Mean Monthly Air Transport over the North Pacific Ocean

by

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Materials and Methods used for the Preparation of the Wind Charts.

1. Pilot charts of the oceans are regularly issued by the Hydrographic Office of the U. S. Navy Department. For the oceans most trafficated, monthly charts are published; for other seas, the charts are published quarterly, each comprising data for three consecutive months. The charts are constructed in Mercator's projection, and contain a great amount of information useful to sailors, for instance, great circle routes, curves of equal magnetic variation, storm tracks, number of days with fog, and wind roses for five degree squares. The barometric pressure is shown on inset charts.

2. This paper being a discussion of the wind conditions on a part of the ocean, the representation of the wind is a point of chief interest. As to the interpretation of the windroses, the following quotation from the text printed on the charts may give the best account (cf. Fig. 1):

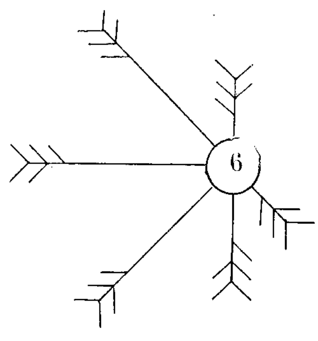

Fig. 1. Wind rose, at 52°30′ N. Lat., 157°30′ W. Long., for March, 1916.

»The wind rose in each 5-degree square shows the frequency, the direction and the average force of the winds that may be expected to prevail within that square. At the request of this office the wind percentages were concentrated upon eight points. In some instances the full length of the arrows cannot be shown, and the line is broken and the total percentage is given between the broken lines. The arrows fly with the wind. The length of the arrow measured from the centre of the circle gives, by means of the attached scale, the number of times in each 100 observations that the wind may be expected to blow from the given point. The number of feathers indicates the average force of the wind according to the Beaufort scale. The percentage of calms, light airs and variable winds is shown by the number within the circle. The number outside the circle represents the gales of force 8 and over.«

3. We may further quote from the charts the following statements as to the sources of meteorological information:

»Publications from the following sources have been used in the preparation of the meteorological features of this chart:

U. S. Weather Bureau, Department of Agriculture; U. S. Hydrographic Office, Navy Department; Central Meteorological Observatory, Japan; Department of Federal Telegraphs, Mexico; Central Physical Observatory, Russia; Weather Bureau, Philippine Islands; Canadian Meteorological Service; Deutsche Seewarte, Germany; Hongkong Observatory; Zi-ka-wei Observatory (located in Shanghai)«.

From a letter by Mr. C. F. Marvin, Chief of the Weather Bureau, U. S. Department of Agriculture, the following passage may be quoted:

»The last tabulation of data for use in constructing the wind-roses was made during the years 1909—1911, and covered available data for the preceding 25 years. Where

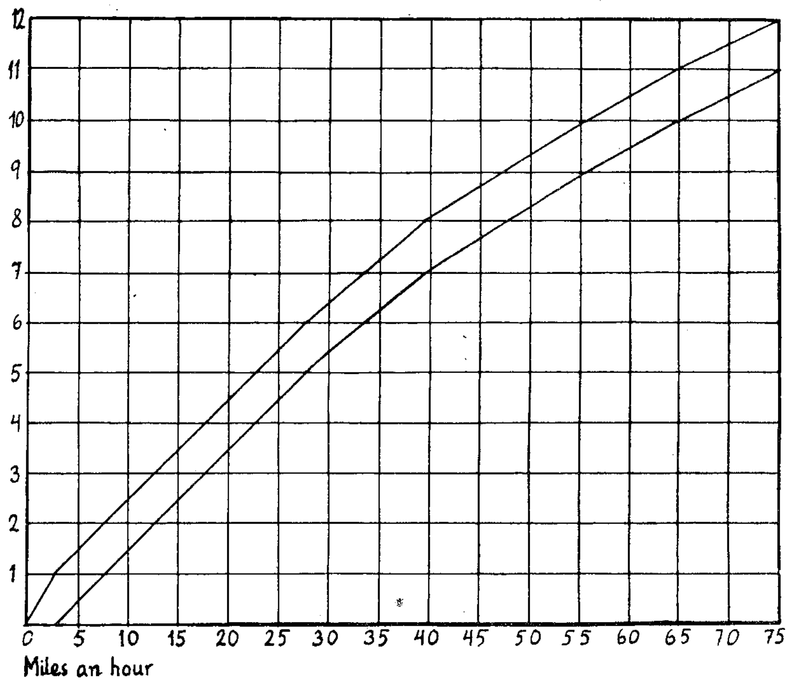

Fig. 2. Graph showing correspondence between wind force, according to the Beaufort scale, and velocity in miles an hour, upper and lower limit.

only a few observations were available Maury's charts, as well as English ones, were consulted. The data appearing on the North Atlantic Pilot Chart were tabulated in 1909.

The collection of ocean meteorological data is effected through the cooperation of vessel captains and others. Necessary forms and books of instruction are supplied to vessels by the Weather Bureau, generally through local offices of the Bureau at various ports. Before the war something like 2 000 vessels were rendering reports more or less regularly to the Bureau. This number is now, of course, considerably reduced.

There are several different scales known as »Beaufort« in use. That adopted by the Hydrographic Office and Weather Bureau is known as the »Scott« scale and differs slightly from the English. It is as follows:

Force	Designation	Miles per Hour
0	Calm	From 0 to 3
1	Light air	Over 3 to 8
2	Light breeze (or wind). . .	Over 8 to 13
3	Gentle breeze (or wind) . .	Over 13 to 18
4	Moderate breeze (or wind) .	Over 18 to 23
5	Fresh breeze (or wind). . .	Over 23 to 28

Force	Designation	Miles per Hour
6	Strong breeze (or wind) . .	Over 28 to 34
7	Moderate gale	Over 34 to 40
8	Fresh gale	Over 40 to 48
9	Strong gale	Over 48 to 56
10	Whole gale.	Over 56 to 65
11	Storm	Over 65 to 75
12	Hurricane	Over 75.«

Fig. 2 gives a graphical representation of this table.

4. The method of presenting the average wind conditions by means of the wind roses is certainly most convenient to the sailors, but it is difficult or impossible to get an immediate perception of the actual average air transport. I have therefore undertaken to reduce the mass of data concerning the wind as presented on the pilot charts, into a chart, that gives the direction and amount of the average air transport for each month. From various reasons I have chosen the charts of the Northern Pacific Ocean for this experiment.

The average wind is then shown by means of stream lines and curves of equal average velocity. The stream lines are not such ones, strictly spoken, but they indicate the direction of the resulting air transport at each place.

5. The meteorologists have usually determined the average wind at some place by treating the data as to the direction and the force separately — for instance, by calculating the mean of all directions observed, without regard to force, and on the other hand, the mean of all velocities without regard to directions. From a purely climatological standpoint this proceeding may be considered rational, but the true means are not obtained in this way. If the average wind is to be found both in direction and force, the different wind observations must be treated as data regarding vectors. From a mathematical standpoint this is the only rational method.

The question is now, whether the wind roses present the observations in such a form, that the resulting vector, representing the average air transport per second, can be found. If all observations could be figured as to direction and force, the case would be clear, but this is of course impossible.

6. For the area here considered, the North Pacific, the complete wind rose has 8 arrows, and the actual directions observed in each instance have been reduced to the 8 points of the compass, N, NE, E, SE, S, SW, W, and NW. Hereby a source of error is introduced, which is diminished on the charts of the North Atlantic, with roses of 16 points. The practice is namely to distribute the observations for intermediate points with equal parts to the neighbouring directions, a method that is not strictly correct; the error is not, however, considerable.

The angle between two neighbouring directions of a rose of 8 points is 45° (Fig. 3). If a vector V, in an intermediate direction between two of these points, say ENE, is distributed with equal parts along the two neighbouring points, NE and E, the two halves represent together, by geometrical addition, a vector $V' = 2 \cdot \frac{1}{2} V \cos 22\frac{1}{2}° = V \cos 22\frac{1}{2}° = 0.924\, V$. In this way, an error of 7.6 % is introduced, of the magnitude of the vector in question.

The average force is given for each of the 8 directions; but as the force according to the Beaufort scale is not strictly proportional to the velocity, the average force does not correspond accurately to the mean of the velocities. The correct procedure is to reduce all observed wind forces to velocities, and then to calculate the mean velocity.

The charts do not furnish materials for this calculation. As the average force only is given, there is no other course left than to identify the average velocity with the velocity corresponding to the average force. The error thus introduced is not important. As a matter of fact, the force is found by estimate, and the accuracy depends upon the judgment of the observer. It has then clearly no meaning to tamper with decimals, when the whole numbers may be inaccurate.

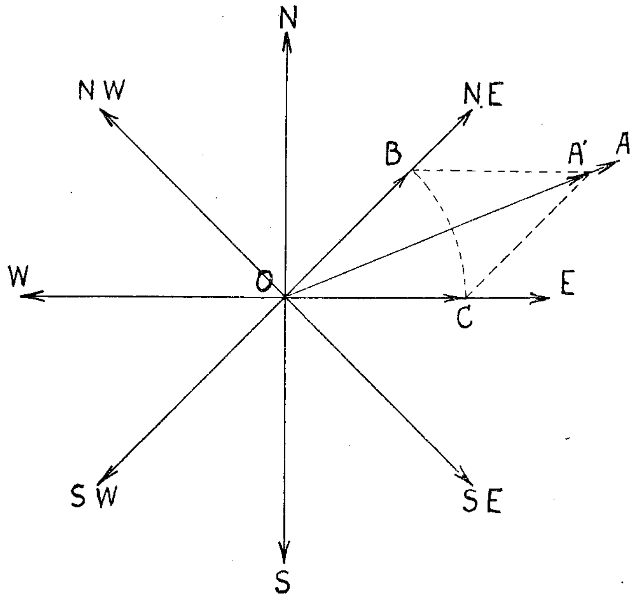

Fig. 3. OA = vector V. $OB = OC = \frac{1}{2} V$.
$OA' = V \cos 22\frac{1}{2}{}^{\circ} = V'$.

It is seen from the table of reduction, communicated by Mr. Marvin, that each force of the Beaufort scale corresponds to a somewhat wide space of veloeities, the intervals being 5 miles an hour in the most common cases. To obtain a definite value. I have simply used the means, and reduced the miles an hour to meter pr. second.

7. We may then interpret the 8 average forces as velocities, and then the resulting mean velocity for the whole rose can be found by vectorial addition.

As this involves a certain amount of labour for each rose, and each chart contains about 330 roses, some special method had to be devised, in order to reduce the amount of labour as much as possible.

