the origin $y = 0$. Multiplying this acceleration by $\cos\chi$ and $\sin\chi$ respectively, we get the components of the acceleration, $\dot{v}$ and $\dot{w}$, along the axes of y and z.

The conditions which restrict this motion are thus the following

$$v = 0\ ,\ w = 0\ ,\ \dot{u} = 0,\ \dot{v} = \frac{u^2}{R}\cos\chi\ ,\quad \dot{w} = \frac{u^2}{R}\sin\chi \tag{a}$$

with a corresponding set of equations with accented letters for the case of the surface of discontinuity. Introducing them in the equations (m) and (n) of the preceding section we get for the angle of elevation of an isobaric surface

$$\operatorname{tg}\theta = -\frac{2\,\Omega u\sin\varphi + u^2/R\cos\chi}{g - 2\,\Omega u\cos\varphi\sin\psi + u^2/R\sin\chi} \tag{b}$$

and of a surface of discontinuity

$$\operatorname{tg}\theta = -\frac{2\Omega u^*\sin\varphi + u^{**}/R\cos\chi}{g - 2\Omega u^*\cos\varphi\sin\psi + u^{**}/R\sin\chi} \tag{c}$$

where

$$u^* = \frac{\varrho u - \varrho' u'}{\varrho - \varrho'}\ ,\ u^{**} = \frac{\varrho u^2 - \varrho' u'^2}{\varrho - \varrho'} \tag{d}$$

When the moving medium is an ideal gas, we may use the gas equation to bring in the absolute temperatures ϑ and ϑ' on the two sides of the surface of discontinuity instead of the densities ϱ and ϱ'. Equations (d) then take the form

$$u^* = \frac{\vartheta' u - \vartheta u'}{\vartheta' - \vartheta}\ ,\ u^{**} = \frac{\vartheta' u^2 - \vartheta u'^2}{\vartheta' - \vartheta}\ . \tag{e}$$

The formulae (b) and (c) apply to any horizontal current of any curvature, but otherwise unaccelerated in a tangential direction. If we wish to apply them to a circular vortex, we must remember that the only true circles on the level surface of our rotating system are the parallel circles. Therefore, a true circular vortex in the relative motion must be centred round the axis of the rotating system and thus be a circular vortex also in the absolute motion. We arrive at this case when the yz-plane coincides with a meridian plane, so that $\psi = \pi/2$, and the angle of inclination χ of the osculating plane with the level surface is complementary to the angle of latitude, $\chi = \pi/2 - \varphi$. Then R becomes the distance from the axis of rotation, and we get for the angle of elevation of the isobaric surfaces

$$\text{(f)} \qquad \operatorname{tg}\theta = -\frac{2\Omega u \sin\varphi + \frac{u^2}{R}\sin\varphi}{g - 2\Omega u\cos\varphi + \frac{u^2}{R}\cos\varphi}$$

and for the surface of discontinuity

$$\text{(g)} \qquad \operatorname{tg}\theta = -\frac{2\Omega u^* \sin\varphi + \frac{u^{**}}{R}\sin\varphi}{g - 2\Omega u^*\cos\varphi + \frac{u^{**}}{R}\cos\varphi}$$

These formulae give thus from the point of view of the relative motion the same results for isobaric surfaces and a surface of discontinuity in circular vortices as those which we have already developed from the point of view of the absolute motion. It is a good exercise to verify it in detail.

If we introduce $\Omega = 0$ in our formulae we return to the case of absolute motion; (b) and (c) then give

$$\text{(h)} \qquad \operatorname{tg}\theta = -\frac{\frac{u^2}{R}\cos\chi}{g + \frac{u^2}{R}\sin\chi}, \qquad \operatorname{tg}\theta = -\frac{\frac{u^{**}}{R}\cos\chi}{g + \frac{u^{**}}{R}\sin\chi}$$

In the case of the true circular vortex we have as in (f) and (g) that $\sin\chi$ has the value $\cos\varphi$ and $\cos\chi$ the value $\sin\varphi$. These formulae (h) give the angle of elevation of the surfaces in question above the level surfaces of the true exterior force, g representing the intensity of this force, while in (b) and (c) g represents the resultant of this force and the centrifugal force.

26. *Special cases.* — From the general formulae 25 (b) and (c) we may deduce a number of simplified cases according as different parameters are great or small.

(A). The radius of curvature R shall be small. The effect is that at the same time χ becomes small, so that $\sin\chi = 0$ and $\cos\chi = 1$. Then 25 (b) and (c) reduce to

$$\text{(a)} \qquad \operatorname{tg}\theta = -\frac{1}{g}\frac{u^2}{R}, \qquad \operatorname{tg}\theta = -\frac{1}{g}\frac{u^{**}}{R}$$

(B). The radius of curvature R is great, the angle of latitude φ different from zero, and the values of u, respectively of u^* and u^{**}, small enough to make the term g predominant in the denominator. Then

(b) $$\operatorname{tg}\theta = -\frac{2\Omega\sin\varphi}{g}u\ ,\quad \operatorname{tg}\theta = -\frac{2\Omega\sin\varphi}{g}u^*$$

(C). The radius of curvature R is moderate, the angle of latitude φ may have all values, including the value 0, but the values of u, u^* and u^{**}, respectively, remain as in the preceding case small enough to make g the predominating term in the denominator. Then

(c) $$\operatorname{tg}\theta = -\frac{2\Omega\sin\varphi}{g}u - \frac{\cos\chi}{gR}u^2\ ,\quad \operatorname{tg}\theta = -\frac{2\Omega\sin\varphi}{g}u^* - \frac{\cos\chi}{gR}u^{**}$$

(D). As in the case (B) the radius of curvature is great, and the angle of latitude is different from zero, but no limit is set on the values of u, respectively u^* and u^{**}. Then

(d) $$\operatorname{tg}\theta = -\frac{2\Omega u\sin\varphi}{g - 2\Omega u\cos\varphi\sin\psi}\ ,\quad \operatorname{tg}\theta = -\frac{2\Omega u^*\sin\varphi}{g - 2\Omega u^*\cos\varphi\sin\psi}$$

Case (A) assumes the angular velocity u/R in the relative motion to be great compared to Ω which is that of the rotating rigid system. The condition is fulfilled by every vortex of sufficiently small radius R, and leads back to the formulae for absolute motion 20 (c). The negative sign is due to the choice of origin of coordinates, not at the axis of the vortex as in 20 (c), but at the specially examined point of the vortex. We return to formulae 20 (c) by introducing $R = -y$. The formulae give the results which are represented by Fig. 14: the isobaric surfaces show depressions, always of the same amount for the same scalar value of velocity. No difference is seen between the cases of cyclonic and anticyclonic direction of the circulation. The surface of discontinuity may have both elevations and depressions, but equally independent of the direction of the motion around the axis.

Case (B) gives the classical approximation formulae used for calculating the inclination of isobaric surfaces or surfaces of discontinuity in the large scale atmospheric motions. The second formula (b) with the value 25 (e) of u^* is due to Margules.[1] The negative sign shows that in the northern hemisphere where $\sin\varphi$ is positive, the angle of inclination is produced by a negative rotation round the vector u or u^*, respectively. The direction of the latter vector is given by the sign of the expression $\varrho u - \varrho' u'$. To shorten the expression we shall call ϱu and $\varrho' u'$ the »strength« of the currents. When we remember that $\varrho > \varrho'$, we see that the vector u^* has the direction of the current which has greater density as long as the two currents are of opposite directions, the common direction of both of them when that of the greater density ϱ has also the excess of strength, the direction opposite to both of them, when that of the smaller density ϱ has the greater strength $\varrho' u'$. We thus easily arrive at the following rules:

(I). *An observer in the northern hemisphere looking in the direction of the current will have the slope of the isobaric surfaces from right to left.*

(II). *An observer in the northern hemisphere looking in the direction of the denser or the underlying of two currents will have the slope of their mutual boundary surfaces from right to left (1) as long as they are of opposite directions, and (2), as long as that of the greater density has the greater strength, — but from left to right when the currents are of the same direction and that of the smaller density has the greater strength.*

[1]) *Margules*: Ueber Temperaturschichtung in stationär bewegter und in ruhender Luft. Hann — Band der Met. Zeitschr. 1906. p. 293. *Exner*: Dynamische Meteorologie p. 155.

In the southern hemisphere we have the same rules, only with the words right and left interchanged. The elementary explanation of these inclinations is well known: A current in the northern hemisphere has the tendency to curve to the right. When, therefore, it is forced to keep to a straight path, this tendency to curve to the right must be counterbalanced by the slope from right to left of the isobaric surfaces given by the rule (I). When the two currents both tend to the right, the strongest attains the position when both are of same direction, while both gain when they are of opposite directions: this gives rule (II).

The annexed tables will facilitate the determination of the inclination of our surfaces. The first gives the horizontal lengths for unit increase of height as function of the latitude φ and the velocity u, respectively the complex quantity u^*. This first table is sufficient for the case of isobaric surfaces. But when we have to do with a surface of discontinuity, we find in the second table approximate values of the quantity u^*, which is then the argument in the first table.

Table II. — Horizontal distances in kilometers for 1 meter rise of an isobaric surface or a surface of discontinuity.

Latitude φ	Velocity u or u^* m/sec. NB. When the argument is multiplied by 10, 100, ... the tabulated quantity is divided by the same number.																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	25
90	68	34	23	17	14	11	10	8.4	7.5	6.8	6.1	5.6	5.2	4.8	4.5	4.2	4.0	3 8	3.6	3.4	2.7
80	69	34	23	17	14	11	10	8.6	7.6	6.9	6.2	5.7	5.3	4.6	4.6	4 3	4.0	3.8	3.6	3.5	2.7
70	72	36	24	18	14	12	10	9.0	8.0	7.2	6.5	6.0	5.5	5.1	4.8	4.5	4.2	4 0	3.8	3.6	2.9
60	78	39	26	20	16	13	11	10	8.7	7.8	7.1	6.5	6.0	5 6	5.2	4.9	4.6	4.3	4.1	3.9	3.1
50	88	44	29	22	18	15	13	11	10	8.8	*8.0	7.3	6.8	6.3	5.9	5.5	5.2	4.9	4.6	4.4	3.5
45	95	48	32	24	19	16	14	12	11	9 5	8.7	8.0	7.3	6.8	6.4	6.0	5.6	5.3	5.0	4.8	3.8
40	105	53	35	26	21	18	15	13	12	11	9.6	8.8	8.1	7.5	7.0	6.6	6.2	5.8	5.5	5.3	4.2
35	118	59	39	29	24	20	17	15	13	12	11	9.8	9.1	8.4	7.8	7.4	6.9	6.5	6.2	5.9	4.7
30	135	68	45	34	27	23	19	17	15	14	12	11	10	9.6	9.0	8.4	7.9	7.5	7.1	6.8	5.4
25	160	80	53	40	32	27	23	20	18	16	15	13	12	11	11	10	9.4	8.9	8.4	8.0	6.4
20	197	99	66	49	40	33	28	25	22	26	18	16	15	14	13	12	12	11	10	9.9	7.9
15	261	130	87	65	52	44	38	33	29	26	24	22	20	19	17	16	15	14	14	13	10
10	389	194	130	97	78	65	56	49	43	39	35	32	30	28	26	24	23	22	20	19	16
5	774	387	258	199	155	129	111	97	86	77	70	65	60	55	52	48	46	43	41	39	31

Ex. At lat. 20° gives u = 8.5 m/sec. a rise of the isobaric surface of 1 m. on 23.5 km.
—»— $u^* = 850$ » —»— surface of discontinuity of 1 m. on 0.235 km.

The exact values of u^* are found from the first of formulae 25 (d) or (e). As these depend upon four variables a complete tabulation is circumstantial. But without introducing greater errors than those due to the wind observations, we can write u^* as a function merely of two variables, the difference of velocity $u - u'$ and the difference of temperature $\vartheta' - \vartheta$, thus:

$$u^* \text{ approximately} = 273 \frac{u - u'}{\vartheta' - \vartheta}$$

The values of u^* according to this approximation formula is found from Table III.

Table III. Approximate values of $u^* \cdot 10^{-2}$.

Discontinuity of temperature $\vartheta' - \vartheta$ °C	Discontinuity of velocity $(u - u')$ m/sec. NB. The tabulated numbers multiplied by 100 give the quantity u^*, which is used as argument in Table II.											
	2	4	6	8	10	15	20	25	30	40	50	100
1	5.5	11	16	22	27	41	55	68	82	109	136	273
2	2.7	5.5	8.2	11	14	20	27	34	41	55	68	136
3	1.8	3.6	5.5	7.3	9.1	14	18	23	27	36	46	91
4	1.4	2.7	4.1	5.5	6.8	10	14	17	20	27	34	68
5	1.1	2.2	3.3	4.4	5.5	8.2	11	14	16	22	27	55
6	0.91	1.8	2.7	3.6	4.6	6.8	9.1	11	14	18	23	46
7	0.78	1.6	2.3	3.1	3.9	5.8	7.8	9.8	12	16	20	39
8	0.68	1.4	2.0	2.7	3.4	5.1	6.8	8.5	10	14	17	34
9	0.61	1.2	1.8	2.4	3.0	4.6	6.1	7.6	9.1	12	15	30
10	0.55	1.1	1.6	2.2	2.7	4.1	5.5	6.8	8.2	11	14	27
15	0.36	0.73	1.1	1.5	1.8	2.7	3.6	4.6	5.5	7.3	9.1	18
20	0.27	0.55	0.82	1.1	1.4	2.0	2.7	3.9	4.1	5.5	6.8	14
25	0.21	0.44	0.66	0.87	1.1	1.6	2.2	2.7	3.2	4.4	5.5	12
30	0.18	0.36	0.55	0.73	0.91	1.4	1.8	2.3	2.7	3.6	4.6	9.1

When the average temperature $(\vartheta' + \vartheta)/2$ differs considerably from 273, the tabulated figures of this table should be altered on the proportion $\vartheta/273$. Thus, when the table is applied to the surface separating stratosphere and troposphere we should subtract about 20 % from all tabulated figures.

When these formulae or tables are applied to the moderately curved air currents of the large scale atmospheric vortices, they give depressions of the isobaric surfaces in a cyclonic vortex, and an equal elevation in an anticyclonic vortex of the same intensity, in striking contrast to case (A). In the same manner (A) gives the same result and (B) different results for the inclination of a surface of discontinuity by cyclonic and anticyclonic direction of the circulation.

Case (C) is a compromise between the two extreme cases (A) and (B). The complete formulae (c) show that the two elementary effects (A) and (B) assist each other in cyclones, but counteract each other in anticyclones.