First the length of the arrows must be measured by the percentage scale. For that purpose the scale was copied on a slab of celluloide, with a hole at the zero point. The centre of the wind rose is not marked, but is occupied by a little circle, for the percentage of calms. Therefore, a similar circle was copied on the slab of celluloide, with its centre at the zero point. The slab was pinned with a needle in each wind rose, and the length of the arrows noted. The numbers giving the percentage were written on each arrow. Including the number indicating calms etc. the sum of these percentages shall of course be exactly 100, but in fact most roses gave a slightly different sum, generally below the correct value. Some few gross errors were, however, detected.

8 The next stage is to get the forces reduced to velocities. It was found expedient to use a graphic reduction scale. As already noted, I have used the simple means of the limits of the velocities corresponding to each force; the table then will be as follows:

Force Beaufort	Velocity Miles an hour	Velocity Metres pr. second
0	1.5	0.67
1	5.5	2.46
2	10.5	4 .69
3	15.5	6.93
4	20.5	9.16
5	25.5	11.40
6	31	13.86

Force Beaufort	Velocity Miles an hour	Velocity Metres pr. second
7	37	16.54
8	44	19.67
9	52	23.24
10	60.5	27.04
11	70	31.29

The graphic table I (Fig. 4 a) gives the mean velocity in m/sec. for different forces and percentages up to 70. The table II (Fig. 4 b) gives the same, but all velocities multiplied by cos 45°. In this way it is possible to find the projections on the N—S and the E—W directions from all 8 arrows. Table I is used for the arrows from N,

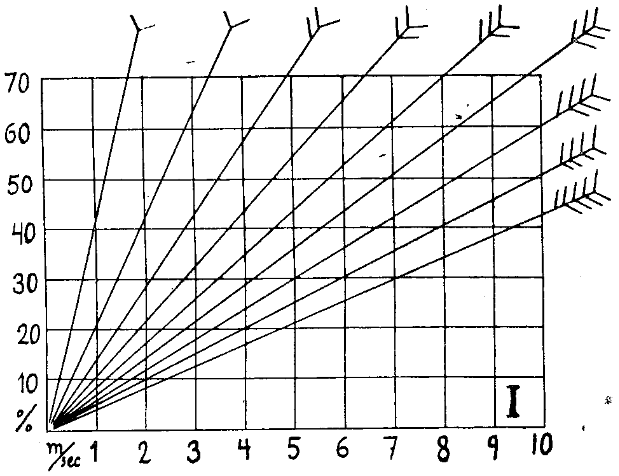

Fig. 4 a. Graph for reduction of wind arrows to velocity in metres per second.

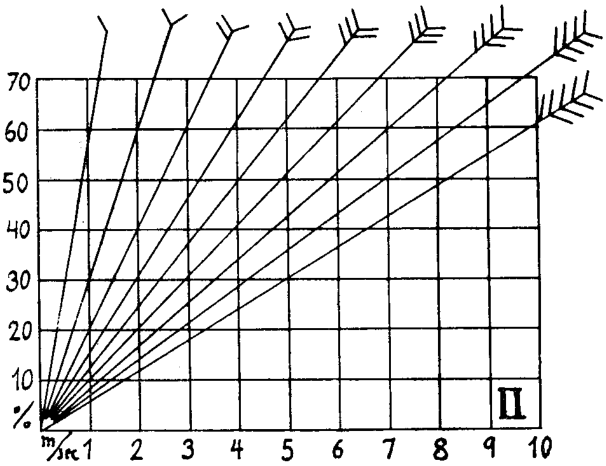

Fig. 4 b. Graph for reduction of wind arrows to velocity in metres per second, multiplied by cos 45°.

E, S, and W, table II for the intervening arrows. The number thus found is written off each arrow, and the resulting projections on the N—S and the E—W line can be found by simple addition and subtraction; the result is written down as two numbers for each rose. The uppermost figure indicates the N—S component, the lower figure, the E—W component of the resulting mean velocity.

9. This accomplished, the resulting velocity might be drawn as an arrow, one for each rose. This has been done for the month of June for the sake of control (see Fig. 31, p. 38). By means of these arrows, a set of continuous stream lines can be drawn, and likewise curves of equal velocity. But this requires much labour, and the curves must be smoothed.

I have found it more convenient to proceed along an other line; the two components of the velocity have been treated separately. For each month two charts have been made, one representing the magnitude of the north-south component, the other, in the same manner giving the east-west component of the mean velocity. The components are given on the charts by means of curves of equal values, like contour maps. On each chart there are areas of positive and negative values of the quantity represented, separated by curves for the value zero. The curves could be called isoboreales and isoaustrales, if erudite names be desired.

The curves will generally have a winding and crooked course, and must be smoothed. This I have done in free-hand. There is no special difficulty in areas of strong and persistent winds, as the trades and the monsoons, but in the northern parts of the ocean the curves are sometimes exceedingly irregular, and the smoothing requires a good deal of tact.

It is of course possible to do the smoothing by calculation, for instance, by calculating the means of four adjacent numbers; but this means extensive and utterly tedious work, without much gain as to the real value of the representation.

The charts of the components were drawn on tracing paper; to avoid mistakes, the chart of the north component was drawn in red ink, that of the east component, in green. The zero curves were drawn with double contours.

From these charts again the direction and force of the mean velocity were found.

10. *Direction.* — The direction of the mean velocity is to be represented by continuous stream lines. These do not indicate the actual path of the air particles, but serve as a means to figure the vectorial field. Let A denote the component towards west, and B, that towards south, then the direction of the vector is determined by:

$$\operatorname{tg} w = \frac{B}{A} = k$$

Here w is the angle between the parallels and the direction of the mean velocity.

The direction can be found by means of isogones. These curves were introduced into meteorology by *Sandström*,[1] but have not been much used. The isogone method does not generally lead to diminished labour or better accuracy than a simple free-hand drawing of stream lines from the observations. But in this special case it is not necessary to construct the directions of the mean velocity at every wind-rose, and the isogones can be found in a simple manner, directly from the two charts of the components A and B.

The two sheets are fixed in place on a table, and the isogones are drawn on a third sheet, placed uppermost. The green and red curves can be seen quite distinctly through the tracing paper.

Two isogones can be drawn immediately,

$$k = 0 \quad \text{and} \quad k = \infty$$

corresponding to $\quad B = 0 \quad \text{and} \quad A = 0$

The intersecting points between the curves $A = 0$ and $B = 0$ are singular points, through which all isogones must pass. In fact, the equation:

$$B - A\,k = 0$$

is satisfied for all values of k, in points where both A and B are $= 0$.

Further it is easy to draw the isogones $k = 1$ and $k = -1$ corresponding to the curves $A - B = 0$, and $A + B = 0$. Starting at the point $A = 0$, $B = 0$, we draw a curve through all points $A = B$, namely: $A = 1$, $B = 1$; $A = 2$, $B = 2$, etc. Likewise, the isogone $k = -1$ passes through all point $A = -B$. Thus the isogones $k = 1$ and $k = -1$ are represented by the diagonals through the squares formed by the two intersecting sets of curves A and B.

In an analogous manner, the isogones $k = \pm \frac{1}{2}$ and $k = \pm 2$ are drawn through all points where $A = \pm 2B$, or $B = \pm 2A$, respectively. The corresponding angles are: $\operatorname{arctg} \frac{1}{2} = 26^\circ\, 33'.9$ and $\operatorname{arctg} 2 = 63^\circ\, 26'.5$.

[1] *J. W. Sandström*, Über die Bewegung der Flüssigkeiten. Ann. d. Hydr. u. d. mar. Met. Berlin 1909.

In this way 8 isogones can be drawn, corresponding to 16 wind directions. From each singular point $A = 0$, $B = 0$, these 16 curves diverge in all directions, and ultimately run into some other singular point, or pass out of the area of the chart. Some isogones form closed curves, without passing through any singular point. Special care must be taken, that these curves do not escape notice.

11. *Drawing of continuous stream lines.* — According to the common practice, a great many short streaks are ruled along the isogones, parallel to the directions of the vector, which is represented by each of these curves, namely: $w = 0, 26\frac{1}{2}, 45, 63\frac{1}{2}, 90°$, and so on. Then the continuous curves are drawn, free hand. I have found the 16 isogones sufficient. Some inconvenience is caused by the fact that the angle intervals are not all alike, the angle of 45° being divided into unequal parts, $26\frac{1}{2}°$ and $18\frac{1}{2}°$, instead of $22\frac{1}{2}°$. But it would be much more difficult to draw the isogones for these directions, $22\frac{1}{2}°$ and $67\frac{1}{2}°$; by means of proper tables it can be done, however, but I consider the method used here to be the simplest one.

12. *Points of inflexion.* — It is possible to make the drawing of the stream lines more easy, and at the same time more elegant, by a little more preparative work.

The stream lines are determined by the differential equation:

$$\frac{dy}{dx} = k \qquad \text{(I)}$$

For inflexion points of the integral curves:

$$\frac{d^2y}{dx^2} = 0$$

By differentiation of (I) we obtain:

$$\frac{d^2y}{dx^2} = \frac{\partial k}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial k}{\partial x} + \frac{\partial k}{\partial y}k$$

For points of inflexion then:

$$k = -\frac{\dfrac{\partial k}{\partial x}}{\dfrac{\partial k}{\partial y}}$$

that is, the tangent to the isogone is parallel to the direction k represented by the same isogone.

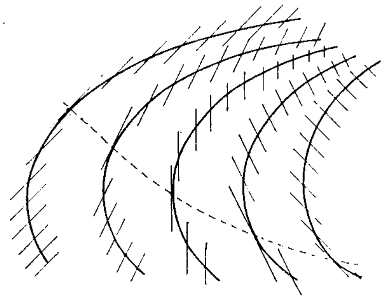

Fig. 5. A series of isogones with short streaks, showing corresponding directions k; the tangent points are connected by a curve.

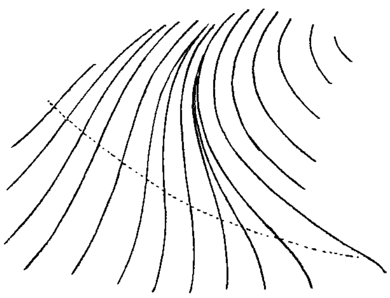

Fig. 6. Integral curves represented by the system of isogones of the preceding figure. The inflexion points are situated on the said curve.

If all points are marked, where the short streaks are tangents to the isogones, and a curve is drawn through these points (Fig. 5), the integral curves have inflexion points at the intersection with this curve (Fig. 6). These inflexion curves pass through the singular points $A = 0$, $B = 0$. We shall return to this point later on[1].

Some patterns of stream lines, represented by simple differential equations.

13. *Graphic integration.* — The drawing of continuous stream lines, when the direction of velocity is given in all points of a certain field, is equal to an integration of the differential equation

$$\frac{dy}{dx} = k$$

the value of k being known for all points x, y of the field in question.

If the two components of the wind velocity, A and B, are known, the equation becomes:

$$\frac{dy}{dx} = \frac{B}{A}$$

The quantities A and B are not known as definite analytical functions of the coordinates x and y, but as the velocity here concerned is defined as monthly means, we may safely suppose that both direction and magnitude of this velocity varies continually, and that both A and B can be developed in series:

$$A = A_0 + \left(\frac{\partial A}{\partial x}\right)_0 x + \left(\frac{\partial A}{\partial y}\right)_0 y + \ldots\ldots$$
$$B = B_0 + \left(\frac{\partial B}{\partial x}\right)_0 x + \left(\frac{\partial B}{\partial y}\right)_0 y + \ldots\ldots$$

The differential equation can then be written:

$$\frac{dy}{dx} = \frac{B_0 + \left(\frac{\partial B}{\partial x}\right)_0 x + \left(\frac{\partial B}{\partial y}\right)_0 y + \ldots\ldots}{A_0 + \left(\frac{\partial A}{\partial x}\right)_0 x + \left(\frac{\partial A}{\partial y}\right)_0 y + \ldots\ldots}$$

The dots represent terms of second and higher degree.

It may then have some interest to compare the results of the graphic integration with the analytical integration of the corresponding equation:

$$\frac{dy}{dx} = \frac{b_0 + b_1 x + b_2 y}{a_0 + a_1 x + a_2 y}$$

which does not contain terms of second or higher degree. The co-efficients a and b are constants.[2]

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[1] *W. Werenskiold,* Et tilfælde av grafisk integration, Norsk Matematisk Tidsskrift, 1. Aarg., 2. Hefte, 1919.

[2] *R. Dietzius,* Die Gestalt der Stromlinien in der Nähe der singulären Punkten. Beitr. zur Phys. d. freien Atm. VIII, I, 1918. See also *Sandström,* l. c. (§ 10).

By a simple parallel translation the expression can be transformed into the following:

$$\frac{dy}{dx}=\frac{b_1'\,x+b_2'\,y}{a_1'\,x+a_2'\,y} \tag{I}$$

when not the determinant:

$$a_1\,b_2-a_2\,b_1=0$$

Is this the case, the equation can be reduced to an expression of the form:

$$\frac{dy}{dx}=p\,x+q\,y+r \tag{II}$$

Both equations I and II can be integrated.

14. Note on the transformation of the equation:

$$\frac{dy}{dx}=\frac{b_0+b_1\,x+b_2\,y}{a_0+a_1\,x+a_1\,y}$$

By the substitution:

$$x'=x\cos u-y\sin u$$
$$y'=x\sin u+y\cos u$$

the equation becomes (indices dropped):

$$\frac{dy}{dx}=\frac{b_0\cos u-a_0\sin u+[b_1\cos^2 u+(b_2-a_1)\sin u\cos u-a_2\sin^2 u]\,x+[b_2\cos^2 u-(b_1+a_2)\sin u\cos u+a_1\sin^2 u)\,y}{a_0\cos u+b_0\sin u+[a_1\cos^2 u+(a_2+b_1)\sin u\cos u+b_2\sin^2 u]\,x+[a_2\cos^2 u+(b_2-a_1)\sin u\cos u-b_1\sin^2 u)\,y}$$

Here we can put:

$$a_1\cos^2 u+(a_2+b_1)\sin u\cos u+b_2\sin^2 u=0$$
$$a_2\cos^2 u+(b_2-a_1)\sin u\cos u-b_1\sin^2 u=0$$

provided that:

$$a_1\,b_2-a_2\,b_1=0$$

In that case, we can put:

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=n$$

The angle u is then determined by:

$$\operatorname{tg} u=-n$$

The transformed equation is:

$$\frac{dy}{dx}=\frac{b_0+n\,a_0+(1+n^2)^{\frac{1}{2}}\,[(b_1-nb_2)\,x+(b_2+nb_1)\,y]}{a_0-nb_0}$$

or:

$$\frac{dy}{dx}=p\,x+q\,y+r \tag{II}$$

where:

$$p = (1 + n^2)^{\frac{1}{2}} \frac{b_1 - nb_2}{a_0 - nb_0}$$

$$q = (1 + n^2)^{\frac{1}{2}} \frac{b_2 + nb_1}{a_0 - nb_0}$$

$$r = \frac{b_0 + na_0}{a_0 - nb_0} \qquad \text{and} \qquad \frac{a_0}{b_0} \gtrless n$$

15. The integral of equation (II) is:

$$y = C e^{qx} - \frac{p}{q} x - \frac{p + rq}{q^2} \qquad (q \text{ not} = 0)$$

where C is an arbitrary constant. The isogones are all parallel and equidistant. The integral curves are all congruent, and the whole set can be obtained by a parallel translation of one of them. This is seen by putting:

$$C = e^{qm} \qquad n = -\frac{p}{q} m + \frac{p + rq}{q^2}$$

The equation then assumes the form:

$$(y + n) + p/q\,(x + m) = e^{q\,(x + m)}$$

All integral curves can be obtained by parallel translation of the curve:

$$y + p/q\,x = e^{qx}$$

parallel to the straight line:

$$px + qy + r = 0$$

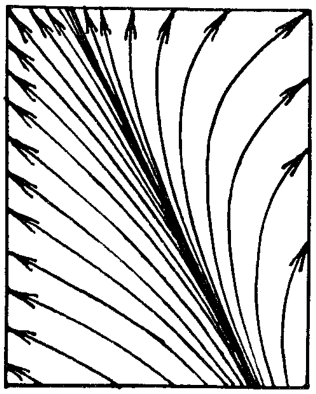

Fig. 7. Integral curves of the equation $\frac{dy}{dx} = p\,x + q\,y$

The integral curves have a common asymptote,

$$px + qy + \frac{p + rq}{q} = 0$$

This asymptote corresponds to a line of convergence or divergence in the vector field (Fig. 7).

If $p = 0$, $r = 0$ the equations are simpler still:

$$\frac{dy}{dx} = q\,y$$

with the integral:

$$y = C\,e^{qx}$$

The whole field is then symmetrically arranged as to the X-axis (Fig. 8). If $q=0$ the equation is:

$$\frac{dy}{dx}=px+r$$

with the integral:

$$y=\tfrac{1}{2}\,p\,x^2+r\,x+C$$

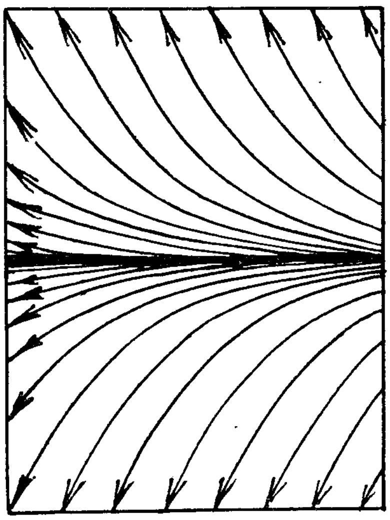

Fig. 8. Integral curves of the equation $\frac{dy}{dx}=q\,y$

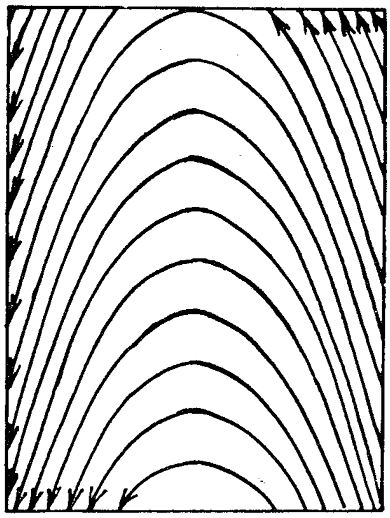

Fig. 9. Integral curves of the equation $\frac{dy}{dx}=p\,x$

representing a set of parabolas, obtained by translation of the curve:

$$y=\tfrac{1}{2}\,p\,x^2+r\,x$$

parallel to the Y-axis (Fig. 9).

This case corresponds to a simple and symmetric bend of the stream lines.

Remains the trivial case $p=0,\ q=0$, or:

$$\frac{dy}{dx}=r,\quad y=r\,x+C$$

But now p and q cannot both be $=0$, except when both

$$b_1=0 \quad\text{and}\quad b_2=0,$$

and consequently $a_1=0$ and $a_2=0$ (cf. § 14)

The original equation is then:

$$\frac{dy}{dx}=\frac{b_0}{a_0}$$

If $b_1 = 0$, $b_2 = 0$, with $a_1 \gtrless 0$, $a_2 \lessgtr 0$, then the quotient n is infinitely great; we have then only to examine the expression:

$$\frac{dx}{dy} = \frac{a_0 + a_1 x + a_2 y}{b_0}$$

which equation can be integrated at once, leading to similar cases as those treated above.

If ultimately:

$$\frac{a_0}{b_0} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = n$$

the equation can be written:

$$\frac{dy}{dx} = \frac{b_0 + b_1 x + b_2 y}{n(b_0 + b_1 x + b_2 y)}$$

with the integral:

$$y = \frac{x}{n} + C$$

and a singulary solution:

$$b_0 + b_1 x + b_2 y = 0$$

The stream lines are parallel straight lines, and along the singulary line the velocity is $= 0$. This case is represented by Fig. 6, B, of the paper by *Hesselberg and Sverdrup.*[1]

16. We shall return to the equation:

$$\frac{dy}{dx} = \frac{b_1 x + b_2 y}{a_1 x + a_2 y} = \frac{B}{A}$$

The integration of this differential equation has been discussed in various text-books, and also by MM. Hesselberg and Sverdrup, from a meteorological point of view. I will here only mention the different cases. — According to the theory of homogeneous differential equations the integral can be written down at once:

$$y = C e^{-\int_0^x \frac{B\,dx}{Ay - Bx}}$$

The calculations are much simplified, if the co-ordinate system is turned an angle u, determined by:

$$\text{tg}\, 2u = \frac{b_2 - a_1}{b_1 + a_2}$$

This is always possible, except when:

$$b_2 - a_1 = 0 \quad \text{and} \quad b_1 + a_2 = 0$$

but then no turning is needed.

[1] *Th. Hesselberg und H. U. Sverdrup,* Das Beschleunigungsfeld bei einfachen Luftbewegungen. Veröffentlichungen des Geophysikalischen Instituts der Universität Leipzig, Zweite Serie, Spezialarbeiten aus dem Geophysikalischen Institute, Heft 5, Leipzig 1914.