To limit ourselves to the consideration of the isobaric surfaces we see that there will be no limit to the depth which the depression may reach in cyclones. But in anticyclones we get a maximum of elevation of the isobaric surfaces for a value of u which fulfils the equation

$$\text{(e)} \qquad \frac{u}{R} \cos \chi = - \Omega \sin \varphi$$

Now $\Omega \sin \varphi$ is the vertical component of the angular velocity of the rotating system (the earth) and $u/R \cos \chi$ the vertical component of the relative angular velocity of the anticyclone. The maximum occurs at opposite equality of these two angular velocities, and is seen to have the value

$$\text{(f)} \qquad \text{tg}\, \theta_{\text{max.}} = \frac{u^2}{gR} \cos \chi$$

where u has the value given by the equation (e). But the same formula (f) only with the opposite sign because of the changed orientation of the axes is obtained when we apply equation (c) to the absolute motion in introducing $\Omega = 0$. It is seen in this way that (f) also represents the angle between the equipotential surfaces of the true exterior force and the surfaces called level by the inhabitants of the rotating system when this has the angular velocity $u/R \cos \chi$ round the vertical. In application to the earth this result is arrived at:

(III). *The isobaric surfaces in an anticyclonic vortex will have a maximum elevation at a point where the earth and the vortex have oppositely equal component angular velocity round the vertical through this point: the maximum angle of elevation is equal to the angle under which at that place the ellipsoidal level surfaces of the rotating earth cut the spherical level surfaces of pure attraction.*

When the anticyclonic angular velocity exceeds the value (e) which leads to this maximum the angle of elevation will again decrease, and become zero when the relative angular velocity round the vertical has become the double of that of the earth, with the opposite sign, $u/R \cos \chi = -2\Omega \sin \varphi$. For still greater angular velocities, depressions instead of elevations of the isobaric surfaces will then follow, and these depressions may increase without limit. Good examples are given by tornadoes and waterspouts, which may have both cyclonic and anticyclonic circulation, with in both cases strong barometric depressions.

Case (D) shows that with sufficiently great values of the velocity u, or of the vector u^* an asymmetric effect will appear, depending upon the azimuth ψ of the current, which was without influence in the cases (A)—(C).

Considering the isobaric surfaces we see that the angle of elevation is greater when $\sin \psi$ is positive and thus we have an east going current than when $\sin \psi$ is negative, and the current is travelling west. Or, under otherwise equal circumstances we get greater elevations when the surfaces rise on the equatorial rather than on the polar side of the current. The ultimate elevation to the one or the other side would be reached for infinite values of u, and amount to $\text{tg}\,\theta = \text{tg}\,\varphi / \sin \psi$, i. e., the tangent plane to the isobaric surface becomes parallel to the earth's axis. In the case $\sin \psi = 0$, for a current along the meridians, the plane would be a meridian plane and thus vertical. In all other cases it will be inclined to the polar side, and most so when $\sin \psi = 1$, i. e., the current directed along the parallels. For the angle of elevation we then find $\text{tg}\,\theta = \dfrac{2\Omega u \sin \varphi}{g - 2\Omega u \cos \varphi}$. This position is reached directly if the current is going west, and thus the surface rises on the polar side, but through the intermediate vertical position if the current is east going, and thus the surface rises from the equatorial side.

The great velocities which would lead to such elevations never occur in the earth's atmosphere, they would lead to such phenomena as the formation of Saturn Rings. But the vector u^* may by small differences reach any value, and a surface of discontinuity therefore reach any of these elevations.

27. Analytical examples of barotropic vortices by relative motion. — We shall confine the degree of approximation to that given by the formulae 26 (b). For the inclination of the isobaric surfaces the deviating force of the earth's rotation $2\Omega \sin\varphi\, u$ plays the same role as the centrifugal force u_a^2/y in the case of absolute motion. If we then in addition confine the extension of the vortex to such a small interval of latitude that we may consider φ as constant, we shall get the same inclination of the isobaric surfaces in a vortex with the absolute velocity u_a and a vortex with the relative velocity u, when u fulfils the relation

(a) $$2\Omega u \sin\varphi = \frac{u_a^2}{y}\ , \quad \text{i. e.,} \quad u = \frac{1}{2\Omega \sin\varphi} \cdot \frac{u_a^2}{y}$$

We can then use the examples worked out in sect. 18, and give the following table of isobaric surfaces and the distribution of velocity producing them in the case of absolute and the case of relative motion.

Isobaric surfaces	Absolute velocity	Relative velocity
(1) Paraboloids	$u_a = u_0\, y/b$	$u = \frac{u_0^2}{2\Omega \sin\varphi \cdot b} \cdot y/b$
(2) Cones	$u_a = u_0\, (y/b)^{\frac{1}{2}}$	$u = \frac{u_0^2}{2\Omega \sin\varphi \cdot b} \cdot 1$
(3) Logarithmic	$u_a = u_0$	$u = \frac{u_0^2}{2\Omega \sin\varphi \cdot b} \cdot b/y$
(4) Hyperboloidal	$u_a = u_0\, (b/y)^{\frac{1}{2}}$	$u = \frac{u_0^2}{2\Omega \sin\varphi \cdot b}\, (b/y)^2$
(5) Hyperboloidal of third degree	$u_a = u_0\, (b/y)$	$u = \frac{u_0^2}{2\Omega \sin\varphi \cdot b}\, (b/y)^3$
(6) Binomic logarithmic	$u_a = \sqrt{2}\, u_0\, y/b\, (1 + y^2/b^2)^{-\frac{1}{2}}$	$u = \frac{2u_0^2}{2\Omega \sin\varphi \cdot b} \cdot y/b\, (1 + y^2/b^2)^{-1}$
(7) Binomic of third degree	$u_a = 2u_0\, y/b\, (1 + y^2/b^2)^{-1}$	$u = \frac{4u_0^2}{2\Omega \sin\varphi \cdot b} \cdot y/b\, (1 + y^2/b^2)^{-2}$

It is seen that in one case only we get the same isobaric surfaces by the same law of velocity distribution: when the velocity increases proportional to the distance from the axis, so that the fluid rotates like a rigid body, we get paraboloidal isobaric surfaces in both cases. Otherwise, we have the characteristic differences, that conical surfaces appear when the absolute velocity increases as the square root of the distance, and when the relative velocity is invariable; the logarithmic surfaces when the absolute velocity is invariable and when the relative velocity decreases in inverse ratio to the distance from the axis, and so on.

The results are illustrated by the diagrams Figs. 10 and 11 which then may be used to represent vortices both for absolute and relative motion.

But one striking difference between the cases of absolute and relative motion should be remembered: opposite and equal values of the velocity give the same isobaric surfaces for absolute, but opposite and inclined surfaces for relative motion. In the case of anticyclonic relative velocity we must imagine the diagrams of Figs. 10 and 11 reversed, remembering, however, that there is a definite limit to the elevations which the isobaric surfaces can reach in the case of anticyclonic circulation.

The formulae we have given may be used in first approximations for the representation of the fields of motion and the correlated fields of pressure in atmospheric vortices. Of course other expressions containing, if needed, a greater number of constants may easily be formed, and greater approximation be obtained by starting from more exact formulae than 26 (b).

28. Simplest models of stationary cyclones and anticyclones. — As an illustration to the preceding principles of absolute and relative motion we shall give what may be considered as the simplest small scale model of a cyclone or an anticyclone, simple enough to be partly realized by laboratory experiment.

The exterior force is constant gravity. The fluid system shall in the state of equilibrium consist of two horizontal strata. The internal equilibrium in each of these shall be stable, and at the mutual surface of separation there shall be a sudden drop of density as we pass through it from below. Then we consider the same system when it has a vortex motion with an angular velocity which is great enough, compared to that of the earth, to allow us to treat the motion relatively to the earth as absolute. Under these suppositions we shall consider two different distributions of velocity in the vortex.

In the first case there shall be increased circulation as we proceed upwards in the lower stratum. A maximum is reached just below the surface of discontinuity, while a sudden drop follows as we pass this surface; higher up we shall no longer have any marked variations. To this distribution of velocity corresponds a field of pressure characterized by increasing inclination of the isobaric surfaces as we proceed upwards in the lower stratum, followed by a sudden decrease as we enter the upper stratum. The corresponding distribution of mass is governed by the pumping effect exerted by the layer which has the maximum circulation, i. e., the upper layer of the lower stratum. In the region of the axis, the surface of discontinuity which separates the two strata is sucked down so that a depression exceeding that of the isobaric surfaces in either stratum is formed (cf. Fig. 14 E). The latter surfaces show the typical refraction as they cut through the surface of discontinuity. But the same pumping effect acts also upon the heavier masses below the stratum of maximum vortex intensity. These masses are sucked up to regions of lower pressure, and by a sufficient intensity even to higher levels than those to which they belong statically. Then the equisubstantial surfaces in the lower strata show elevations (cf. Fig. 17 A, corresponding to the depressions presented by the surface of discontinuity.

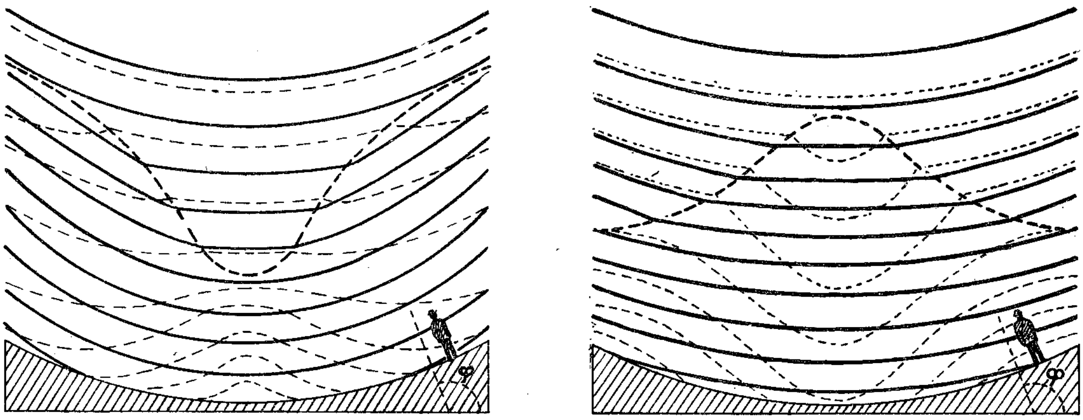
If the fluid medium is an ideal gas, the density will depend upon pressure and temperature. The surface of discontinuity is characterized by a sudden increase of temperature as we pass through it from below, while the stable stratification in the lower layer involves a lapse rate of temperature smaller than adiabatic. The masses lifted from below will then get the required equilibrium density by adiabatic cooling and thus lower temperatures than those previously existing in the attained layer. Then in the lower stratum the vortex will have a *cold core*, and the isothermal surfaces will show corresponding depressions within the central part of the vortex.

The vortex described shows characteristic features of a cyclone in the atmosphere. Ascents in the free atmosphere have shown that the stratosphere has a marked depression above the cyclonic area, and further that within the troposphere we have in general cold masses round the axis. Our model gives the simplest dynamical arrangement which may account for these two phenomena: both may be explained by the pumping effect due to a sufficiently intensive circulation in the upper part of the troposphere.

In the second case, the intensity of the vortex motion shall first decrease as we proceed upwards in the lower stratum, until a minimum is reached just below the boundary surface; then a sudden increase shall follow as we pass through the surface, and afterwards a very slow increase or constancy.

The isobaric surfaces will then be relatively steep near the ground, flatten as we proceed upwards in the lower layer, and then suddenly become steeper again as we pass into the upper layer. The pumping effect due to the strong circulation in the upper stratum makes the boundary surface rise and present an elevation in the central region (cf. Fig. 14 A). In the same manner the intensive circulation near the ground, sucks down lighter masses from the upper parts of the lower stratum. Thus depressions are formed in the surface of equal density exceeding the depressions presented by the isobaric surfaces (cf. Fig. 17 E). In case of a gas, the equilibrium density of the masses thus sucked down is reached by adiabatic heating. The vortex has in its lower part a *warm core,* and if we draw the isothermal surfaces they will present elevations in this part of the vortex.

This vortex shows characteristic features of what is called an *anticyclone* in the atmosphere. The ascents have shown that above the anticyclonic area the boundary sur-



A. Cyclone. B. Anticyclone.

Fig. 19. Simplest models of cyclone and anticyclone.

face between troposphere and stratosphere shows a marked elevation, and statistics made upon the recorded temperatures show that within the troposphere the anticyclone has in general a warm core.

Returning from temperatures to pressures, our vortex still differs in one important point from the anticyclone. For in the latter the isobaric surfaces present elevations, while in our vortex they show depressions. But this aspect is changed when we place ourselves in the position of an observer who participates in the motion of the rotation.

Let us suppose the two vortices of Fig. 19 A and B to have the same direction of circulation while the intensity of their motion is different, that of vortex A being strongest as indicated by the stronger inclination of its isobaric surfaces. Then let the observer circulate with an intermediate angular velocity, smaller than the smallest occurring in A, and greater than the greatest occurring in B. The relative circulations observed by him will then be of opposite directions in the two vortices, and in accordance herewith he may call the one, A, a cyclone, and the other, B, an anticyclone. In the anticyclone he will ascribe the strongest anticyclonic circulation to that stratum which in the absolute motion has the weakest cyclonic circulation: the stratum just below the surface of discontinuity.

The level surfaces for this observer will be paraboloids of revolution, which are flatter than the flattest isobaric surfaces of the cyclone A, but deeper than the deepest isobaric surfaces of the anticyclone B. He will then find the isobaric surfaces of the cyclone concave. But those of the anticyclone will appear convex, the strongest convexity being found by the surfaces just below the surface of discontinuity, where the anticyclonic circulation appears strongest.

In all cases he will find the angles of elevation represented by the formulae 25

(f) and (g), where it is to be remembered that the angle of latitude φ for the parallel circles on the paraboloids is in the second quadrant. The simpler formulae 26 (b) may be used when the angles are sufficiently small. It will be found that there exists a definite maximum of relative elevation for the isobaric surfaces in the anticyclone: they may rise to coincidence with the horizontal planes of exterior gravity: this is the case when the relative anticyclonic circulation gives equilibrium from the point of view of the absolute motion.

In these cyclone or anticyclone models we have not yet any ascending or descending motion. But these will appear as soon as the supposed steady state of motion be slightly modified by friction at the ground.

In the case of the cyclone, the friction against the rotating ground will, both absolutely and relatively reckoned, retard the circulation in the lower strata. This gives a lack of centrifugal force here, the horizontal pressure gradient is in excess, and the air nearest the ground is forced inwards towards the centre. Then the increased deficit of circulation in the lower strata relative to the higher leads to an increased pumping effect. This facilitates the motion upwards in the central region which must continue the motion inwards along the ground. In the higher levels then a motion outwards will follow, in reality facilitated in the same way.

This slow vertical circulation does not essentially effect the temperature distribution which we have already deduced; the ascending air in the cyclone remains cold and heavy, being forced up dynamically. But as the formation of this vertical circulation is entirely due to a destruction of kinetic energy by friction, it can be permanently entertained only on the condition that the horizontal circulation in the upper strata does not lose its energy. The ultimate effect of this vertical circulation is that the entire cyclone becomes filled with masses which always have slower horizontal circulation, and the cyclone must die. Therefore the cyclone can persist only on the condition that it propagates to places where it finds new kinetic energy to annihilate.

In the anticyclone the friction against the rotating ground will reduce the circulation in the lowest layers, reckoned relatively, but increase them when reckoned absolutely. This gives an excess of the centrifugal force over the gradient, and a motion outwards in the lower layer. At the same time the increased circulation in the lowest layer gives an increase of the pumping effect. This facilitates the motion downwards which must feed the motion outwards in the central region. We get the well known anticyclonic vertical circulation: inwards in the upper levels, downwards in the central region, and outwards along the ground. The descending air in this case is varm and dynamically moved downwards in spite of its bouyancy.

This anticyclonic vertical circulation will be maintained as long as this circulation does not bring down to the ground masses which, absolutely reckoned, have stronger circulation than the ground.

The theory which has thus been developed for circular vortices may be developed independently of any supposition of circular form when we use the general theory of circulation relative to the earth. This method has been used by Sandström, in papers referred to (on page 5).

V. The Earth's Atmosphere as a Circular Vortex.

29. The main atmospheric surfaces of discontinuity. — When we consider the motion of the atmosphere in reference to coordinates which do not participate in the rotation of the earth, we may in the first approximation consider the conditions of the circular vortex as fulfilled. The motion is considered as going along the parallels, with a constant angular velocity for each parallel, differing in most cases slightly from that of the earth.

This vortex is characterized first of all by its great surfaces of discontinuity. These are partly the external boundary surfaces, partly internal surfaces, dividing the atmosphere in more or less distinct divisions.

The first external surface is that separating the atmosphere from the lithosphere and hydrosphere. The boundary surface of the lithosphere is given and rigid, but that of the hydrosphere takes equilibrium inclination in accordance with formula 24 (n), or with sufficient approximation 26 (b). But the elevations are exceedingly slight, and of importance only in geodesy.

We may also have to acknowledge an upper boundary surface, outside of which we have to put $\varrho' = 0$ in formula 24 (n). This formula becomes identical with 24 (m), i. e., the boundary surface towards empty space must be an isobaric surface. It is also probable that at this distance from the earth's surface all internal differences of velocity have been smoothed out, so that the atmosphere here rotates as a rigid body, with the angular velocity of the solid earth. In that case we get the motion relative to the earth equal to zero, and the formula gives $\theta = 0$. The boundary surface of the atmosphere, if there is any, will then be one of the level surfaces of the apparent gravity resulting from gravitational attraction and centrifugal force, either the lenticular surface itself, or one of the surfaces inside it on Fig. 12. But if we have other angular velocities than that of the earth, other forms of the boundary surface may be present, including also Saturn Rings.

Passing then to the *internal* surfaces of discontinuity, we remember that the idea of an abrupt change at a surface is merely a convenient abbreviation for the idea of a transitional layer, of finite thickness, within which we have a rapid but continuous variation of velocity and density.

The most important internal surface of discontinuity in this sense of the word is *that separating troposphere and stratosphere* or the *tropopause* as it has been called by English meteorologists.[1]) Passing through it from below we have a sudden increase of temperature or at least a marked decrease of the vertical temperature gradient. As to the air motion we find in general westerly winds below, and more or less indeterminate motions above it at all latitudes from the poles down to about 20° N or S. From the poles to these latitudes we have thus over-normal angular velocity below the surface and more or less normal above it. This leads to the oblate form of the surface as in Fig. 15, E, central part of the figure. The surface is more oblate than the isobaric surfaces, even of the troposphere, which in their turn are more oblate than the level surface of the earth. In the equatorial belt between N 20° and S 20°, we have easterly winds which seem to reach great heights without showing the same marked discontinuity in the passage from troposhpere to stratosphere as at higher latitudes. Here the surface will have more the character shown by Fig. 15 C. Thus, when we proceed from the poles towards the equator, the boundary of the stratosphere will be found at an increasingly greater height above sea-level until we reach the latitude of 20°. Within the equatorial belt itself, it may go down again. But we lack sufficient data to determine its course more accurately. But even if it goes down, the depression will be slight on account of the factor $\sin \varphi$ occurring in the formula 26 (d). The main result will therefore be that the surface will be low over the polar and high above the subtropical and equatorial regions.

The heights actually found are about 9 km. near the poles and 17 in the inter-

[1]) Meteorological Glossary **1918**.

tropical zone. This gives an average rise of the surface of 1 : 1000 from the poles to the subtropics. Now the first table of sect. 26 shows that the rise 1 : 1000 will, at an average latitude of 50°, correspond to $u^* = 90$. As the figures tabulated in the second table of sect. 26 are 20 % too great in case of the tropopause, we have, by the reversed use of the table to add 20 % to the above value 90. For $u^* = 110$ this second table gives then the following possible combinations of correlated discontinuities when we pass through the surfaces from below:

Decrease of velocity:	2	4	6	8	10 m/sec.
Increase of temperature:	5	10	15	20	25 °C.

Thus, if the tropopause were a real surface of discontinuity, and not a diffuse transitional layer, its average inclination of 1 : 1000 would correspond for instance to an increase of temperature from — 60 °C. to — 50 °C., and a corresponding decrease of westerly wind from 6 m/sec. to 2 m/sec. The inclination of the isobaric surfaces for these winds would be 1 : 15000 and 1 : 44000 respectively.

The winds actually met with in the upper part of the troposphere are much stronger than 6 m/sec. But as they have all possible directions a resultant to the east of this order of magnitude may be very reasonable.

Nothing is known positively as yet about further surfaces of discontinuity within the stratosphere, though there may be theoretical reasons to believe in layers of rapid change of the chemical constituents of the atmosphere. But in the troposphere we meet with two internal boundary surfaces of high meteorological importance.

First, there is a *marked transitional layer between the trade and antitrade winds*, which is well known, for instance, in the meteorology of Teneriffe. Thus, *K. v. Fritsch* reports from this island:[1]) »Bei meinen Wanderungen fand ich in September 1862 die obere Grenze des Passats meist bei 2000 bis 2400 Meter, bisweilen schien dieselbe aber bedeutend auf- und abwärtz zu schwanken. Über dem Passatwind folgt in der Regel eine 300 bis 600 Meter mächtige windstille Zwischenregion, über welcher erst der Antipassat aus Südwesten weht, — ein Wind der fast stets auf dem Teyde herrscht, oft auch, während in der Nähe der Küste noch der Passat fühlbar ist, herabsteigt bis zu den Höhen von Canaria (1800 bis 1900 Meter) und Palma (2000 bis 2200 Meter).»

To avoid false estimations of the thickness of this layer it is important to note that lower down in the trade wind another inversion is found, namely at the upper limit of the trade wind clouds. v. Fritsch reports of these (ibid. p. 219). »Diese Wolkenschicht durchschritt ich oft bei Bergwanderungen. Sie ist in der Regel 300 Meter und mehr mächtig. Keineswegs bezeichnet sie die obere Grenze des Passats, der erst 600 bis 1000 Meter höher durch die fast windstille Zone vom Antipassat getrennt ist». But as another expert of Teneriffe, Prof. *R. Wenger* remarks, it is not always easy to distinguish between the two layers. For, »Wenn die Passatschicht weniger mächtig ist, fallen beide Inversionsschichten wohl auch zusammen und erscheinen als *eine* mächtige Schicht von 1000 Meter und mehr Dicke«.[2]) That the thickness of the true transitional layer should be of the order of magnitude indicated by v. Fritsch, 300 to 600 meters, seems to be in good accordance also with the results of aerological ascents, when we have the opportunity to examine them individually[3]). Thus we are concerned with a relatively thin

[1]) *K. v. Fritsch*: Meteorologische und klimatographische Beiträge zur Kenntniss der Canarischen Inseln. Petermanns Mitteilungen 1866, p. 218.

[2]) *R. Wenger*: Untersuchungen über die Mechanik und Thermodynamik der freien Atmosphäre im nordatlantischen Passatgebiet. Beiträge zur Physik der freien Atmosphäre, Bd. 3, p. 198.

[3]) Cf. two different curves of pilot-balloon ascents given in the quoted paper of *Wenger*, and the table of original observations in *H. U. Sverdrup*, Der Nordatlantische Passat«, Veröffentlichungen des Geophysikalischen Instituts der Universität Leipzig, II. Ser., 1917.

transitional layer. And we are entitled to substitute for it the mental picture of a surface of simultaneous thermal and kinematical discontinuity: *the sliding surface of the trades.*

The aspect is of course fully changed when from individual observations we pass to statistics made for finding average conditions at different heights. The results will then not be a determination of the true thickness of the layer, *but of the space in which the layer oscillates.* For this space, which *Hergesell* has called »Mischungsschicht«, *H. U. Sverdrup's* statistics give an average thickness of about two kilometers. It is represented in the annexed diagram which is copied from Sverdrup's paper. Disregarding the thickness of the sheet, we direct our attention to its inclination: this striking inclination proves that there must have been a corresponding inclination of the sliding surface, of sufficient strength and constancy to survive the process of averaging.

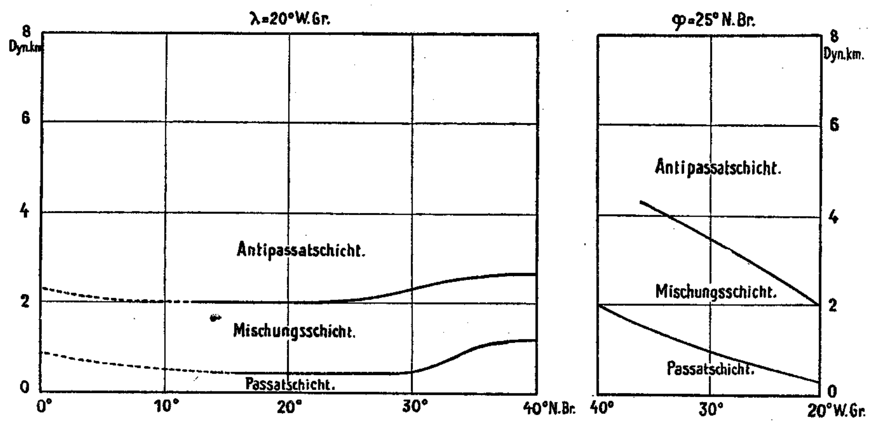


Fig. 20. Space in which the sliding surface of the North Atlantic trades oscillates.

For the discussion of this inclination it should be noticed that if the atmosphere was a true circular vortex without friction, the trade wind would be a pure east and the antitrade wind a pure west wind. This would give undernormal angular velocity below and overnormal above the mutual boundary surface. Consequently, this surface would be of the type represented by Fig. 15, A, i. e., sloping from the pole to the equator.

Under the actual conditions a marked slope in this direction is to be expected only in the region where the antitrade wind, returning from the equator, has gained a west component, i. e., for latitudes between 20° and the subtropical highs. In this region Sverdrup's diagram also shows a slope attaining the maximal value of about 1 : 700. For the latitude of 30° the first table of sect. 26 gives then $u^* = 200$, and this value of u^* used in the second table of sect. 26 leads to the following possible combinations of correlated discontinuities:

Difference of wind:	2	4	6	8	10	15 m/sec.
» » temperature:	3	6	8	12	14	21 °C.

A wind discontinuity of 6 m/sec. is very probable according to the observations used by Sverdrup, but the corresponding increase of temperature of 8 °C seems rather great. I. e., the condition of the trades are not so constant that the method of direct averaging can give the full value of the inclination of the sliding surface.

Besides this slope towards the equator, Sverdrup has found, that the same »Mischungsschicht« has a striking slope from the American to the African continent. The angle of this slope is not in itself greater, but the phenomenon is more conspicuous because it continues over a much greater distance. This slope is due to the north and south components of the trade and the antitrade winds. As they are of the same order of magnitude as the corresponding east and west components, with the same difference of temperature, an angle of inclination of the same order of magnitude must come out.

In case of a true circular vortex, the north and south components producing this latter slope would not exist. The discontinuity would define a surface of revolution all round the earth, uninterrupted by the continents. It might also extend laterally to the boundaries of the troposphere, dividing this part of the atmosphere into departments of eastern equatorial winds and western winds of the temperate zone. But as under actual conditions a vertical circulation is entertained round it, it must necessarily have free borders, both to the equatorial and to the polar side, and has in all probability borders also in the vicinity of the continents.

Then we have within the troposphere a second surface of discontinuity, the existance of which has been predicted from theoretical reasons by *Helmholtz* as early as 1888,[1]) while the full empirical evidence for its existence as well as for its meteorological importance is of more recent date. It separates the cold polar air from the warmer air of more equatorial origin, and cuts the earth along the line which has been called the »polar front«. On the meteorological maps this line is seen to be in continuous motion, sweeping over the entire zone called the temperate. But in the mental picture which we construct of the atmosphere in the form of a circular vortex the polar front must be a steady line, following a certain parallel, say that of 60 °. *The sliding surface separating the western equatorial and the eastern polar winds would then in the defined planetary system be a surface of revolution cutting the earth along this parallel.*

Within this surface, we should then have undernormal, and outside it overnormal angular velocity, leading to a surface like that of Fig. 15 A, forming a well-defined calotte over each pole. At present we know this surface, or the corresponding finite layer, best in the cyclones, where we have the strongest wind and temperature discontinuities. The layer may have a similar thickness as that of the trades,[2]) and an inclination of the order of magnitude of 1 : 100. Then the first table 26 gives, for the latitude of 60 °, $u = 800$ and the second table 26 leads to the following possible combinations of correlated discontinuities:

Difference of wind:	4	6	8	10	15	20	25	30	50 m/sec.
» » temperature:	1.5	2	2.8	3.7	5	7	8.5	10	17 °C.

which are all reasonable according to our experience concerning shifts of wind and contrasts of temperature in cyclones.

While there is no doubt that this surface cuts the ground, we have at present no direct empirical data to decide if it really reaches the full height of the tropopause, dividing the troposphere into two distinct divisions, or if it ends with a more or less

[1]) *Helmholtz*: Ueber atmosphärische Bewegungen. Sitzungsberichte der k. preuss. Akademie der Wissenschaften 31. Mai 1888. Meteorologische Zeitschrift 1888, p. 329—340.

[2]) Meteorological ascents through it are at hand in great number, but have not yet been examined systematically in connection with the weather charts. Examples of ascents under well defined conditions are represented by Fig. 8 of J. Bjerknes' and H. Solberg's quoted paper, leading to an average thickness of 300 m. Similar results are found by Captain *Douglas:* Temperature variations in the lowest four kilometers, Quarterly Journal of the Roy. Met. Society, January 1921. See especially the example on p. 33—34.

indeterminate border, as the corresponding surface in the zone of the trade winds. The question is of fundamental importance for deciding the nature of the atmospheric circulation between the higher and the lower latitudes, and will be discussed in connection with this subject below.

With some reserve as to the completeness of the divisions we may say that the

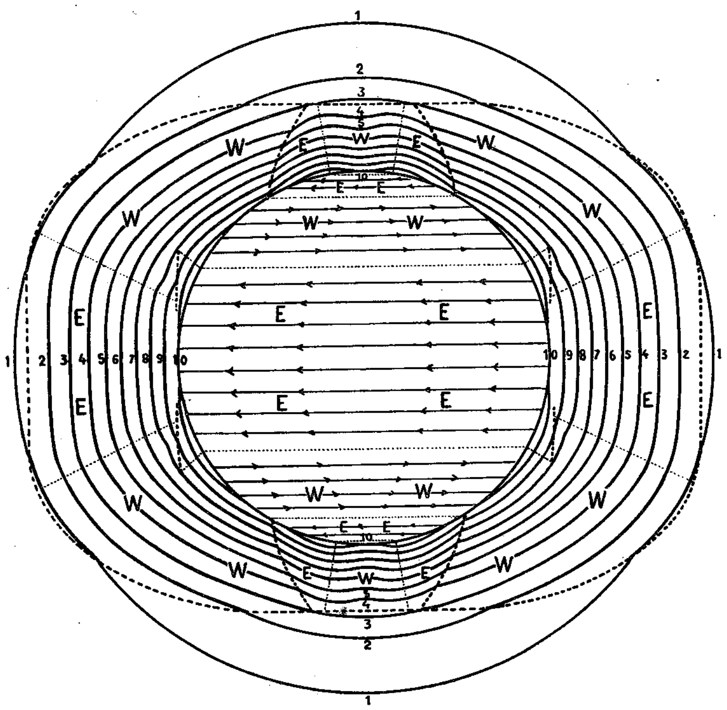


Fig. 21. Surfaces of discontinuity (dotted), isobaric surfaces, and wind zones of the planetary vortex.

tropopause and the polar sliding surfaces divide the atmosphere into four parts: (Fig. 21 stratosphere, the equatorial part of the troposphere, and its two polar caps.