Dropping the indices for x and y, we get:

$$\frac{dy}{dx} = \frac{\alpha x + \beta y}{\beta x + \gamma y} \qquad \text{III}$$

where:

$$\alpha = b_1 - a_2 + D$$
$$\beta = a_1 + b_2$$
$$\gamma = a_2 - b_1 + D$$
$$D^2 = (a_2 + b_1)^2 + (a_1 - b_2)^2$$

The integral is:

$$y = C e^{-\int_0^x \frac{\alpha x + \beta y}{\alpha x^2 - \gamma y^2} dx}$$

Here three cases must be treated differently as:

$$\alpha \gamma \gtreqless 0$$

1) $\alpha\gamma > 0$. We can choose both positive. The integral is:

$$y = C \frac{y}{\sqrt{\alpha x^2 - \gamma y^2}} \left[\frac{\sqrt{\alpha}\, x - \sqrt{\gamma}\, y}{\sqrt{\alpha}\, x + \sqrt{\gamma}\, y}\right]^{\frac{\beta}{2\sqrt{\alpha\gamma}}}$$

Putting:

$$\sqrt{\alpha}\, x + \sqrt{\gamma}\, y = Y \quad , \quad \sqrt{\alpha}\, x - \sqrt{\gamma}\, y = X \quad , \quad \frac{\beta}{\sqrt{\alpha\gamma}} = m$$

the integral can be written in the following form:

$$Y^{m+1} = C\, X^{m-1}$$

This equation represents all kinds of parabolas and hyperbolas in the co-ordinate system X, Y, viz.:

parabolas for $m^2 > 1$
hyperbolas for $1 > m^2 > 0$

For $m^2 = 1$ the result is either $\begin{cases} Y^2 = C \\ X = 0 \end{cases}$ or $\begin{cases} X^2 = 1/C \\ Y = 0 \end{cases}$

representing parallel straight lines, with another straight line as a singular integral.

2) $\alpha\gamma = 0$. If $\alpha = 0$, γ different from 0, the integral is:

$$y = C e^{\frac{\beta}{\gamma} \frac{x}{y}}$$

Is $\gamma = 0$, α different from 0, the integral must be found from the equation (III) directly:

$$x = C e^{\frac{\beta}{\alpha} \frac{y}{x}}$$

Are both $\alpha = 0$ and $\gamma = 0$, the integral is:

$$y = C x$$

3) $\alpha\gamma < 0$. Let for instance $-\gamma = \delta$, and α and δ both positive. The integral is:

$$y = C \frac{y}{\sqrt{\alpha x^2 + \delta y^2}} e^{-\frac{\beta}{\sqrt{\alpha\delta}} \text{ arc tg } \sqrt{\frac{\alpha}{\delta}} \frac{x}{y}}$$

By the substitutions:

$$\sqrt{\alpha}\, x = \xi \qquad \sqrt{\delta}\, y = \eta \qquad m = \frac{\beta}{\sqrt{\alpha\delta}}$$

and further:

$$\xi = r \cos\left(\varphi + \frac{\pi}{2}\right)$$

$$\eta = r \sin\left(\varphi + \frac{\pi}{2}\right)$$

the integral can be written:

$$r = C e^{m\varphi}$$

In case $m = 0$, or $\beta = 0$, the equation is:

$$r = C$$

The integral of the differential equation (III) when the conditions 3) are fulfilled, represents all kinds of logarithmic spirals in the coordinate system ξ, η, corresponding to similar flattened curves in the system x, y. In special cases the integral represents a series of concentric circles in the system $\xi\,\eta$, corresponding to a series of concentric ellipses, with the same shape and orientation, in the system x, y.[1]

17. We have now treated the different cases following from the equation:

$$\frac{dy}{dx} = \frac{b_0 + b_1 x + b_2 y}{a_0 + a_1 x + a_2 y}$$

and we shall proceed to consider some forms of the equation

$$\frac{dy}{dx} = \frac{B}{A}$$

where the expression B begins with terms of second degree.

[1] In the paper by *Hesselberg and Sverdrup*, five different cases are treated and figured, viz: A, corresponding to 1, a. B, to 1, c; C, to 1, b; D, to 2; E, to 3.

Let the differential equation be:

$$\frac{dy}{dx} = \frac{b_{11}x^2 + 2\,b_{12}\,xy + b_{22}\,y^2 + \ldots\ldots}{a_0 + a_1 x + a_2 y + a_{11} x^2 + 2\,a_{12}\,xy + a_{22}\,y^2 + \ldots\ldots}$$

Dots represent terms of third or higher degree.
For the sake of comparison, let us consider the equation:

$$\frac{dy}{dx} = \frac{c_1 x^2 + 2\,c_2\,xy + c_3\,y^2}{a}$$

The isogones are concentric conic sections:

$$c_1 x^2 + 2c_2\,xy + c_3\,y^2 - ak = 0$$

The integral curves form a system with a centre of symmetry, viz. the point *0,0*. The most characteristic feature is the occurrence of two singular curves, one representing a line of divergence, the other, a line of convergence. We shall only consider one special case, viz.: $c_1\,c_3 = c_2{}^2$; the equation can then be written:

$$\frac{dy}{dx} = (p\,x + q\,y)^2$$

The equation is easily integrable by putting:

$$p\,x + q\,y = z$$

This substitution brings the differential equation into the form:

$$\frac{dz}{dx} = q\,z^2 + p$$

We have now three distinct cases: $pq \gtreqless 0$

1) $pq > 0$ (both positive).

The integral is then:

$$\frac{1}{\sqrt{pq}}\,\mathrm{arctg}\sqrt{\frac{q}{p}}\,z = x + C$$

or:

$$p\,x + q\,y = \sqrt{\frac{p}{q}}\,\mathrm{tg}\,(\sqrt{pq}\,x + C)$$

The integral curves can all be obtained by translation of the curve

$$px + qy = \sqrt{\frac{p}{q}}\,\mathrm{tg}\,\sqrt{pq}\,x$$

parallel to the straight line

$$px + qy = 0$$

The curves represent a double bend of parallel stream lines, see Fig. 10.

2) $pq = 0$. Now either $p = 0$ or $q = 0$.
a) Let $p = 0$. The equation is:

$$\frac{dy}{dx} = q^2 y^2$$

which leads to the integral:

$$y(C - q^2 x) = 1$$

The integral curves are obtained by translation of the hyperbola

$$q^2 xy = -1$$

parallel to the X-axis, and represent a combined line of convergence and divergence (Fig. 11). This case may not be without representation on wind charts.

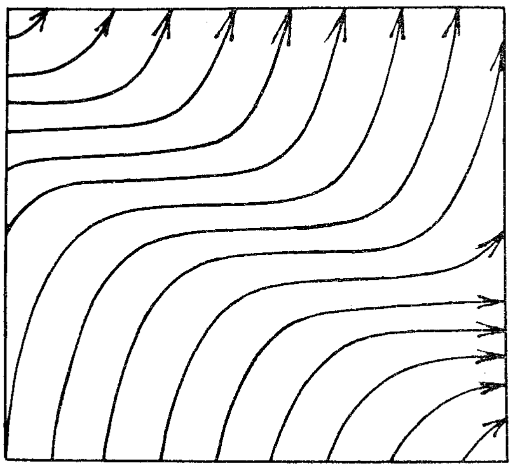

Fig. 10. Integral curves of the equation
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{3} + \frac{y}{1.5}\right)^2$$

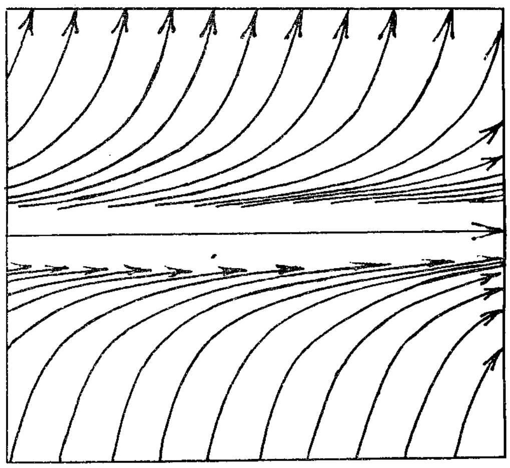

Fig. 11. Integral curves of the equation
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{1.5}\right)^2$$

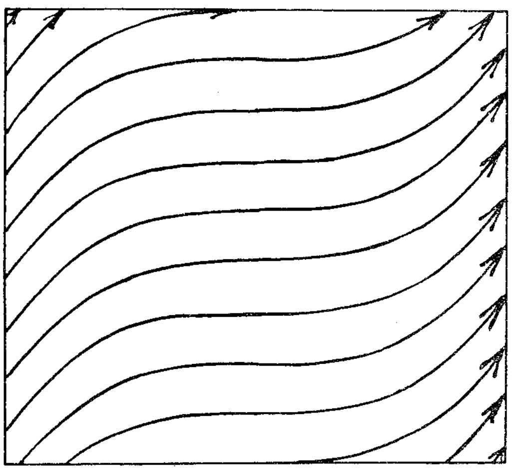

Fig. 12. Integral curves of the equation
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{3}\right)^2$$

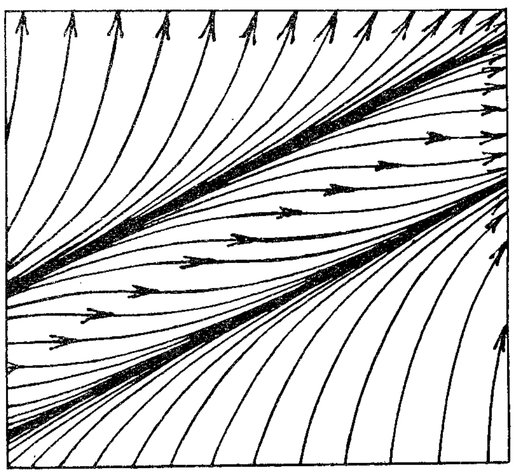

Fig. 13. Integral curves of the equation
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{3} - \frac{y}{1.5}\right)^2$$

b) Let $q = 0$. We have then:

$$\frac{dy}{dx} = p^2 x^2$$

with the integral:

$$y = \tfrac{1}{3} p^2 x^3 + C$$

representing a series of cubic parabolas, obtained by translation of the curve

$$3\, y = p^2 x^3$$

parallel to the Y-axis (Fig. 12).

3) If $pq < 0$, let $p = -r$, and r and q both positive. The (transformed) equation is then:

$$\frac{dz}{dx} = q\, z^2 - r$$

with the integral:

$$x + C = \frac{1}{2q} \operatorname{lognat} \frac{z - \sqrt{\frac{r}{q}}}{z + \sqrt{\frac{r}{q}}}$$

or:

$$q\, y - r\, x = \sqrt{\frac{r}{q}} \operatorname{Tghyp}(qx + C)$$

This equation represents a series of congruent curves, obtained by the usual translation parallel to the straight line $px + qy = 0$. The curves have two common asymptotes, which may be interpreted as lines of convergence and divergence (Fig. 13).

If these are brought into coincidence, case 2 a, appears. (This can be arranged by proper turning of the system of isogones, until $p = 0$.) The combination of a line of convergence with a line of divergence has been noted by *Bjerknes*.[1]

18. *Complex singular points.* — If in the differential equation:

$$\frac{dy}{dx} = \frac{b_1 x + b_2 y + b_{11} x^2 + 2\, b_{12}\, xy + b_{22}\, y^2 + \ldots\ldots}{a_1 x + a_2 y + a_{11} x^2 + 2\, a_{12}\, xy + a_{22}\, y^2 + \ldots\ldots} = \frac{B}{A} \qquad \text{I}$$

the determinant:

$$a_1 b_2 - a_2 b_1 = 0$$

the curves $A = 0$ and $B = 0$ have a common tangent in the point $0,0$, and the integral curves have a peculiar shape in the neighbourhood of this point, not resembling any of the types treated above.

If $b_1/a_1 = b_2/a_2 = n$, and the co-ordinate system is turned an angle u determined by

$$\operatorname{tg} u = n$$

[1] *V. Bjerknes*, Dynamische Meteorologie. Zweiter Teil: Kinematik. Braunschweig 1913. Fig. 20, D—F, page 51.

the equation assumes the form:

$$\frac{dy}{dx} = \frac{c_1 x^2 + 2c_2 xy + c_3 y^2 + \ldots .}{g_1 x + g_2 y + \ldots\ldots\ldots\ldots} = \frac{B'}{A'} \qquad \text{II}$$

The curve $B' = 0$ has a singular point in $0,0$.

Equation I can be written:

$$\frac{dy}{dx} = \frac{n(a_1 x + a_2 y) + b_{11} x^2 + 2\, b_{12} xy + b_{22} y^2 + \ldots .}{a_1 x + a_2 y + a_{11} x^2 + 2\, a_{12} xy + a_{22} y^2 + \ldots .}$$

According to the $0/0$ method, the following values for $k = \frac{dy}{dx}$ are found in the point $0,0$:

$$k_1 = n$$

$$k_2 = -\frac{a_1}{a_2}$$

The equations I or II are not generally integrable by quadratures, but by means of the isogones it is possible to get a graphical, approximative solution of the problem. In some cases, however, the equations can be integrated.

In order to study the character of the singular points of the integral curves belonging to the equations I or II, it is sufficient to regard expressions of second degree. The isogones are then conic sections, with a common tangent $a_1 x + a_2 y = 0$ in the point $0,0$. The equation of the isogone k is:

$$(b_{11} - ka_{11}) x^2 + 2(b_{12} - ka_{12}) xy + (b_{22} - ka_{22}) y^2 + (n - k)(a_1 x + a_2 y) = 0$$

representing a conic section. The centres of the isogones are situated on a conic section through the point $0,0$.

The isogone for $k = n$ has a singular point in the origin.

19. We shall not enter upon a discussion of the equation

$$\frac{dy}{dx} = \frac{c_1 x^2 + 2c_2 xy + c_3 y^2}{g_1 x + g_2 y}$$

as the different types of the complex singular point are illustrated by an equation representing a more special case:

$$\frac{dy}{dx} = \frac{(px + qy)^2}{rx + sy} = \frac{B}{A}$$

The isogones are then found by the equation:

$$(px + qy)^2 - k(rx + sy) = 0$$

which represents a series of parabolas with a common tangent

$$rx + sy = 0$$

in the point $0,0$. The curve $A = 0$ coincides with this tangent. The curve $B = 0$ is the (double) straight line

$$px + qy = 0$$

(see Fig. 16).

For the sake of convenience we shall treat two special cases first, in order to make the case clearer.

1) Let $p = 0$ and $s = 0$. The equation is:

$$\frac{dy}{dx} = \frac{q^2 y^2}{r x}$$

with the integral:

$$-\frac{r}{y} = q^2 \operatorname{lognat} x + C$$

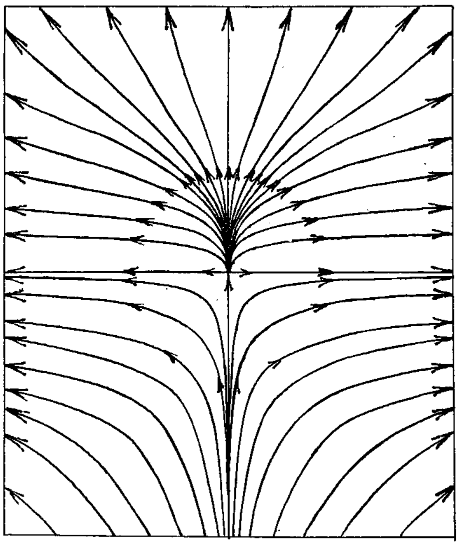

Fig. 14. Integral curves of the equation
$$\frac{dy}{dx} = \frac{y^2}{ax}$$

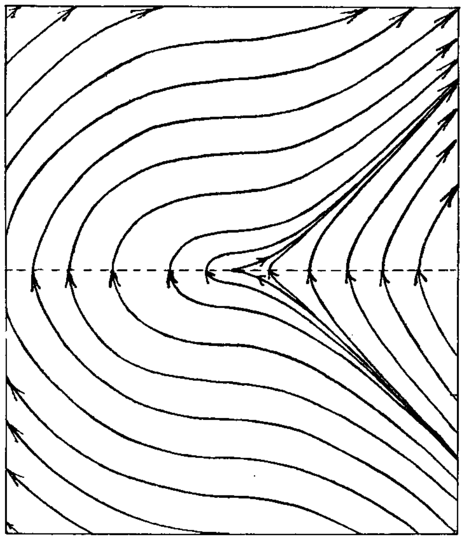

Fig. 15. Integral curves of the equation
$$\frac{dy}{dx} = \frac{x^2}{by}$$

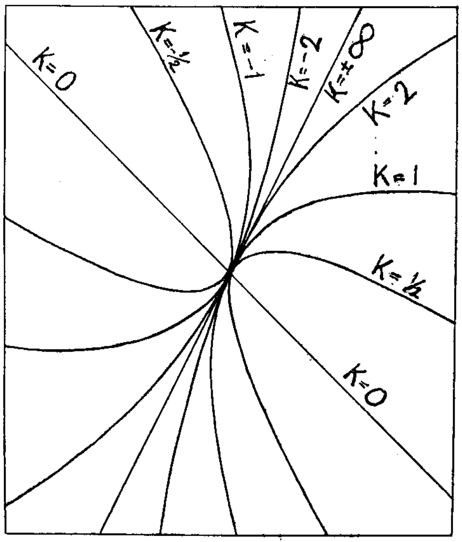

Fig. 16. Isogones represented by the equation
$$\frac{dy}{dx} = \frac{(px+qy)^2}{rx+sy}$$

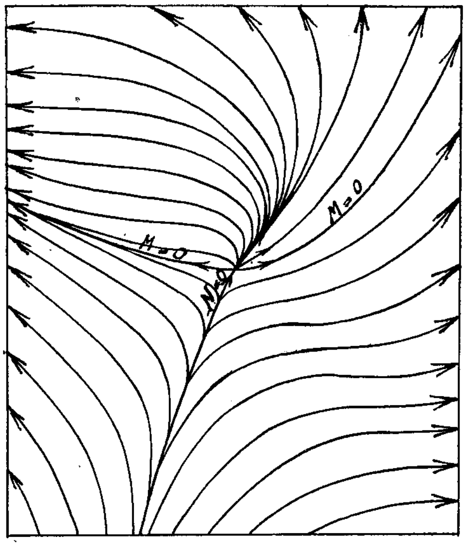

Fig. 17. Integral curves of the equation
$$\frac{dy}{dx} = \frac{(px+qy)^2}{rx+sy}$$

$(p = q = 1,\ r = 12,\ s = -6)$

By putting $C = q^2 \operatorname{lognat} t$ the expression can be written:

$$y = \frac{r}{q^2 \operatorname{lognat} \frac{t}{x}}$$

In the area below the X-axis, for $y < 0$, only one integral curve runs into the singular point $0,0$. On the other side of the X-axis, the same line $x = 0$ is a common cusp-tangent to an infinite number of integral curves. Besides, the X-axis itself is an integral curve (Fig. 14).

It may be noted that all inflexion points are situated along the straight line:

$$2\, q^2\, y = r$$

2) Now let:

$$q = 0,\ r = 0$$

The equation is then:

$$\frac{dy}{dx} = \frac{p^2\, x^2}{s\, y}$$

with the integral:

$$2\, p^2\, x^3 = 3\, s\, y^2 + C$$

For $C = 0$ this equation represents the common semicubic parabola. The cusp of this curve is the only integral curve passing through the point $0,0$. The whole set of curves is symmetrically arranged on both sides of the X-axis (Fig. 15).

3) Other values of the co-efficients lead to assymmetric figures. (We may put aside the two degenerate cases: $p = 0,\ r = 0$, and $q = 0,\ s = 0$.) These figures form transitions between the two special cases 1) and 2) (Figs. 14 and 15). In the point $0,0$ always two values of k are found, viz.

$$k = 0 \quad \text{and} \quad k = -\frac{r}{s}$$

An integral curve $M = 0$ passes the singular point with the tangent $y = 0$ and has here a definite curvature, $\frac{1}{R} = \frac{p}{\sqrt{r}}$. This curve divides the field in the immediate neighbourhood of $0,0$ in two parts. One of these areas is occupied by a »broom« of cuspate curves, with a common tangent $rx + sy = 0$. On the other side of the curve $M = 0$, only one integral curve, $N = 0$, runs into the point $0,0$, with the tangent $rx - sy = 0$ (Figs. 16 and 17).

20. If we return to the equation:

$$\frac{dy}{dx} = \frac{n\,(a_1\, x + a_2\, y) + B_2}{a_1\, x + a_2\, y + A_2}$$

A_2 and B_2 designating terms of second or higher degree, the character of the singular point is substantially the same as for the special case just treated. One integral curve $M = 0$ passes through the origin with a tangent

$$y - n\, x = 0$$

and divides the field in the immediate neighbourhood of the origin into two parts. One part is occupied by a »broom« of cusps, with a common tangent

$$a_1 x + a_2 y = 0$$

On the other side of the curve $M = 0$ only one integral curve, $N = 0$, runs into the point $0,0$, with the same tangent $a_1 x + a_2 y = 0$. If, however,

$$a_1 + a_2 n = 0$$

the two tangents coincide, and the curves $M = 0$ and $N = 0$ form branches of one cuspate curve; this is the only curve that runs into the point $0,0$.