We shall than consider more in detail the different meteorological elements within these different parts.

30. Velocity and pressure in the atmosphere. — The general features of the air motion have already been considered for deriving the course of the surfaces of discontinuity. Going more in detail, we shall derive the course of the isobaric surfaces within the different atmospheric departments.

The winds in the higher part of the troposphere are, as mentioned already, eastern in an equatorial belt between the parallels of about 20 ° N and S, and western from these latitudes so far into the polar regions as we have any knowledge. The corresponding course of the isobaric surfaces will be simple: they are lowest above the poles, rise gradually above the level surfaces as we approach the equator, reaching their highest elevation above sea level at about 20 °. There they decline again to the equator, but very slightly as the sin φ occurring in the formula for the inclination approaches here to zero.

Near the ground we have a more complex distribution. The equatorial zone of easterly winds is larger, extending below the sliding surfaces of the trades up to latitudes of 30°. Then westerly winds follow up to the polar front at 60°, and from this latitude of shift easterly winds so far towards the poles as we have sufficient observations for making statistics. But theoretical reasons will be given below for a tendency to a last shift to westerly winds again sufficiently near the poles, due to the descending air in the central polar regions. As this air arrives from lower latitudes, it cannot easily have lost its westerly motion before it has come into contact with the ground and begun its motions to lower latitudes again: To accentuate this we may introduce the idea of a zone of westerly winds nearest the poles in our planetary vortex.

The isobaric surfaces in the lower strata should then begin by a depression near the pole itself. Then they should rise to the parallel of the maximum easterly winds, and decline again to the latitude of the polar front. Along this we should have a trough of low pressure in case of the undisturbed planetary vortex. As the isobaric surfaces cut through the sliding surface of discontinuity they are refracted, rising again as they enter the region of the westerly winds. Then they are elevated highest at the latitude of the subtropical calms, about 30°, in order to fall slightly again, forming a shallow trough in the region of the equatorial east winds.

We do not know the motions sufficiently well in the stratosphere to give details on the course of the surfaces here. Our main empirical result is the sudden drop of velocity when we enter it from the troposphere. The isobaric surfaces of the troposphere, which are in general more oblate then the level surfaces, will consequently be refracted towards parallelism with these as they cut through the tropopause.

Concerning the greater heights in the stratosphere, we can only mention the two principal theoretical possibilities. If the atmosphere is limited we must suppose that the internal differences of velocity will be smoothed out as we move away from the surface of the earth, which is the source of the disturbances. The final result should then be that the outer parts of the atmosphere rotate as a rigid body, with the angular velocity of the solid earth. In this case the isobaric surfaces would approach more and more to the surfaces represented by Fig. 12, and the limit of the atmosphere would be one of them, in the extreme case the lenticular surface itself. If, on the other hand, the atmosphere is unlimited, its motion of rotation will cease with increasing distance from the earth. In this case the isobaric surfaces in the stratosphere will more and more approach towards spheres round the center of the earth.

Fig. 21 gives the scheme of surfaces of discontinuity, the different wind departments, and of isobaric surfaces according to the results developed above. The dimensions in the vertical direction are exaggerated 300 times, and the isobaric surfaces of the troposphere are for simplicity drawn as spheres.

31. Temperature distribution in the atmosphere. — The theorem 22 (A) takes the following form when we apply it to the case of the atmosphere:

(A) *Increase of absolute velocity (linear or angular) upwards along a parallel to the earth's axis gives isobaric temperature gradient directed from the equator to the pole; decrease gives isobaric temperature gradient directed from pole to equator.*

This law, combined with the empirical data concerning the increase or decrease of temperature along the vertical gives the course of the isobaric surfaces. It is easily seen that we arrive at the following rules:

(B) *In atmospheric layers with decrease of temperature upwards along the vertical: the isothermal surfaces have their slope from the equator to the poles in regions where the velocity increases, from the poles to the equator where it decreases upwards along a parallel to the earth's axis.*

(C) *In atmospheric layers of temperature inversion: the isothermal surfaces have their slope from the pole to the equator where the absolute velocity increases, from the equator to the poles where it decreases upwards along a parallel to the earth's axis.*

We shall first apply these rules to the *zone of westerly winds of the intermediate latitudes.* Ascending parallel to the earth's axis we here find increasing intensity of the westerly winds from the ground to the vicinity of the stratosphere, and then decrease as soon as the retarding frictional effect from the stratosphere begins. This decrease will in all probability continue all through the stratosphere itself as the velocity converges to its ultimate value, namely zero if the atmosphere is unlimited, and that given by the angular velocity of the earth if the atmosphere is limited. The rule (A) for the isobaric or practically horizontal temperature gradient gives then the well known decrease of temperature from equator to pole in the greater part of the troposphere for all heights from the ground up to the level of maximal westerly winds. But vice versa, it leads to decrease of temperature from pole to equator in the highest layer of the troposphere and in the stratosphere.

In general, these horizontal temperature gradients are small. The greatest values occur in the region of the rapid variation of wind intensity, i. e., on the one side near the ground, and on the other side in the highest layer of the troposphere and the lowest of the stratosphere. An isobaric surface which cuts through this layer will be in the stratosphere on the polar and in the troposphere on the equatorial side. Then following it in the direction from the pole to the equator we have a rapid fall of temperature passing from the stratosphere to the troposphere.

Conditions will be the same in the admitted *polar zone of westerly winds* as in that of the intermediate latitudes. The distribution of velocity along a parallel to the earth's axis is here the same, and must consequently lead to the same decrease of temperature towards the pole in the troposphere and towards the equator iu the stratosphere.

The regions of the pure westerly winds in the troposphere are bordered by *zones with easterly winds near the ground undercutting the westerly*: the zone of the trades, and the zone of the eastern polar winds. But considered as absolute velocities, those at the ground are still westerly, only weaker. Ascending along the parallel to the earth's axis we find à fortiori decreasing absolute velocities as from the layers of easterly winds we approach and enter those of westerly. The general result will therefore be the same as in the regions of pure westerly winds: decrease of temperature from equator to pole within the troposphere, and from pole to equator in the stratosphere.

But a certain reserve must be applied to the very lowest strata, in the immediate vicinity of the ground: here the east winds are retarded by the friction. Ascending from the ground along a parallel to the earth's axis we have then to begin with not decreasing but increasing east wind, i. e., decreasing absolute velocity, and should by dynamical reasons expect a decrease of temperature towards the equator. But just in these layers the adjustment to the equilibrium conditions is most retarded by the friction. It is therefore to be expected that the temperature distribution will remain determined principally by the conditions of the solar radiation, though the increase of the temperature towards the equator may be somewhat reduced.

The zone of the trades is bordered on the equatorial side by *that of pure east winds.* Central in this zone we have the equatorial calm, with an average width of 4 °, while the winds of the higher levels remain easterly up to latitudes of about 20 ° on either sides.

We then consider a line parallel to the earth's axis which touches the earth at the equator. Travelling along it we first have the calm, with the full angular velocity of the earth, and then the east winds, with undernormal angular velocity. Then our line will reach the stratosphere about 4—500 kilometers or not quite 4—5 degrees north and south of the equator, thus completely within the zone of the equatorial east winds.[1] Up to this limit the characteristic feature of the velocity distribution is decrease of the

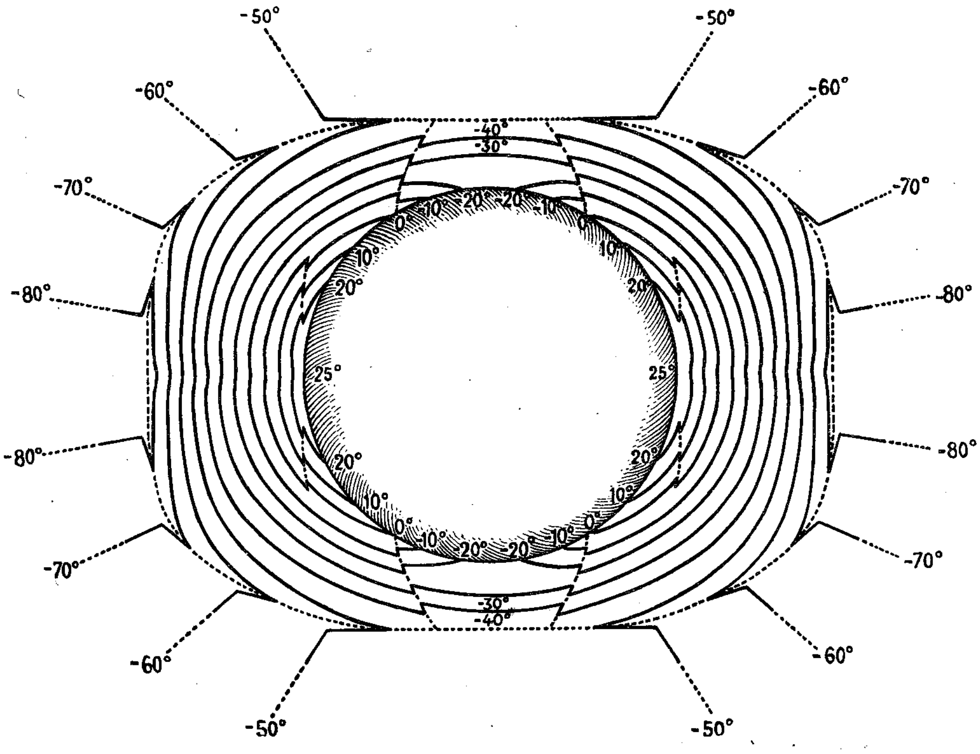


Fig. 22. Isotherms of the atmosphere.

angular velocity as we proceed upwards along the line. Consequently the isobaric temperature gradient should be directed towards the equator: we should have a minimum of temperature within the equatorial calm. Too near the ground the adjustment of the temperature to the dynamical equilibrium conditions may be retarded by friction. But at slightly higher levels this equatorial minimum of temperature ought to be a fact. And it should exist upwards as long as the angular velocity of the air in the equatorial plane itself is greater than to both sides of it. Within the troposphere, where the distribution of velocity along the parallels is determined by the convection of air masses which have received their velocity by friction at the ground, we can hardly look for any other place for maximum of absolute velocity than in the equatorial plane, where the air masses have their motion from the fastest moving part of the solid earth.

The isothermal surfaces of the circular atmospheric vortices may easily be drawn in accordance with the results thus derived for the isobaric temperature gradient, and guided by the rules (B) and (C). This leads to the diagram of Fig. 22. Here the atmospheric

[1] The vertical dimensions of the atmosphere in Figs. 21 and 22 must be reduced to their real proportions if this is to be seen from the diagram.

surfaces of discontinuity present themselves as temperature inversions seen by the folding of the isothermal surfaces.

The great features of this diagram are known to be true: the general slope of the isothermal surfaces in the lower part of the troposphere from the equator towards the poles; the inversion at the passage from troposphere to stratosphere or at least a corresponding decrease of the vertical temperature gradient; the inversions at the sliding surface separating the trade and the antitrade wind, and at that separating polar and equatorial air.

But this picture would not come out when averages are formed in the ordinary way. The characteristic discontinuities would then to a great extent be smoothed out, on account of the motions of the surfaces of discontinuity, especially in the zone of the

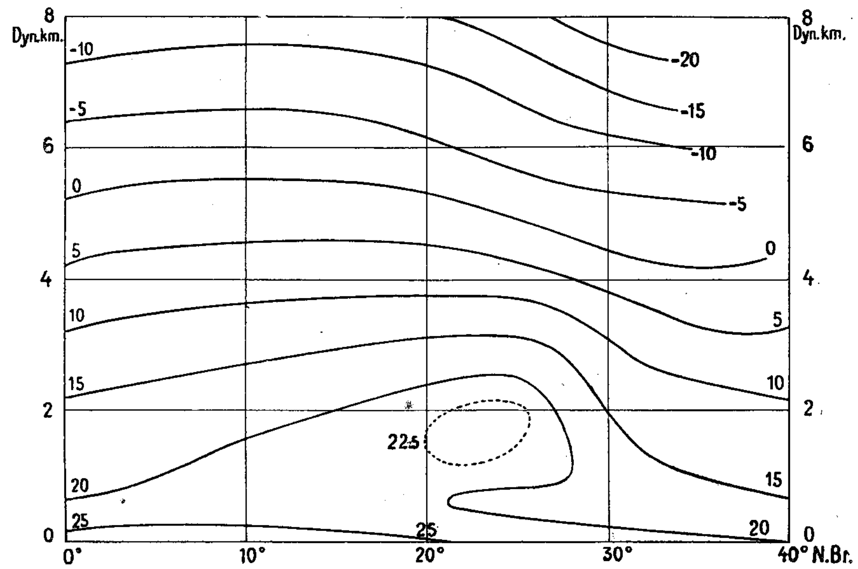


Fig. 23. Average isotherms of the North Atlantic trade according to Sverdrup.

cyclones. But in the trade wind zone the relatively stationary conditions may make the method less disadvantageous. The annexed diagram Fig. 23 according to Sverdrup gives the average isotherms in the North Atlantic trade wind zone, along the meridian 20° W. This diagram seems to give a remarkable verification of theoretical conclusions: all isotherms have depressions at the equator, rise towards north, and have their highest points more or less north of the equator: the 25° isotherm reaches its highest point between 5° and 10° N. The 20° isotherm reaches its highest point as far north as 23° N, and shows then the typical folding in the region where it cuts the sliding surface of the trades. Then for the following isotherms the highest point is displaced to the south again. But still for the isotherm — 10°C it is north of the 10° parallel.

Still a single verification as this must be taken with some reserve, especially as the temperature maximum is so marked, as shown by the isotherms for 20 and 15°C, and displaced so far to the north. A reason for this may be the nearness of the African continent. The best place for testing the theory would be in the central part of the Pacific. The wind charts show that here the trade winds are much more directed along the parallels than in the narrow channel of the Atlantic, which favors the north—south components of the wind, and thereby departures from the true conditions of the planetary vortex.

It may be subject to discussion how the isothermal surfaces should be properly continued in the higher part of the stratosphere. If we suppose that the temperature for increasing height converges to a uniform value between — 60° and — 70°, the isotherms of — 50° and — 60° would curve to the poles, bounding relatively warm air masses concentrated round the earth's axis and situated above the coldest areas of the earth's surface. And the isotherms — 70° and — 80° would curve towards the equator, and form closed annular surfaces, defining a Saturn-Ring of cold air surrounding the earth, and situated above the warmest part of the earth's surface.

32. *Stability and disturbance.* — We have thus described a type of »planetary« circulation, which gives already a close relation to certain general conditions of the atmosphere.

The main feature of this vortex is that it separates from each other air masses of different temperatures. If radiation and conduction keep the field of temperature invariable, and if no frictional resistance or other disturbances interfere, this vortex will persist invariable. But as in reality always disturbing effects come into play, the next step will be to see how the vortex behaves in case of disturbances of different types.