This last case is illustrated by the equation:

$$\frac{dy}{dx} = \frac{c_1 x^2 + c_3 y^2}{2y}$$

which can be integrated by putting $y^2 = z$, The solution is:

$$c_1 x^2 + c_3 y^2 + 2\, c_1/c_3\, x + 2\, c_1/c_3^2 = C\, e^{c_3 x}$$

For $C = 2\, c_1/c_3^2$ the curve passes through the origin. By developing in series, we obtain:

$$y^2 = \frac{1}{3} c_1 x^3 + \ldots\ldots$$

showing the character of the cusp. — These complex singular points are obtained by the fusion of a hyperbolic point and an elliptic point. — Two such cases are represented on Figs. 18 to 25. Figure 18 shows the two curves (here circles) $A = 0$ and $B = 0$, intersecting in two points; these are singular points in the field of the stream lines, (Figure 19), namely, a hyperbolic point, and an elliptic one. (V. § 16, 3). This peculiar feature is due to the symmetry of the field. — If now the two curves $A = 0$ and $B = 0$ are drawn apart, the two points of intersection approach each other, and ultimately coincide. The curves have then a tangent in common. In the corresponding symmetrical field of stream lines, the two singular points have fused into one complex point, a cusp. — Another case is represented by Figure 22. The curves (circles) $A = 0$ and $B = 0$ here also intersect in two points, and the stream lines have here corresponding singular points, namely, a hyperbolic one, and a spiral point (Fig. 23). — If the two curves $A = 0$ and $B = 0$ have a common tangent, the two singular points are fused into one, a complex »broom« (Figure 25). Figures resembling these complex singular points are not infrequently met with on wind charts.

A »broom« is located at appr. 35° N. Lat., 160° W. Long, on the chart for April (Fig. 29).

A cuspate point appears on the chart for May, near equator, at 125° E. Long (Fig. 30).

21. *Note.* — *V. Bjerknes* has observed that a neutral point and a point of convergence can unite to form a complex singular point — appearing as a branching stream line. His remark, that the branches may unite with definite angles, does not in general hold good.[1]

[1] Dynamische Meteorologie und Hydrographie von *V. Bjerknes*. Zweiter Teil, Kinematik. Braunschweig 1913, p. 47.

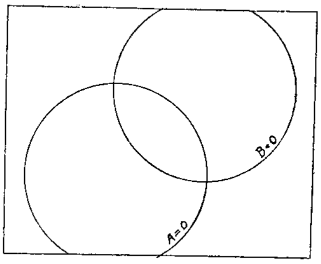

Fig. 18. Curves $A = 0$ and $B = 0$, where:

$$B = (x - a)^2 + (y - a)^2 - r^2$$
$$A = (x + a)^2 + (y + a)^2 - r^2$$

and $r^2 > 2a^2$

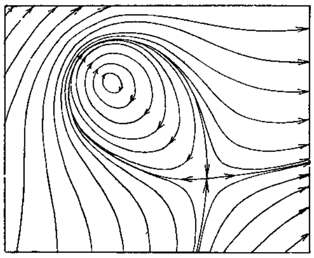

Fig 19. Integral curves of the equation

$$\frac{dy}{dx} = \frac{B}{A}$$

(A and B the same as for the preceding figure).

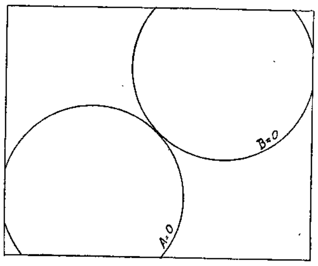

Fig. 20. Curves $A = 0$ and $B = 0$, where:

$$B = (x - a)^2 + (y - a)^2 - 2a^2$$
$$A = (x + a)^2 + (y + a)^2 - 2a^2$$

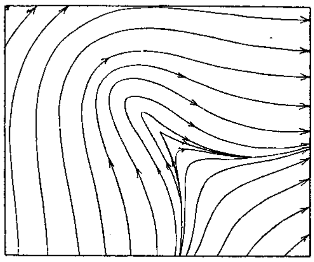

Fig. 21. Integral curves of the equation

$$\frac{dy}{dx} = \frac{B}{A}$$

(A and B the same as for the preceding figure).

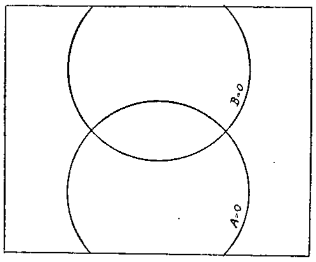

Fig. 22. Curves $A = 0$ and $B = 0$, where:

$$B = x^2 + (y - a)^2 - r^2$$
$$-A = x^2 + (y + a)^2 - r^2$$

and $r^2 > a^2$

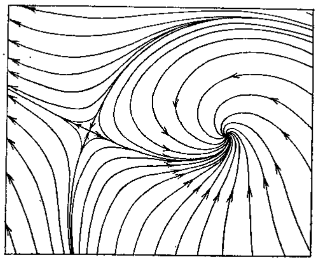

Fig. 23. Integral curves of the equation

$$\frac{dy}{dx} = \frac{B}{A}$$

(A and B the same as for the preceding figure).

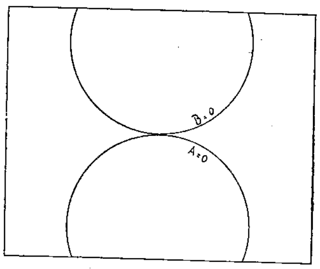

Fig. 24. Curves $A = 0$ and $B = 0$, where:

$$B = x^2 + (y - a)^2 - a^2$$
$$-A = x^2 + (y + a)^2 - a^2$$

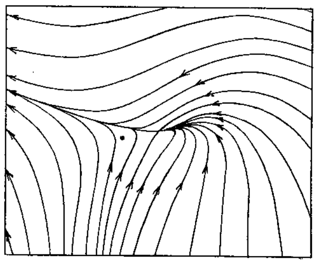

Fig. 25. Integral curves of the equation

$$\frac{dy}{dx} = \frac{B}{A}$$

(A and B the same as for the preceding figure).

Ivar Bendixon[1] has treated the equation

$$x^m \frac{dy}{dx} = ax + by + P(x,y)$$

$P(x,y)$ being a Taylor series with no terms of lesser degree than the second, and a different from zero. If m is an equal number, the integral curves form a »broom« in the point $0,0.$[1])

If the equation is written on the form:

$$\frac{dy}{dx} = \frac{ax + by + P(x,y)}{x^2}$$

it appears as a special form of our equation I, § 18.

Supplementary Investigations of the Field of Velocity.

22. *The inflexion curve.* — The integral curves defined by the differential equation:

$$\frac{dy}{dx} = \frac{B}{A}$$

have points of inflexion along a curve defined by:

$$\frac{d^2y}{dx^2} = 0$$

or:

$$A\,B \frac{\partial A}{\partial x} - A^2 \frac{\partial B}{\partial x} + B^2 \frac{\partial A}{\partial y} - A\,B \frac{\partial B}{\partial y} = 0$$

This is the equation of the inflexion curve. It passes through all singular points, where $A = 0$ and $B = 0$, and has singular points itself in these same points.

If the origin is such a point, we have: $A_0 = 0$ and $B_0 = 0$. The equation of the tangents of the inflexion curve in this point $0,0$ is found as usual; after some reckoning we have:

$$\frac{\partial A}{\partial y} (y')^2 + \left(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right) y' - \frac{\partial B}{\partial x} = 0$$

Putting $y' = h$, the equation of the tangents is then:

$$(y - h_1)\ (y - h_2) = 0$$

h_1 and h_2 being the two values of h, obtained from the quadratic equation.

---

1) *Ivar Bendixon.* Sur les points singuliers des équations différentielles. Öfversigt af Kongl. Vet. Akad. Förh. Febr., 9, 1898.

Ivar Bendixon. Sur les courbes définies par des équations différentielles. Acta Mathematica, B. 24, Stockholm 1901. — (Spec. p. 45.)

To determine the tangents of the integral curves in the point *0,0,* we must use the *0/0* method, and obtain:

$$\frac{dy}{dx} = \frac{B}{A} = \frac{\partial B/\partial x + \partial B/\partial y \cdot y'}{\partial A/\partial x + \partial A/\partial y \cdot y'}$$

Putting $dy/dx = y' = k$, we obtain:

$$\frac{\partial A}{\partial y} k^2 + \left(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right) k - \frac{\partial B}{\partial x} = 0$$

The tangents of the integral curves in the singular point are determined by the same equation as the tangents of the inflexion curve in the same point.

The inflexion curve has the same tangents as the integral curves in the singular point $A = 0$, $B = 0$.

In a spiral point, the inflexion curve has an isolated point; in hyperbolic and parabolic points (case 1, § 16) the inflexion curve has double point and ultimately in case 2, § 16, the inflexion curve has a cusp.