We have to take two types into consideration: on the one side a sudden impulse, which in short time produces finite departure from the conditions of the circular vortex, and on the other side the friction, which continuously and permanently affects and changes the motion.

The first case is the easier to treat from the mathematical point of view. It leads in the first approximation to a purely hydrodynamical problem, in which thermodynamics does not interfere explicitly. The classical theory for the motion of a system in the vicinity of a state of equilibrium may serve as model. Still the full analytical solution will be as difficult as it is important.

The second problem is undoubtedly still much more difficult from an analytical point of view. The frictional terms in the hydrodynamic equations increase enormously the difficulty of their integration. Further, the friction causes progressively increasing departures of the particles from the circles of equilibrium along which they moved in the undisturbed planetary vortex. Therefore, we cannot, as in the previous problem, introduce any condition of a limited departure of the particles from these circles. In the hydrodynamical equations we can then no longer neglect the second order terms in the expressions of the acceleration, nor can we simplify by excluding thermodynamics from our considerations.

We shall take a general view of both problems to the extent possible without entering into mathematical details.

VI. Disturbances in the Atmospheric Surfaces of Discontinuity. Wave Theory of Cyclones and Anticyclones.

33. *Historical remarks.* — When a state of stable equilibrium of any system is disturbed there will always result an oscillating motion. If the system moves steadily satisfying certain conditions of stability, a disturbance will in the same manner produce

oscillations which are superposed upon the steady motion. When the system is a continuous medium, the oscillations produced will have the character of standing or of propagating waves. These general considerations lead naturally to the idea that wave motions necessarily exist and play a more or less important part in a moving medium like the atmosphere, where there are so many causes of disturbance. This idea naturally suggests another one: should not the great propagating atmospheric disturbances have the feature of waves? No doubt, many investigators must have directed their thoughts to this question, and especially Helmholtz seems to have studied atmospheric disturbances from this point of view,[1]) though without arriving to a developed theory of the disturbances of greatest scale as cyclones and anticyclones.

For my own part I took up the theory of gravity waves in compressible heterogeneous fluids,[2]) led rather by the negative idea that it was difficult to understand why more or less important wave motions should not exist in the atmosphere, than by any positive hypothesis of their role. Then the weather service organized in Norway in the year 1918, gave the connection with actual meteorological phenomena. The empirical facts collected after that time have given always increasing evidence for the view that cyclones may be said to be a kind of waves, though relatively shortlived, and of a type differing so much from those ordinarily seen that it was not easy at once to recognize their mechanism. Some of the empirical facts supporting this view will be found in the papers of *J. Bjerknes* and of *J. Bjerknes* and *H. Solberg* quoted on page 1, and further evidence will be brought in following publications of the meteorologists of the Norwegian Weather Service as soon as their official work will give them time to publish their results. Meanwhile, I may refer also to the Norwegian daily weather maps, which have appeared since the summer 1919 and implicitly contain most of these results.

In the following the main points of the wave theory will be discussed without entering into analytical details. Contributions to the mathematical theory are to be given later. As the full theory will be certainly as difficult as it is important, it is very desirable that the subject should be taken up an a broader base by mathematicians.

34. Inclined orbital motion in waves. — The reader has been reminded earlier of some of the general features of wave motions according to the classical theory. The orbit of a fluid particle is an ellipse in that vertical plane which gives the direction of propagation. The mayor axis is horizontal and the minor axis vertical.

But the conditions which allow the orbital plane to be vertical are in reality exceptional. The symmetry required for this exists, for instance, in the case of a horizontal bottom and infinitely extended fluid layers. It will exist also in a canal with horizontal bottom and vertical walls. But by other configurations of the boundaries the orbital plane will in general be more or less inclined. By finite amplitudes the orbit will also in general cease to be a plane ellipse, and take the form of a more general curve in space. Little has been as yet accomplished in the mathematical theory of waves under

[1]) *Helmholtz*: Ueber atmosphärische Bewegungen, zweite Mitteilung: Zur Theorie von Wind und Wellen. Sitzungsberichte der Berliner Akademie July 25, 1889. Wiss. Abh. T. III.

[2]) *V. Bjerknes:* Ueber Wellenbewegung in kompressiblen, schweren Flüssigkeiten. Abhandlungen der math.-physischen Klasse der K. sächsischen Gesellschaft der Wissenschaften XXXV, No. 2. Leipzig 1916.

these conditions. But the qualitative laws can to some extent be extrapolated from the simpler cases, or derived from observations or experiments.

Thus, when waves propagate parallel to an inclined rigid plane, the orbit must become always more inclined as we approach this boundary, in order finally to coincide with it. In this way a transverse horizontal component motion occurs, which will exceed the vertical component when the angle with the vertical exceeds 45°. The vertical component will remain the fundamental from a dynamical point of view, as it gives the potential energy which underlies the propagation. But geometrically the motion may appear practically horizontal. Then we arrive at the geometrical representation of the motion in its essential features simply by using in the horizontal plane the diagrams which under ordinary circumstances are used to represent the motion in a vertical plane.

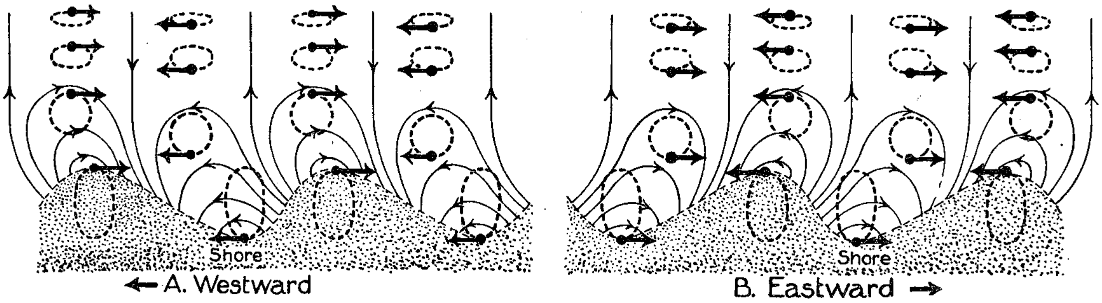


Fig. 24. Waves propagating parallel to a very flat shore. Horizontal projection of streamlines and orbits.

As an example waves propagating parallel to a very flat shore are represented by the diagrams of Fig. 24. The shaded area represent the shore which is supposed to extend east-west, and to have a very slight slope to the north. The waves of diagram A propagate westward, those of diagram B eastward. And, as it is seen, by the given northward slope of the shore we have cyclonic orbital revolution in the east going, anticyclonic in the west going waves.

As to their main structure, the diagrams are obtained by lying horizontally a diagram as that of Fig. 9. But further adjustments to the new conditions may also be necessary when we shall go into details. Thus, we know by formula 17 (s), or the well known special cases of it, that the velocity of propagation decreases with decreasing depth of the water. Therefore, the waves will be retarded in the vicinity of the shore, the wave ridges are bent backwards, and the tongues of water washing over the shore lag behind. This gives the asymetry of the diagrams of Fig. 24, as contrasted with the full symmetry of Fig. 9. We have no immediate use for these details, but shall have them in mind for later applications.

But an important point to emphasize for our immediate purpose is that the inclination of the orbital plane has the character of a stable position of equilibrium. If by external impulses the inclination is changed, it will return to this stable position. A corresponding inclination of instable equilibrium for the orbital plane may naturally also be imagined, but will have no chance of coming up by actual wave motions.

Another cause producing inclination of the orbital plane in waves is *the deviating force of the earth's rotation*. In the classical wave theory this force is neglected, as it is not seen to produce any sensible effect by motions of unquestionable wave nature. But we have no right to neglect it a priori when we take up new problems. To examine the character of its influence we consider separately the vertical and the horizontal component motion in the waves. The vertical component will not be sensibly affected: in this direction the same strong forces which determine the equilibrium also regulate the motion, and the small force due to the earth's rotations can exert no appreciable influence. But in a horizontal direction the equilibrium is under ordinary conditions indifferent. Then even the smallest force must manifest its effect if it has sufficient time to produce a sensible impulse, and the well known deviation to the right (on the northen hemisphere) occurs. Consequently the horizontal projection of the motion must change from rectilinear to elliptic form.

As the period for the orbital motion does not exceed a few minutes even in the greatest oceanic waves, the effect will remain insensible in all ordinary cases. But we must reckon with periods of quite another order of magnitude if there exist internal atmospheric waves of great dimensions. The table p. 28 gives an idea of the velocities of propagation of such waves. An average value from the table as 20 m/sec. may correspond to the most common velocity of propagation of a cyclone in our latitudes. Counting with a wave length of 2000 km., which may be the average distance between consecutive cyclones in a series, we get a period of about 30 hours. As the time of revolution of a particle in the »circle of inertia« is only 15 hours at the latitude of 60 °, this time of 30 hours will certainly be sufficient to produce a very pronounced ellipticity of the horizontal motion.

With periods of this order of magnitude, the orbital plane will take a certain inclination of stable equilibrium. As easily seen this inclination is such that the orbital motion in projection on the horizontal plane becomes anticyclonic, just as the revolution in the »circle of inertia«. To this inclination of stable equilibrium will correspond another which gives instable equilibrium, and which is characterized by the cyclonic direction og revolution in the horizontal projection. As long as the deviating force of the earth's rotation has any noteworthy influence, waves which in projection upon the earth's surface have cyclonic orbital revolution can only exist on condition that the effect from the boundary surfaces gives the required stability to the motion.

Thus, we meet with a striking asymmetry as soon as the deviating force of the earth's rotation becomes important. Boundary surfaces give rise to orbital motions in both directions, while the deviating force of the earth's rotation favours those which are anticyclonic in horizontal projection. This leads to an important consequence: *cyclonic orbital revolution can only occur with waves which are under sufficiently strong control of the boundary surfaces.*

35. Waves in the polar sliding surface. — Waves may originate and propagate in any of the atmospheric surfaces of discontinuity. But they will only then strikingly influence the weather when the undulating surface cuts the ground, and causes masses of air of different properties to wash over the earth's surface. The most important waves from this point of view will therefore be those in the polar sliding surface.

To get a view of their character let us first disregard the light air above this surface and consider only the heavy air below it. It has a relative motion from east to west, or an absolute motion to the east which is slower than that of the earth. This makes the heavy air float down the »geoidal slope« — as Marvin has recently called it [1]) — to the pole. For this mass of heavy air the earth's surface is like a shore which is slowly elevated toward the equator. Waves may propagate both eastward and westward parallel to this shore, as the shallowness of the sea compared to wave lengths of the order of

[1]) *Marvin*: The Law of the Geoidal Slope. Monthly Weather Review, October 1920.

magnitude of 2000 km., brings the waves under the full control of the boundaries, and makes both cyclonic and anticyclonic orbital revolutions possible.

But the upper layer of lighter air is not in the same degree under the control of the boundaries: here the anticyclonic orbital motion favoured by the earth's rotation will have the greatest chance of coming up, or will be the only possible. But then the correlated motion in the lower layer will be cyclonic as in Fig. 24 B: *this gives propagation to the east.*

The correlated motions in the two layers on the two sides of the boundary surface are then obtained simply by laying horizontally the diagram of Fig. 9: the »polar front« takes an undulating form, separating northward projecting tongues of warm air and southward projecting tongues of cold air, all propagating from west to east. The streamlines describe the motion as consisting in a series of propagating vortices, those centred round the northern ends of the warm tongues having cyclonic and those centred round the southern ends of the cold tongues having anticyclonic circulation.

But the wave motion represented by this diagram is only a part of the true motion. We must add the general easterly drift north of and westerly drift south of the polar front. This will give a great variety of different diagrams according to the relative strength of these currents in reference to each other and in reference to the wave motion.

The construction must, however, be performed with some care if we are to avoid contradictions with the hydrodynamical equations. The diagram Fig. 9 has for perspicuity been drawn with finite amplitudes of the waves, while the corresponding integral satisfies the hydrodynamical equations only for the case of infinitely small amplitudes. This extrapolation from infinitely small to finite amplitudes is in itself legitimate. But when we perform mathematical operations with the diagram, it is preferable first to return to the infinitely small amplitudes, and only after the performance of the operations apply again the result qualitatively or quantitatively, to finite amplitudes.

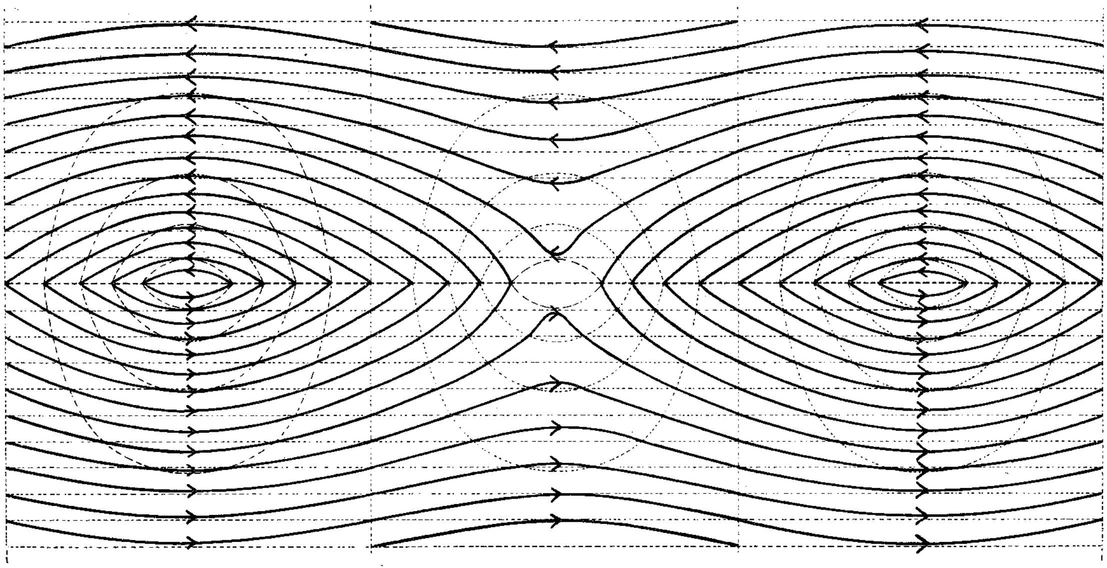


Fig. 25. Wave motion along the polar front. Cyclones and high-pressure ridge.

In accordance with this principle the boundary line — or polar front — is drawn in Figs. 25 and 26 as straight lines, corresponding to really infinitely small amplitudes of the waves. Then the dotted curves give the solenoidal representation of the pure wave motion, and the dotted horizontal lines represent in the same way the easterly current north of the polar front, and the westerly current south of it. These opposite currents are supposed for simplicity to have equal velocity. But in Fig. 25 both of them have greater velocity than the greatest occurring in the waves, in Fig. 26, on the other hand, smaller velocity than the greatest occurring in the waves. This is directly seen from the diagram, the velocity being by the solenoidal representation in inverse ratio to the mutual distances between the succesive stream lines. In both cases the stream lines of the resultant motion are found by drawing the diagonal curves. In virtue of the principle of superposition, with such small motions that we can disregard the quadratic terms in the hydrodynamic equations, we know that the resulting motion will also fulfill the equations.

Then the stream lines may at the same time be considered also as isobars. The dotted rectilinear stream lines, which represent the two opposite currents, are isobars which give the trough of low pressure extending along the polar front. The dotted curved stream lines give the partial pressure due to the wave motion, and the resultant stream lines are the isobars of the resultant motion.

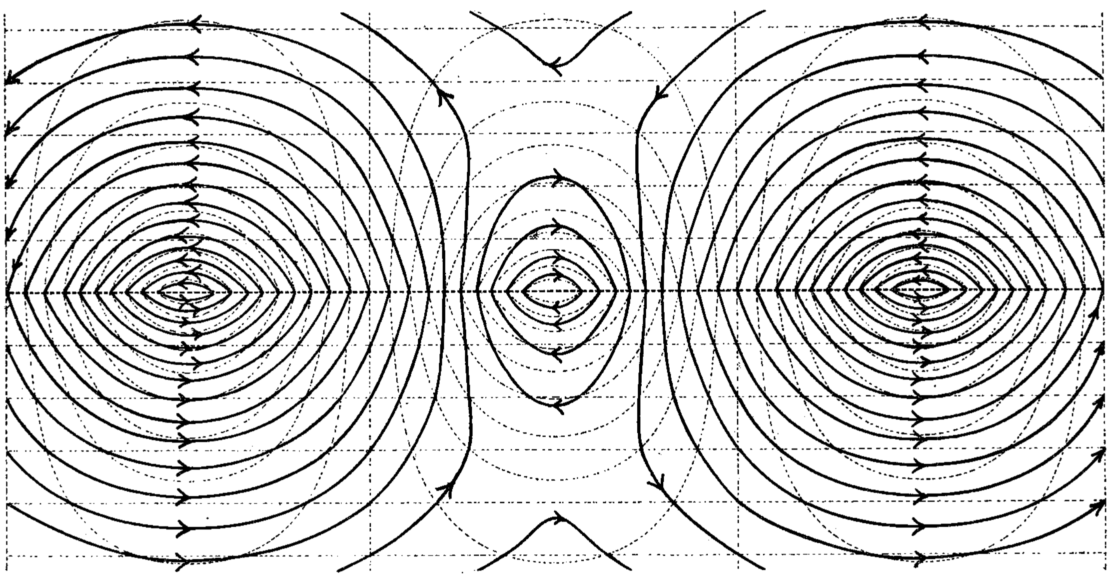


Fig 26. Wave motion along the polar front. Cyclones and anticyclones.

The inspection of the diagrams leads to the following results and considerations, *in considering which we must remember that the straight central lines of the diagrams may henceforth be conceived as a wave line as that of Fig. 9 and separating from each other warm tongues which project northward and cold tongues which project southward:*

1°. The uniform trough of low pressure along the polar front is by the wave motion dissolved in a series of alternately lower and higher pressures. The low pressures come at the northern ends of the warm tongues, the higher at the southern ends of the cold tongues.

2°. If the wave motion is weak relatively to the westerly and easterly drift, the high pressure will be merely a relative maximum, a saddlepoint on the ridge of high pressure separating the successive lows (Fig. 25). But if the wave motion is strong enough, the high pressure will be a real maximum, surrounded by closed isobars (Fig. 26).

3°. Round the Lows we have cyclonic circulation, round the true Highs anticyclonic circulation; in the relative High a hyperbolic field of motion, with a neutral point.

4°. From the distance between the successive curves it is seen that we have the strongest winds in the cyclones and much weaker winds in the anticyclones or the high pressure ridges.

5°. When we ascend to higher levels, we get the same diagrams as at the ground, only displaced to the north in accordance with the inclination of the sliding surface. The axis of the cyclones is therefore inclined to the north, with an angle of inclination of the order of magnitude of 1 to 100. The same should be the case with the much more indeterminate axis of anticyclones of the type which come in question here, viz., anticyclones which follow the cyclones in their propagation.

6°. An interesting feature which is not seen on the diagram, but which would be presented at once had the diagram been constructed for visible amplitudes, is this: the anticyclones have their centres at more equatorial latitudes than the cyclones.

7°. To calculate the exact velocity of propagation of this systems of waves is a difficult problem. Provisionally we can only estimate the gravity term, which must be considered as the principal one, and discuss qualitatively the modifications originating from the other effects. The waves propagate in an oblique surface, and the higher up this surface takes part in the motion, the greater will the velocity of propagation be which is given by the table of sect. 15. The motion in the lower part of the surface adjusts itself to this velocity by the bending backwards of the wave ridges which is illustrated in Fig. 24 for waves in shallow water. Then let the surface participate in the motion up to heights, say, from 1000 to 7000 meters. The corresponding pressures being from 90 to 40 centibars, we get velocities from 0 to 60 m/sec. according as the dicontinuities of temperature range from 0° to 20° C. These velocities will be more or less diminished, occasionally even down to zero, as a consequence of the sliding motion at the surface. The amount of this effect will depend also upon the earth's rotation. But at the same time the two currents give a convective effect, equal to a certain average of their velocities, and thus generally to the east, in the direction of the strongest current. As thus the two effects have the tendency to compensate each other, the resulting velocity of the waves will in all probability remain in the field given by the pure gravity effect, i. e., from zero up to 60 m/sec., with average velocities in the region of 20 m/sec., — thus in good accordance with the result found empirically for the propagation of cyclones.

36. Thermodynamics of cyclones and anticyclones. — As long as friction is considered, the theory of these cyclonic and anticyclonic waves remains purely hydrodynamic. But the thermodynamical side of the question is important as soon as the frictional resistance is taken into account.

We have already shown it by the discussion of the simplest model of stationary cyclones or anticyclones (sect. 28) when they were represented as continuous stationary vortices. The same considerations are easily seen to be valid when they now present themselves as discontinuous, propagating vortices.

In our propagating cyclones and anticyclones the circulation near the ground is retarded by the friction, just as in the stationary cases. This disturbs the equilibrium which in the absence of friction existed between the gradient and the forces of inertia (centrifugal force, deviating force of the earth's rotation). Through the excess gradient the air in the lowest strata has a tendency to move inwards to the centre in the cyclone, outwards from the centre in the anticyclone. The previously closed stream lines tend to take a spiral shape, inwards to the centre of the cyclone, outwards from the centre of the anticyclone.

When the warm and the cold current compete for the place in the central region of the cyclone, the warm must necessarily climb the cold one, and feed the ascending current. But the ascension does not originate merely from the buoyancy of the air. For while the circulation is retarded by the friction at the ground, it goes on with full intensity in the higher levels. This gives the centrifugal pumping effect which has often been alluded to. It has a twofold result: the tropopause, which has already a slight depression in consequence of the wave-motion, is sucked down; and the air masses from the lower strata are lifted. The ascending air, though initially warm, is therefore transported to a higher level than that which it would have reached by its buoyancy alone. It suffers the corresponding adiabatic cooling, and gets undernormal temperature relatively to the pressure or to the level which it has attained.

In the anticyclone we have in the same manner excess of anticyclonic circulation in the higher levels, with the effect described in sect. 28. The excess of anticyclonic circulation relatively to the earth is, absolutely reckoned, a deficit of cyclonic circulation. In the absolute motion we have therefore excess of circulation both at the ground and in the stratosphere. It follows that the tropopause is sucked up and reaches a still more marked elevation above the anticylonic area than that which it has already in consequence of the wave motion. And within the troposphere the air masses are subject to a downward directed sucking effect. The cold masses which, tend downward by their weight, are therefore at the same time forced downward dynamically, and thus brought to lower levels than those which they would have reached by their own gravity. I. e., the descending air in the anticyclones, though coming from a cold source, present overnormal temperatures relatively to the levels which they have attained.

Statistical investigation must therefore of necessity lead to a high correlation between pressure at the ground and temperature in the higher part of the troposphere, high pressure at the ground leading to high temperature, and low pressure at the ground to low temperatures.[1])

This originally unexpected temperature distribution in cyclones and anticyclones simply shows that both of them are thermodynamical engines going inversely, transforming mechanical energy to heat. The mechanical energy is delivered in kinetic form by the great westerly current on the equatorial side and the great easterly current on the polar side of the sliding surface. The energy of the two currents is always renewed, as always new masses are conveyed from both sides to the sliding surface, in virtue of the general circulation which we shall consider below. The role of the cyclones and the anticyclones is to reduce the energy of these two currents. This view is, I believe, in full accordance with that advanced by Helmholtz in his remarkable paper of 1888, to which we have often referred.

37. *Further consequences of the wave theory of cyclones and anticyclones.* — The theoretical diagrams of Figs. 25 og 26 are constructed under suppositions which are still relatively far from the true conditions in the atmosphere. But it is easy to see how they must be modified qualitatively according as we approach actual realities.

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[1]) Cf. *Dines:* The Characteristics of the Free Atmosphere. Geophysical Memoirs No. 13. London 1919.

First, the analytical difficulties, which complicate the integrations, do not prevent natural waves from having finite amplitudes. We may then draw the polar front with waves of any size, separating warm tongues projecting southward and cold tongues projecting northward.

But at the same time we must remember that the farther south we come the more shallow is the lower cold stratum, and the more must wave ridges be bent backwards to follow the propagation. In consequence, the southern ends of the cold tongues are displaced backwards relatively to the northern ends of the warm tongues. The polar front which is drawn as a straight line in Fig. 25, will then not be a sinusoid, but deformed as in Fig. 24, and this deformation will increase with increasing amplitudes.

An other important fact must be remembered which will progressively influence the form of the curve. There is, as we shall see more in detail below, a continuous supply

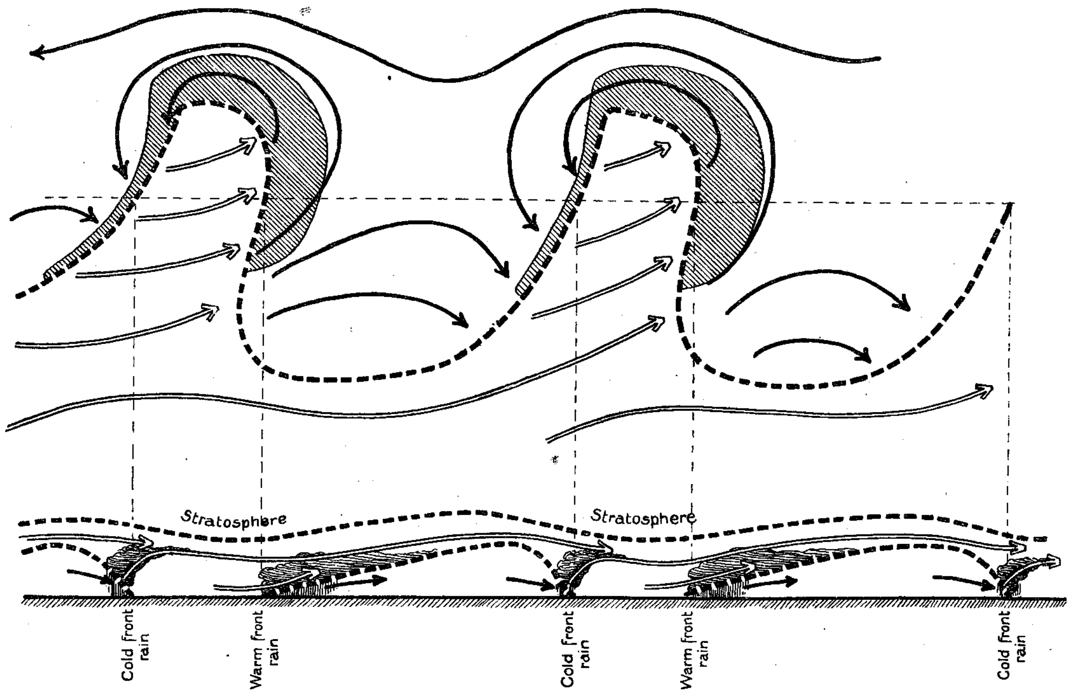

Fig. 27. Succession of waves in the polar front.

of warm air from south west, and of cold air from north east. The warm air escapes upwards, — its further motion in the higher levels will be discussed below. But the cold air can only spread out along the ground. Consequently the cold tongues always swell, and the warm tongues shrink. Therefore, we shall after the development has gone on for some time in general have broad cold tongues and narrow warm tongues.

A fact of importance is also that the westerly drift south of the polar front is in general stronger than the easterly north of it. This also destroys the symmetry of the theoretical Fig. 26, especially by turning the east winds south of the cold tongues into west winds.

Then, absolute discontinuities never exist in reality. The too sharp corners in the theoretical figures must therefore be smoothed out in diagrams representing real conditions.

Finally, we have to remember the changed direction of the stream-lines in consequence of the friction: inwards to the lowest pressure in the cyclonic vortices, outwards from the highest pressure in the anticyclonic vortices.

Taking these circumstances into consideration in modifying the theoretical diagram of Fig. 26, we arrive at a diagram like that of Fig. 27. And here it must be remembered that the stream lines are no longer isobars, as the friction has made the two sets of lines deviate from each other. The isobars still remain closed curves centered round the Lows at the northern end of the warm tongues and the Highs at the southern end of the cold tongues.

The velocities of propagation of the waves of 20 m/sec on the average is smaller than the wind velocity of the warm southwestern current at slightly higher levels. The cold tongues then remain obstacles to this wind. They will be deformed like oceanic waves, get slighter and more even slopes on the windward and more abrupt and irregular slopes on the leeward side, as indicated by the vertical section below in the diagram.

The meteorological phenomena accompanying the propagation of this system of waves is then easily seen. The eastern border of a warm tongue forms an advancing *warm front*, the eastern border of a cold tongue an advancing *cold front*. At the warm front the warm and moist southwestern wind begins its ascension of the windward slope of the polar air. It is cooled, its humidity condensates, clouds and precipitation are formed: this gives the *warm front rain* which precedes the arrival of the warm front itself, and which is represented by the shaded area preceding the varm front.

The following cold front is preceded by descending warm air. Here, therefore, clearing weather is in general to be expected. But the descending motion down the relatively steep slope is combined with instability and a rolling mass of warm air is often formed under the steep leevard slope of the cold tongue. *A narrow stripe of rain and heavy squalls,* therefore, often accompanies the propagation of the cold front. The process is illustrated in the diagram Fig. 20 of J. Bjerknes' and H. Solberg's quoted paper, to which I refer for a more complete discussion.

38. *The life cycle of a cyclone.* — Some not yet published results which the same investigators have found concerning the development of a cyclone from its birth to its death are also naturally discussed from the point of view of the wave theory. With the permission of the authors they will therefore be indicated here in anticipation.

Then Fig. 28 represents schematically a cyclone in four stages of its development. It begins as a little wave in the polar front, usually at the extreme end of the cold front line of a preceding cyclone. The new cyclone is thus in general a »secondary« to the preceding one.[1]) The formation takes place when the front has become stationary, with west wind on its southern and east wind on its northern side. The wave formed grows rapidly, and begins to propagate. As the dimensions of the disturbance increases, so does the velocity of propagation. But while the warm tongue continues to extend northwards, it begins to narrow in laterally in consequence of the continuous growth of the cold tongues which we have already alluded to. Finally, the warm front is handicapped by the following cold, and the warm tongue is cut off somewhere near its root. From that time the collapse of the cyclone begins. The velocity of propagation falls off. The secluded mass of warm air by and by disappears upwards, giving off its last rain, and its place is taken by cold air which inundates the whole

[1]) This formation of secondaries as waves on the cold front line of preceding cyclones has been discovered independently by *C. G. Andrews*: The Application of Bjerknes' Lines to the Development of Secondary Lows. Monthly Weather Review, January 1921.