If the differential equation

$$\frac{dy}{dx} = \frac{B}{A}$$

is lineary, it can be brought on the form:

$$\frac{dy}{dx} = \frac{\alpha x + \beta y}{\beta x + \gamma y}$$

The inflexion curve is then simply:

$$\alpha x^2 + \gamma y^2 = 0$$

representing either two straight lines, one (double) straight line, or the point *0,0,* according to the condition:

$$\alpha\gamma \lesseqgtr 0$$

23. The inflexion curve passes through all singular points of the isogones. The equation of an isogone may be:

$$\frac{dy}{dx} = K = \text{constant.}$$

If the isogone has a singular point, *0,0,* then both:

$$\frac{\partial K}{\partial x} = 0 \quad \text{and} \quad \frac{\partial K}{\partial y} = 0$$

The tangent directions h_1 and h_2 of one isogone are the roots of the equation:

$$\frac{\partial^2 K}{\partial y^2} h^2 + 2 \frac{\partial^2 K}{\partial x \, \partial y} h + \frac{\partial^2 K}{\partial x^2} = 0$$

The two tangents constitute together a degenerate conic section with the equation:

$$\frac{\partial^2 K}{\partial y^2} y^2 + 2 \frac{\partial^2 K}{\partial x \, \partial y} x y + \frac{\partial^2 K}{\partial x^2} x^2 = 0$$

Two diameters $y = m\,x$ and $y = n\,x$ are conjugated with respect to the conic section if:

$$\frac{\partial^2 K}{\partial y^2}\, m\, n + \frac{\partial^2 K}{\partial x\, \partial y}(m + n) + \frac{\partial^2 K}{\partial x^2} = 0$$

Now the equation of fhe inflexion curve is:

$$K\frac{\partial K}{\partial y} + \frac{\partial K}{\partial x} = 0$$

where $K = B/A$. The tangent direction of the inflexion curve, H, is then found from the equation:

$$\frac{\partial^2 K}{\partial y^2} K\, H + \frac{\partial^2 K}{\partial x\, \partial y}(K + H) + \frac{\partial^2 K}{\partial x^2} + \frac{\partial K}{\partial y}\frac{\partial K}{\partial x} + \left(\frac{\partial K}{\partial y}\right)^2 H = 0$$

If now both $\frac{\partial K}{\partial x} = 0$ and $\frac{\partial K}{\partial y} = 0$ the equation is reduced to:

$$\frac{\partial^2 K}{\partial y^2} K\, H + \frac{\partial^2 K}{\partial x\, \partial y}(K + H) + \frac{\partial^2 K}{\partial x^2} = 0$$

By comparison with the equation at the end of the preceding paragraph, it is seen, that: *When an isogone $B/A = K =$ a constant, has a singular point, the inflexion curves passes through this point, and its tangents in the same point is conjugated to the direction K belonging to the isogone, with respect to the degenerate conic section represented by the two tangents to the isogone in the singular point.*

The last rule may be of some use for the correct drawing of the iuflexion curve, and consequently, for the right presentation of the stream lines.

If the isogone has an isolated point, the two tangents are imaginary and the isogones form small ellipses in the neighbourhood of the singular point; it is often possible to determine the conjugated directions from the chart of the isogones, with approximation.

24. *The magnitude of the velocity.* — From a pure analytical point of view it is more rational to present the field of the square of the velocity, V^2, than that of the scalar magnitude of the velocity, $|V|$. But in order to render the clearest and most comprehensible picture of the distribution of velocity, the latter procedure is no doubt to be preferred.

When the two rectangular projections A and B of the velocity are known, this latter is easily found from the relation

$$V^2 = A^2 + B^2$$

By means of a simple device this equation can be solved graphically, as set forth by *Bjerknes,* using a set of circles with radii = 1, 2, 3, 4, &c., drawn on a millimeter paper.

25. *Singular points in the field of V^2.* — The singular points of the field of the square of the velocity coincide with the singular points of the field of the magnitude of the velocity, and accordingly we have only to examine the first field in this respect. — If now:

$$V^2 = \text{maximum}$$

then

$$\frac{\partial V^2}{\partial x} = 0 \quad \text{and} \quad \frac{\partial V^2}{\partial y} = 0$$

If the terminology of the vector analysis is used, we can express this in the following way: The gradients of the scalars A^2 and B^2 are like and opposite in direction. It follows that the curves $A^2 = \text{constant}$ and $B^2 = \text{constant}$ are parallel in the point of maximum of V^2. Then the curves $A = \text{constant}$ and $B = \text{constant}$ are parallel in the same point too. The curves $V = \text{constant}$ in the immediate neighbourhood of such a point are either ellipses, corresponding to a maximum value of the velocity V, or hyperbolas, corresponding to a saddle point in the field of V.

As the magnitude of the velocity always is positive, the minimum of V is zero, which is the case in all singular points in the field of the direction of the vector V.

The curves of equal velocity are:

$$V^2 = A^2 + B^2$$

For values of A and B near zero, these curves are approaching to ellipses, granted that the curves $A = 0$ and $B = 0$ are not parallel. A curve $V = \varepsilon$ thus forms an ellipse round the ordinary singular point.

26. *Error in magnitude.* — The values of the quantities A and B cannot be supposed to be accurate. Is δ_A the mean error of A, and δ_B the mean error of B, then the mean error δ_V of $V = \sqrt{A^2 + B^2}$ is found:

$$\delta_V = \frac{\sqrt{A^2\delta_A{}^2 + B^2\delta_B{}^2}}{\sqrt{A^2 + B^2}}$$

We may assume that

$$\delta_A = \delta_B = \delta$$

Then simply

$$\delta_V = \pm \delta$$

27. *Error in direction.* — The direction W of the velocity is found from the equation:

$$\operatorname{tg} W = \frac{B}{A}$$

Under the same suppositions as in the preceding chapter, we find:

$$\delta_W = \pm \frac{1}{V} \cdot \delta$$

The curves of equal velocity are thus curves of equal mean error in direction too. For $V = 1$ the error in W is $= \pm \delta$ or $\pm \frac{180^\circ}{2\pi} \delta$ in degrees.

28. *Error in the position of the singular points.* — The co-ordinates of the singular points are determined by the equations:

$$A = 0 \quad \text{and} \quad B = 0$$

If now the mean error of A and B is $= \pm \delta$, then the equations are:

$$A = \pm \delta \quad \text{and} \quad B = \pm \delta$$

and the position of the singular point cannot be accurately fixed. An examination according to the theory of least squares leads to the conclusion that the singular point in question lies somewhere inside the ellipse:

$$A^2 + B^2 = \delta^2$$

or, inside the curve $V = \delta$, granted that the curve $A = 0$ is not parallel to the curve $B = 0$. If, for instance, the mean error is put $= \pm \frac{1}{2}$, the singular point lies inside the ellipse $A^2 + B^2 = (\frac{1}{2})^2$, or $V = \frac{1}{2}$.

According to *V. Bjerknes* (Kinematik, 1913, p. 46) the numerical value of the velocity has a relative minimum along the singular curves (that is, curves of convergence and divergence). This thesis does not in general hold good. On the present charts, the numerical value of the vector is obviously lowest, where the curvature of the stream lines is most prominent. This fact may be explained as a result of the forming of monthly means, but may also be a more primary character.

A field with one line of divergence is represented by the following equations: $B = mx + ny$, $A = a$, $\frac{dy}{dx} = \frac{B}{A}$. If we put: $\frac{m}{a} = p$, $\frac{n}{a} = q$, we obtain the equation already treated before (§§ 14, 15) viz. $\frac{dy}{dx} = px + qy$, with the integral: $px + qy + p/q = C e^{qx}$. The integral curves are represented on Fig. 7. The common asymptote is the line: $px + qy + p/q = 0$, corresponding to the line of divergence, while the minimum of velocity is along the line: $px + qy = 0$. These two lines only coincide, when $p = 0$. Then the field is symmetrical as to the singular line. Are not the stream lines symmetrically arranged on both sides of the singular line, the line of minimum velocity is situated on the side, where the curvature of the stream lines is greatest. — By a change of sign, a line of convergence is obtained. —

29. *Value of maximal velocity.* — The curves of equal magnitude of velocity are drawn for whole numbers of V, expressed in m/sec. The point of maximum is then inclosed in an ellipse-shaped area, bounded by a curve $V = n$. The value of the maximum is then:

$$V_{max} = n + f$$

where f is some fraction. To get an estimate of the magnitude of this fraction f, we can rule a straight line through the approximate point of maximum; this line intersects the curve $V = n$ in two points, say, A and B, and likewise the curve $V = n - 1$ in two points, C and D. Now let the distance

$$AB = b_0$$

and

$$CD = b_1$$

then (approximately)

$$V_{max} = n + \frac{b_0^2}{b_1^2 - b_0^2}$$

General Description of the Air Transport.

30. Although the picture of the stream lines vary much from month to month, some general traits are common to all seasons, especially in the eastern part of the area. In the western part, the conditions are radically changed from winter to summer; the summer months have much in common, and the winter months are also very like between themselves, in many respects. The spring and autumn months are characterized by instable conditions, as a consequence of the change of the whole system of air transport (see Figs. 26—38).

At about Lat. 10° N. a strongly marked line of convergence runs westward from some point near the Isthmus of Panama. To the south of this line, the *Southern Trades* continue across the equator. The area of this wind system is divided into two parts; the eastern branch turns to the east, the western branch turns westward. The boundary line between the two branches passes through a neutral point near the western limit of the »doldrums« off Panama. The *Northern Trades* have their origin in the *Subtropic Anticyclone* in about 30° N. Lat., 140—150° W. Long., which is present on the charts of all months (with a seeming exception for April). From this centre generally two distinct lines of divergence run towards SE and W. The first runs roughly parallel to the coast of California and Mexico, and ends mostly in the neutral point mentioned above. The west-going line of divergence marks the northern limit of the trades.

So far, the wind systems are rather steady, and the charts may be taken to represent actual wind conditions. Farther north the conditions are very changeable, and the charts represent only the result of the air transport of the various months. But there are some features common to most months here too. Somewhere near the Aleutian Islands, or in the great bight of Alaska, a cyclonic centre generally appears. To the west and north of this point is found an area with northerly winds, coming down the Bering Sea. Another feature, common to most months, is a line of convergence along the coast of British Columbia and Southern Alaska. To the west of this line the wind blows towards the continent, to the east of the line the wind is off-shore. A neutral point is generally situated somewhere near Puget Sound, marking the limit between the southerly winds of British Columbia and the northerly winds of California. This point seems to indicate the most common gate of entrance of the cyclonic disturbances of the North American continent. — A similar line of convergence will certainly be found to exist along the west coast of Norway, if an investigation of that part of the seas is made along the same lines as in this paper.

In the western half of the North Pacific, the change between the winter and summer conditions is absolute.

In the winter months — say, January — the Southern Trades are deflected to the west and SW, and, joining the extension of the Northern Trades, pass on towards the Australian continent as a North East Monsoon. The Indonesian Seas have northerly winds. The air is streaming out of the Asiatic continent towards SE; but while the stream lines of the southern part are deflected towards S and SW, the stream lines of the coast north of Tsugaru Strait continue across the whole northern half of the Pacific, until they are ultimately lost in the Aleutian cyclonic centre. The east wind north of the Trades turn to north and north east, and joins the general west wind; the stream lines end in the line of convergence which runs along the coast of Vancouver Island, &c. — The Bering Sea north wind reaches as far south as to about 45° N. Lat.