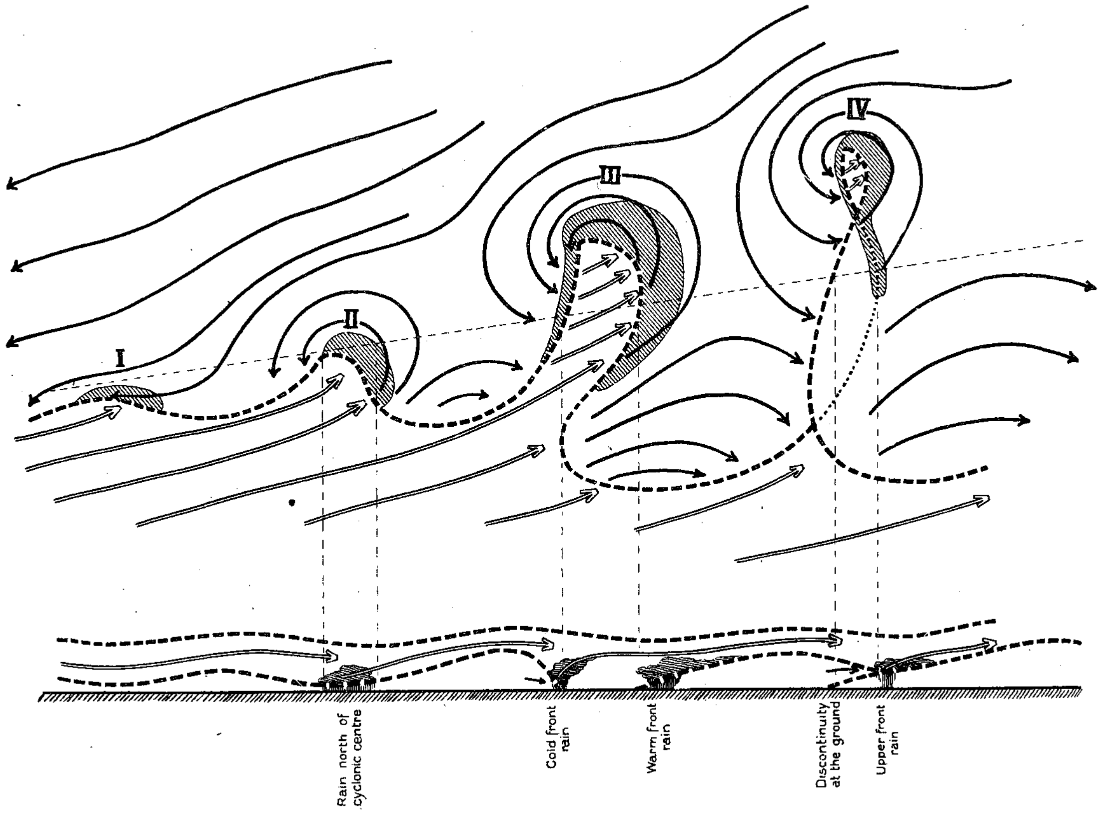

Fig. 28. Life cycle of a cyclone.

area.[1]) The propagation ceases, and the cyclone dies. But an irregularity, which is left in the polar front where the old warm tongue had its base, often develops to the warm tongue of a new secondary, which is born simultaneously with the death of the primary cyclone.

For the sake of argument we may imagine the birth of our cyclone to have taken place in the Atlantic, west of the British Isles, its death outside the Norwegian coast, and the birth of the new secondary in the Meditterranean. To give the full mathematical theory of this life cycle of a cyclone will be difficult. But the different phases of the process seem very intelligible from the point of view of the wave theory.

When the cold front has become stationary, but with strongly different tangential velocities on its two sides, an unstable state of motion exists for a moment. This instability leads to the formation of a wave, possibly by the same principle which underlies the formation of wind waves on the sea surface (cf. the end of sect. 15). The process seems to begin at or near the intersection of the sliding surface with the ground, where the

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[1]) This process of seclusion and the phenomena accompanying it are exceedingly important from the point of view of the forecasts, and will therefore be made the subject of detailed treatment in later papers issuing from the Norwegian Weather Service. Here attention shall be directed merely to the »upper front rain« indicated in the diagram. The two cold tongues which join by process of seclusion have in general different temperatures. When this difference is sufficiently great, it leads to the formation of an »upper front«, which continues to give rain, though it becomes separated from the discontinuity observed by thermograph and barograph at the ground. It may be an upper warm front as in the winter situation of figure 28, giving a stripe of rain up to 50 or 100 km. ahead of the discontinuity afterwards registered at the ground. Or in summer situations it may be an upper cold front: then the passage of the discontinuity at the ground is first registered, and the stripe of rain may follow up to 50 or 100 km. behind this discontinuity.

layer of cold air below the surface is shallow. Therefore, the propagation is initially slow, but increases as the motion spreads higher up where the greater depth of the cold layer below the surface gives greater velocity of propagation. The limit which the velocity may attain is given by the height which the sliding surface reaches in the troposphere.

The development which follows after the velocity of propagation has had its maximum, seems very analogous to the degeneration of a wave into a stationary vortex described in sect. 16. The degeneration must follow if the differences of temperature between the two sides of the sliding surface converge to zero as we proceed eastward. In order to discuss the process we must consider the different ways in which this convergence to zero can take place.

First, let the temperatures on the two sides decrease symmetrically as in Fig. 29 A.

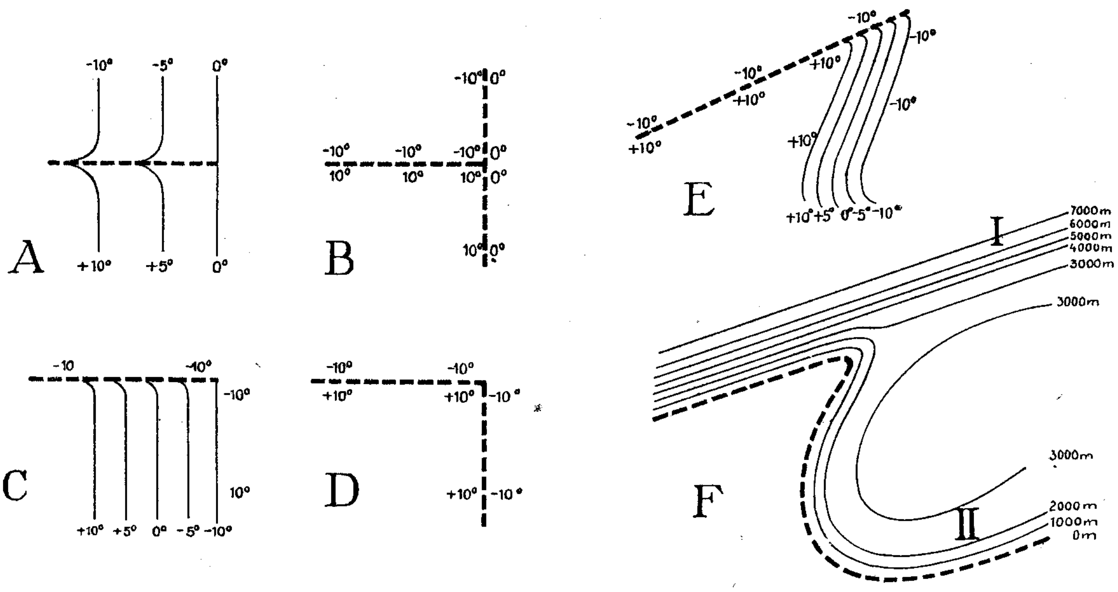

Fig. 29. Thermal conditions at the eastern coast of a continent in winter.

The process will then go precisely as described, and be completed in the region of the isotherm of 0 °C., where every discontinuity ceases. Then, as in Fig. 29 B, let the isotherms for + 10° and — 10°, for + 5° and — 5°, which branch out from the line of discontinuity, run very near to, or in the ultimate case, coincide with, the isotherm of 0°C. This gives a more sudden change from wave to vortex. But in the case of sufficiently abrupt change, we must have in mind the possibility of certain complications. A reflected wave would certainly not be formed, as a wave motion propagating westward in the polar front would be unstable, or at least have very limited stability (cf. sect. 35). But as the discontinuity is split up into a north going and a south going branch, a wave which is not completely destructed might divide itself into two minor waves, one along each of these branches.

From the symmetric arrangements A and B we pass to corresponding cases of asymmetry *C* and *D*, which give somewhat greater approach to actual conditions. The case *C* gives rise to no special remark, the process will develop continuously as in *A*. And in case *D* conditions are as in *B*, with the difference that no northward continuing wave comes into question. But under sufficiently favourable conditions a southward continuing wave might be realized.

The real conditions under which a winter cyclone as that of Fig. 28 propagates tomena. On the other side the intertropical sliding surfaces do not extend to such great wards the anticyclone covering Europe, may be represented by the diagrams of 29 E. It is a compromise between *C* and *D*, in as much as the isotherms run close together in the transitional layer between the oceanic air and the continental anticyclone, but without defining an absolute discontinuity. But it differs from both of them by the acute instead of right angle under which the polar front meets the boundary of the homogeneous mass of continental air. The conditions must in general lead to a rapid collapse of a cyclone propagating towards the European coast, and give only a small chance for the production af a wave continuing along the coastal discontinuity.

The conditions under which a cyclonic wave from the Atlantic approaches Europe, may be considered from a more general point of view when we try to form a mental picture of the configuration of the polar sliding surface as it passes from the ocean to a continent covered by a stationary winter anticyclone. The intersection of the sliding surface with the ground, the »polar front«, must follow the southern border of the anticyclone, which contains polar air. But as the anticyclone is shallow, the sliding surface above it must be nearly flat and relatively low. The entire surface may then have a configuration that is represented by the topographical sketch of Fig. 29 F. Over the Atlantic it presents a single slope, but on the continent it has two separate sloping parts: a lower I, following the border of the anticyclone, and a higher II, continuing across the anticyclonic area.

Then, when we shall consider the propagation of waves in this surface it must be borne in mind that the contrasts of temperature and wind, which characterize the surface, are very marked above the Atlantic, but becomes less perspicuous already as we approach the coast. And this change of condition continues for some distance along the two separated slopes I and II as we proceed inwards over the continent. (The reduced contrast between the masses of air on the two sides of the surface does not involve reduced inclination of the surface). A wave which is bound to follow the sloping part of the surface must under these conditions begin to degenerate already in approaching to the coast, and must have difficulty in passing the point of bifurcation. But the upper part II of the slope gives the cyclone a chance occasionally to find its way across the Eurasian continent.

The process of the seclusion of the air in the warm tongue is the combined effect of the growth of the cold tongues due to the continuous supply of cold air, and of the decreasing velocity of propagation as we approach to the coast. In this process of seclusion we have another independent cause for transformation of a propagating wave into a stationary vortex. As soon as the cold air has surrounded the varm core completely, we have lost the asymmetry of the wave and attained the symmetry of the vortex. Together with this asymmetry even the potential energy is lost which underlies the propagation. The cyclonic wave is changed into a stationary vortex which must be destroyed by the friction in that way which we have described already (end of sect. 28). The degeneration of the cyclones by this reason takes place independently of the orographical conditions, and may explain that so many cyclones arrive in half degenerated state to the European coast.

39. On tropical cyclones. — One of the first discoveries of the modern synoptical meteorology was the recognition that the extratropical cyclones were vortices, as the tropical. Now the more detailed synoptical investigations have shown that the extratropical cyclones are waves as well as vortices. A question then naturally presents itself: is the same the case with the tropical cyclones?

We have no synoptical investigations which give the anatomy of the tropical cyclones with sufficient detail to immediately dicide the question. But still it may be of interest to direct the attention to a number of facts which may throw some light upon the problem.

First, attention is called to the fact that surfaces of discontinuity exist in the regions where the tropical cyclones originate and propagate, namely the sliding surfaces of the trades, and the corresponding sliding surfaces of the monsoons. These sliding surfaces do not in general cut the ground, as the polar sliding surface always does. This may be the reason why the extratropical cyclones are frequent while the tropical are rare pheno-heights in the troposphere as the polar sliding surface. This is seen from Figs. 21 and 22 in as much as the sliding surface of the trades is concerned. And those of the monsoon are known to be still lower. This might be the reason why the tropical cyclones cannot with the same ease pass across mountain ranges of any appreciable height.

While the sliding surfaces of the trades do not in general descend to sea level, there are places where some of them constantly cut the ground. Observations from such places will of course be of great importance for the question which we discuss here. K. v. Fritsch reports in connection with the calm layer of 300 to 600 m. thickness, which at the island of Teneriffe is generally observed between the trade and the anti-trade wind (continuation of the quotation on p. 121).

»Die windstille Zwischenzone ist offenbar die Folge der Reibung beider entgegengesetzter Luftströmungen, wenn dieselben sich gleichmässig bewegen. Ist jedoch die Bewegung eine ungleichmässige, dann wird gerade diese Zwischenzone der Kampfplatz beider Winde, es machen sich dann in raschem Wechsel entgegengesetzte Windstösse bemerkbar oder wohl auch Wirbelwinde. Letztere konnte ich nicht selbst beobachten, auf Palma sind es aber gewiss solche, die, wie man erzählt, Felsblöcke, Baumstämme und bisweilen Wanderer von den Andenes in die Tiefe der Caldera stürzen sollen und die man im Winter sehr fürchtet.«

Observations as these naturally suggest the idea that disturbances of greater scale, as true tropical cyclones, might originate when the sliding surfaces of the trades or monsoons occasionally descend to sea level, and thus cut the ground for longer distances.

As to the sliding surfaces of the trades, we know that they slope towards the equator (see Figs. 21 and 22) so that the equatorial border has the greater chance of occasionally cutting the ground. This would then give the natural explanation of the fact that the tropical cyclones generally originate on the equatorial border of the trade wind zones.

From Sverdrup's investigation we further know that the sliding surface of the North Atlantic trades has a marked slope not only from N to S but also from W to E. The southerly border of the surface will therefore have its lowest point near the African coast: this is just the place where the cyclones of the North Atlantic are said generally to originate. We do not know if the sliding surfaces of the trades go continuously round the earth, but it is more probable that they exist only over the oceans, and end with more or less marked borders near the coast of the continents. If we suppose this to be the case, the well known parabolic motion of the tropical cyclones from the African coast to the West Indies and then to the North East would be a propagation along the border of the sliding surface of the trades.

The conditions which, in addition to the slope towards the equator, cause a slope from E to W, are certainly present in the Pacific as well as in the Atlantic. The surface should therefore even there have its greatest chance of descending to sea level near the equator and rather far East, while it should rise to a greater height and be bordered in the vicinity of the Asiatic continent. The path of the cyclones in the Pacific along a parabola with its apex in the region of the Philippines would then also be a propagation along the border of the sliding surface.

The slow westward motion of tropical cyclones in the lowest latitudes would even from the point of view of the wave theory be due principally to the convective effect which, as both currents go westward, may be strong enough to overcompensate an eventually opposite propagation in the proper sense of the word. But according as higher latitudes are reached the antitrade wind changes from SE through S to SW, and we get the more rapid extratropical propagation as the sum of the convective effect and the proper propagation.

Quite independently of the question of a wave nature of all cyclones, tropical as well as extratropical, the investigation of the conditions under which the tropical cyclones transform into extratropical, will be of the highest interest. Before we can describe the transformation, as well as the structure of the tropical cyclones at every stage of their development, detailed synoptical charts must be produced, similar to those which have revealed the nature of the cyclones of the higher latitudes.

VII. The General Atmospheric Circulation.

40. Thermodynamical circulation. — The second disturbing effect, friction, (sect. 32) prevents a particle of air from permanently remaining on or near the parallel circle along which it moves in the undisturbed planetary vortex, the further development depending upon combined dynamical and thermodynamical conditions. In order to discuss the kind of motion then produced, we shall first consider the simplest type of circulation produced thermodynamically.

Then, let the earth be at rest and let it be heated by a sun, which, to give full symmetry, has the form of a ring surrounding the earth in the equatorial plane. If the particles of the air be artificially kept at rest we should get a temperature distribution determined by the equilibrium of the insolation, the radiation and the thermal conduction. The features of this distribution would be high temperatures in the equatorial and low in the polar regions. The air will then have the same specific volume in low levels, i. e., under high pressure, at the equator as in high levels, i. e., under low pressure, in the polar regions. Thus, the isosteric surfaces are inclined relatively to the isobaric, and a great number of isobaric-isosteric solenoids are formed, which surround the earth as parallels.

Then, as soon as the particles are let loose, a vertical circulation will set in, in accordance with the theorem 6 (B), and distributed symmetrically round the earth's axis in the meridian planes. This circulation will convey cold air along the ground from the poles to the equator, and in the higher levels potentially warmer air from the equator to the poles.

This gives a reduction of the temperature in the equatorial region and a corresponding increase in the polar regions, and reacts thereby upon the dynamic and the thermodynamic processes: the number of isobaric-isosteric solenoids is reduced, the corresponding circulation still increases in intensity but at a slower rate. Further, the air at the poles will lose more heat and the air at the equator will lose less heat by radiation than before. This gives in the equatorial zone an excess of insolated over radiated heat, and vice versa, in the polar zone an excess of radiated over insolated heat. The amount of heat taken up by the air in the equatorial region is divided into two parts, one which yields the kinetic energy of the atmospheric circulation, and one which is brought convectively to the polar regions to be lost there by radiation. The ultimate result will be a certain steady state, determined by a finite convection of heat from the equator to the poles, a steady circulation of the atmosphere, sufficiently rapid to yield this convection, steady fields of temperature, of radiation, and of conduction. At the equator, there is a certain excess of insolation over radiation. This excess is divided into two parts: one which is first transformed to kinetic energy of the moving air, and then by friction to heat again, and another which is carried convectively to the polar regions.

41. Modification of the planetary vortex by friction. — When our circular vortex is disturbed by friction, the result must be in the nature of a compromise between the pure circular vortex motion which tends to separate air masses of different temperature, and the thermodynamical circulation which tends by convection to reduce the same differences of temperature. The quantitative deduction of the resulting steady motion is certainly a difficult problem from a mathematical point of view. And even in the qualitative discussion so many difficulties are met with that it is rather easy to explain the great number of different theories advanced for the general atmospheric circulation. Without entering into details we shall therefore give only a few general remarks concerning the possible or probable result.

The effect which the internal friction exerts upon the motion is difficult to estimate. But the effect of the external friction at the ground is clear. It retards the motion relatively to the earth. But this retardation has opposite dynamical effects in the regions of east and of west wind. When we take the point of view of absolute motion, the friction at the ground is accelerated in the east wind zones and gives rise to greater centrifugal force. But in the west wind zones it acts as a retarding agent, and reduces the centrifugal force. The equilibrium which this force had with the pressure gradient and the gravitational attraction is disturbed. A resultant is formed which is directed outward from the earth's axis in the east wind zones, and towards this axis in the west wind zones. We may resolve this resultant into two components, one tangential and one normal to the sea-surface. On account of the stability of the stratification, the effect of the last one will soon be counterbalanced by statical forces. But the component tangential to the sea-surface will tend to produce a steady flow of the lowest stratum of the air towards the equator in the east wind zones, and towards the poles in the west wind zones. And in the measure in which this flow along the ground is realized, we get corresponding vertical circulations: in the east wind zones towards the equator at the ground, and back to the poles in higher levels; in the west wind zones towards the poles at the ground, and back to the equator in higher levels.

But the conditions for the realisation of this circulation are different in the two kinds of zones. As soon as they have come up, they will be assisted thermodynamically in the east wind zones, counteracted in the west wind zones. Therefore, we have to expect direct thermodynamical circulations strongly developed in the zones of the east winds, but slow opposite circulations in the west wind zones.

In the equatorial zone of east winds this leads to the well known circulation of the trades: along the ground the air moves from the subtropical towards the tropical calms, then it ascends, assisted thermodynamically by the heating; it returns in the higher levels towards the subtropical Highs where it again descends, assisted thermodynamically by the cooling due to radiation. That the descending motion is really assisted thermodynamically is fully proved by the fact that the temperature in corresponding levels is higher above the equator than above the subtropical Highs. The quantity of solar energy used to maintain this circulation may be directly calculated from these temperature differences.[1])

In the polar east wind zone we have the same conditions, tending to produce a circulation outwards from the pole at the ground, upwards along the polar sliding surface, inwards below the tropopause, and down at the polar border of the east wind zone. This circulation does not attain as full development as that of the trades. But the tendency towards its realisation must be remembered.

[1]) Sverdrup: l. c. p. 87.

In the zones of the prevailing west winds when the frictional effect is continuously checked by the opposite thermodynamical effect, the circulations must be slow and indeterminate. Still, the motion of the air from south west at the ground, and the corresponding motion from north west of the higher clouds [1]), seems to indicate this opposite circulation as a fact in the temperate zone. We suppose provisionally the same to be the case in the polar zone of westerly winds which we have introduced by theoretical reasons.

The complete scheme of the meridional circulations should then be as shown in the diagram of Fig. 30. On each hemisphere there are four circulations, running as toothed wheels, the circulation of the trades, the circulation of the temperate zone, the circulation of the polar east wind zone, and the circulation of the theoretically introduced polar west wind zone. The first and the third of these circulations represent thermodynamically direct cycles, in which the motion is maintained by heat energy. But that of the temp-

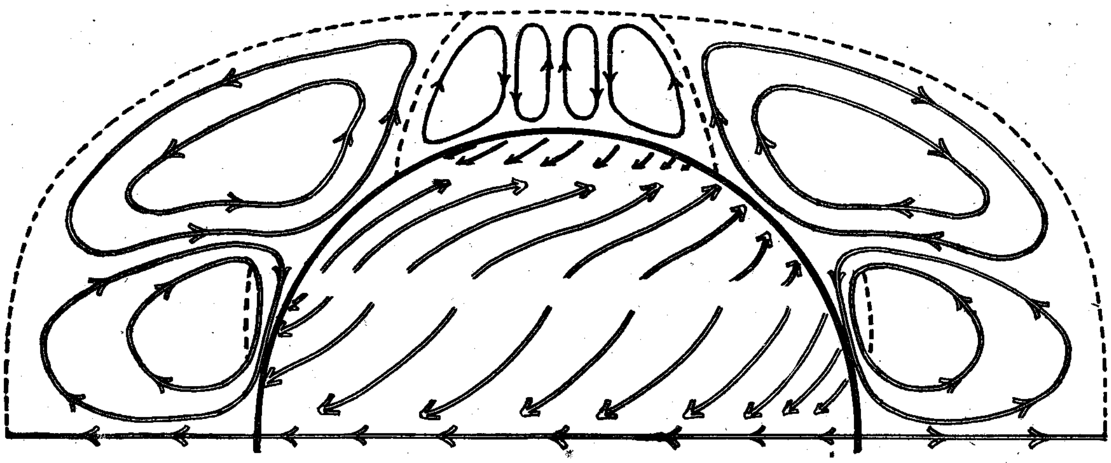

Fig. 30. Planetary schema of the general atmospheric circulation.

erate zone is an indirect cycle, by which kinetic energy is transformed into heat, and the same should be the case with the polar west wind zone.

These circulations give two zones of descending motion, where very limited precipitation should be expected, namely a zone round the pole, and the zone of the subtropical calms; and further, two zones of ascending motion and great precipitation, one along the equator, and one along the polar front, situated on the polar side of it. Finally, a third theoretical zone of precipitation should be at the pole itself, but of no phycical importance as shown below. But the reality of the two great zones of rain along the equator and the polar front is a good verification that a circulation of the described character must exist on the average.

42. Interchange of air through the polar sliding surface. — We have no reason to attribute great stability to this system of four independent circulations, of which two are opposite to the general thermodynamical tendency of producing a single uninterrupted circulation between the poles and the equator. This tendency must be very effectively supported when the wave motions considered occur in the polar sliding surface which separates two of these circulations.

[1]) *Hildebrandsson*: Resultats des recherches empiriques sur les mouvements généraux de l'atmosphère. — Nova Acta Reg. Soc. Scient. Upsaliensis. Upsala 1918.

The air masses of the south west winds tend upwards along the polar sliding surface. When this surface comes into wave motion, channels are formed which collect and convey the ascending masses. They take their way upwards through the troughs extending polewards from the northern ends of the warm tongues.

At the same time the cold masses coming from the north east spread along the ground, causing the cold tongues to swell and the warm to shrink. This gives a sustained advance of the polar front in the direction to the equator. The advance is partly continuous, and partly discontinuous, when namely a warm tongue is cut off and a new more southerly front is formed.

But this advance of the front towards the equator is counteracted by another process, which is seen to occur on the synoptic charts, and which is easily understood from the wave theory. The far southward projecting cold tongues are anticyclones which originally follow the propagation eastward of the wave system. But with a rich supply of polar air, it advances too far to the south. Then it lags behind, and constitutes itself as an independent anticyclone: this is the mode of formation of the great, slowly moving anticyclones of the lower latitudes. By and by the air of this anticyclone is heated, and drawn into the circulation of the lower latitudes, while at the same time a more retired polar front is formed more to the north, behind the old one.

In this way polar air is intermittently expelled along the ground, and brought into the circulation of the lower latitudes. When thus manifestly polar air leaves the space bounded by the polar sliding surface, there can no longer be any doubt that the air masses of more equatorial origin which ascend through the cyclones must at least in part flow in over the polar region for compensation. Details concerning this supply of air to the poles in the higher levels are not yet available, and can only be obtained by aerological investigations at higher latitudes. Two possibilities present themselves: the polar boundary surface may be bordered, and the inflow of equatorial air over the polar regions go continuously in the space left between this border and the tropopause. Or the surface continues straight up to the tropopause dividing the troposphere into two different departments as in Figs. 21, 22, and 30. Then, the equatorial air must break through it intermittently, just below the tropopause, as the polar air breaks through it at the ground.

In whatewer way the air from the south enters the polar region, continuously or intermittently, it must arrive with a westerly motion which it cannot lose before it has descended to the ground and begun its motion southwards: this leads to the consequence often referred to, of a tendency to form a west wind region nearest the pole. But it is not probable that this ever leads to independent development of a central polar cyclone. The two circulations in the polar region given schematically in Fig. 30 will be mixed, and the stronger will gain, namely that which is assisted thermodynamically and gives the outflow of polar air along the ground. But the effect will to some extent be checked by the tendency towards the formation of the polar cyclone: We have entroduced it to remind the reader of this checking effect.

Geographical conditions also exert a considerable influence upon the exchange of air through the polar front. The places where the great outbreaks of polar air come are geographically determined. These outbreaks lead to the formation of the far penetrating cold tongues, which may develope to independent anticyclones south of the polar front. Under favourable geographical conditions also a continuous canal may be formed and exist for a time conveying polar air direct into the tropics. A canal of this description is occasionally seen to extend from Spitsbergen along the Norwegian coast, continuing west of Europe directly down into the North Atlantic trades. The irregular supply of polar air to the tropics, must lead to varying equilibrium conditions of the sliding surfaces of the trades, and may thereby be of importance for producing the circumstances under which the tropical cyclones are formed.

The diagram of Fig. 31 gives a schematic picture of the general atmospheric circulation thus arrived at. It is developed from that of Fig. 30, of which it has retained the essential features, but with the following important changes: The two polar circulations are joined into one, which has the direction of the thermodynamic tendency. The cyclones

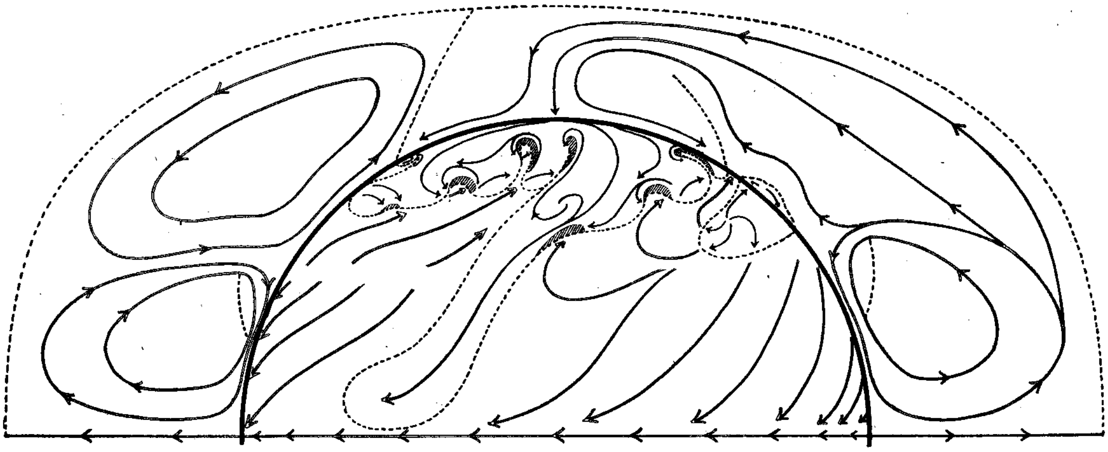

Fig. 31. General atmospheric circulation.

and anticyclones are introduced in their proper places as essential links in the mechanism of circulation, drawing their supply of cold air from the polar circulation and their supply of warm air from the inversely going circulation of the temperate zone. And an example is given of the occasional rushes by which the tendency of the circular vortex to keep the air masses of different temperature separated, is overcome by the general thermodynamic tendency, to produce a continuous circulation between the poles and the equator.

Errata.

P. 1, L. 11 fr. b. (L. 1 in the bottom-note): .. and 1921. —	Read: and 1921. — I
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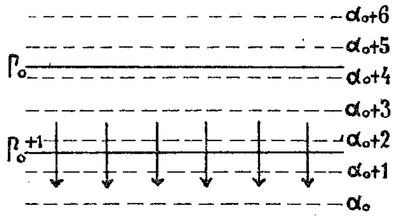

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- 21, - 4 — : u_z	» : u_z
- 26, - 19 fr. b.: value of	» : value
- 30, - 17 fr. t.: diagram	» : diagram, Fig. 8,
- 31, - 21 fr. b.: runin	» : run in
- 38, - 5 fr. t.: dicontinuity	» : discontinuity
- 40, - 6 fr. b.: dicontinuity	» : discontinuity

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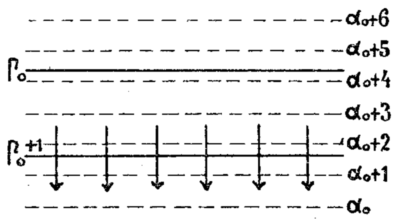

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- 31, - 21 fr. b.: runin . . . . » : run in
- 38, - 5 fr. t.: dicontinuity . . . . » : discontinuity
- 40, - 6 fr. b.: dicontinuity . . . . » : discontinuity