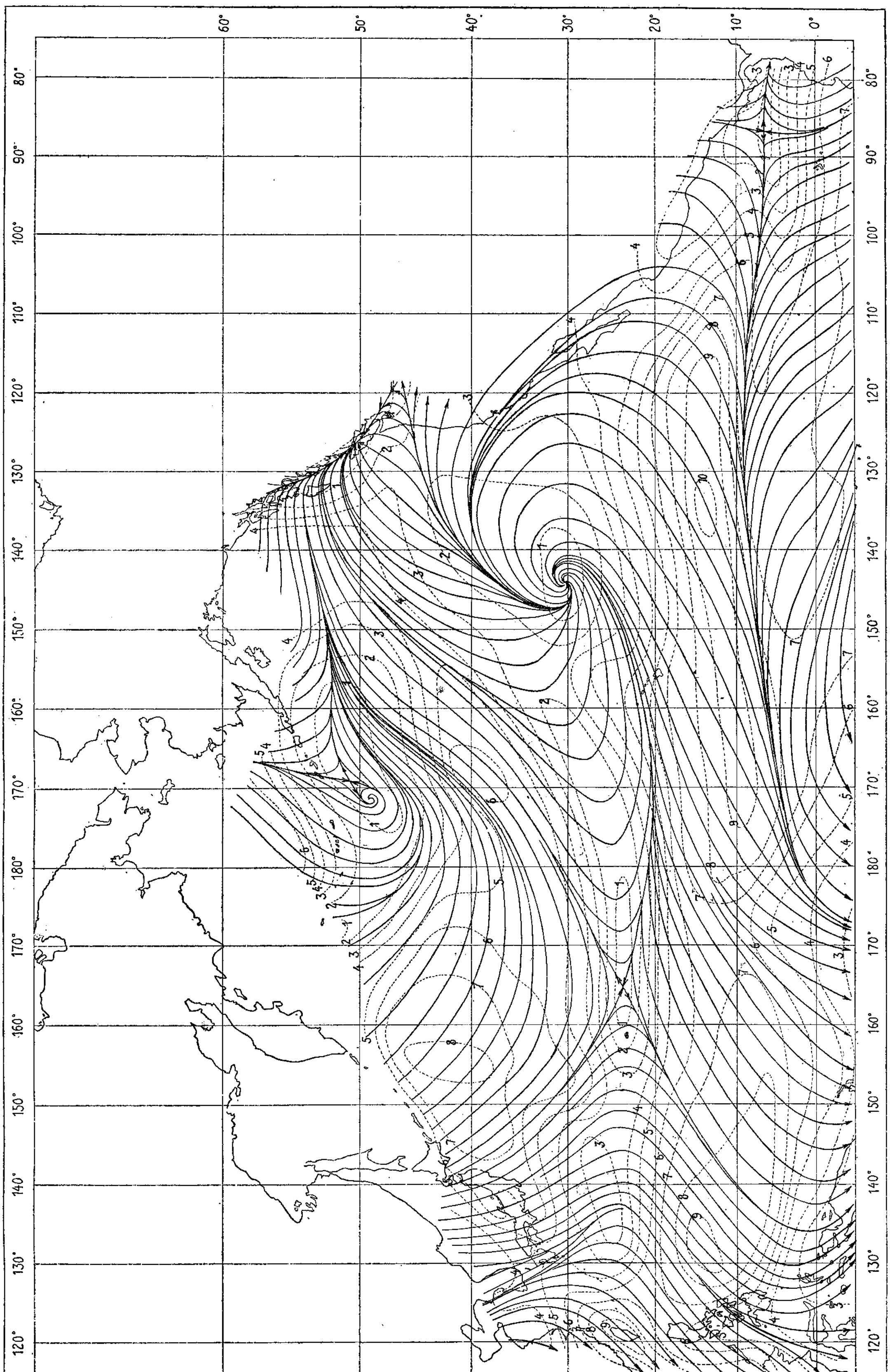

Fig. 26. Mean air transport, January.

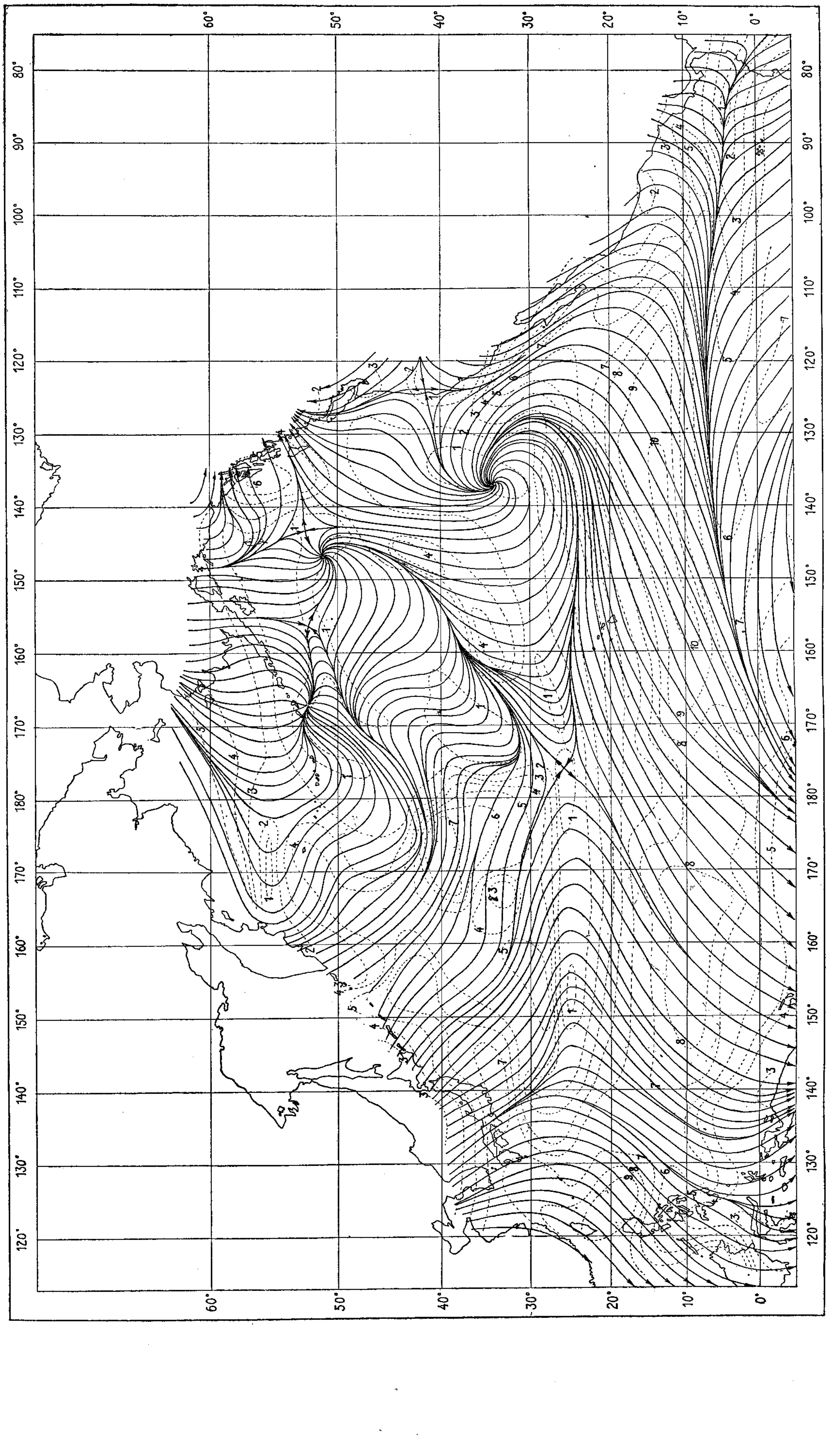

Fig. 27. Mean air transport, February.

The summer conditions (July) are in general as follows. The equatorial line of convergence is displaced northwards, to about 15° N. Lat., and the area of the doldrums is greatly extended westwards, as compared with the winter state of things. The subtropic anticyclone lies farther north in the summer than in the winter, too. Both trade winds continue westwards, turn to NW and N, and unite to form the East Asiatic Summer Monsoon. The air from the polar side of the »Horse Latitudes« north of the trades, turn in the same way, to NW and N, and all coasts of the Northern Pacific have on-shore winds, except parts of California and Central America.

The transitional months are characterized by unruly and unstable conditions, giving rise to all sorts of singularities of the stream lines on the charts.

Description of the Monthly Charts.

31. *January* (Fig. 26). — The equatorial calms occupy an area about 85° W. Long., 7° N. Lat. A line of convergence runs from the Gulf of Panama and westwards to about 180° Long. The subtropical anticyclone lies at 30° N. Lat., 145° W. Long. From here a marked line of divergence runs towards SW, indicating the northern limit of the trades. To the north of this line the wind turns to N and NE and continues towards British Columbia, terminating in a line of convergence which runs along the coast. The wind is on-shore as far south as 35° N. Lat., farther south the wind is off-shore, in Southern California.

The western half of the chart is characterized by the strong Asiatic winter monsoon. The monsoon is split in two parts at a hyperbolic point in 25° N. Lat., 165° E. Long. The southern branch of the monsoon joins the northern Trades and blows towards the Sunda Islands. Both Southern and Northern Trades converge towards the Australian continent. The north branch of the monsoon continues as a west wind across the sea, towards the south coast of Alaska; a north wind is streaming down the Bering Sea, with the result that a cyclone and a neutral point are formed in appr. 170° W. Long., 50° N. Lat.

32. *February* (Fig. 27). — The situation is much like that of January. The equatorial line of convergence begins in the east at the Equator, in about 77° W. Long., and runs in a gentle curve towards 7° N. Lat., in 110° to 140° W. Long. Doldrums, with mean velocities less than 1 m/sec., along the easternmost part of the line of convergence, to about 90° W. Long. Only the northernmost part of the Southern Trades is shown on the chart; towards the west the trades turn to W and SW and join the extension of the Northern Trades, forming the Australian North East Monsoon. The Subtropic Anticyclone is situated in 35° N. Lat., 137° W. Long. A line of divergence runs from here towards the east, and turns strongly to the south and west, thus marking the limit of the Northern Trades. West and south of the neutral point in 25° N. Lat., 175° W. Long., the trades join the East Asiatic Monsoon, and continue towards Australia. The East Asiatic Monsoon is split in two parts at the above mentioned neutral point. One branch turns to the south, and forms part of the Indonesian NE Monsoon. The northern branch is rather irregular, perhaps owing to the relative scarcity of observations. It turns to NE and terminates in some cyclonic points and lines of convergence in about 52° N. Lat.; here northerly winds blow out from the Bering Sea. A point of divergence is being formed at appr. 138° E. Long., 32° N. Lat. (S. Japan). — The area to the west and north of the Subtropic Anticyclone is occupied by southerly winds, which turn to the east and blow against the coasts of British Columbia and Alaska. A line of convergence runs along the Californian coast; a slight displacement of this line to the east will be followed by on-shore winds along this coast.

33. *March* (Fig. 28). — The equatorial line of convergence runs in 6° N. Lat. and turns to SW in 170° W. Long. The doldrums extend to about 87° W. Long. The

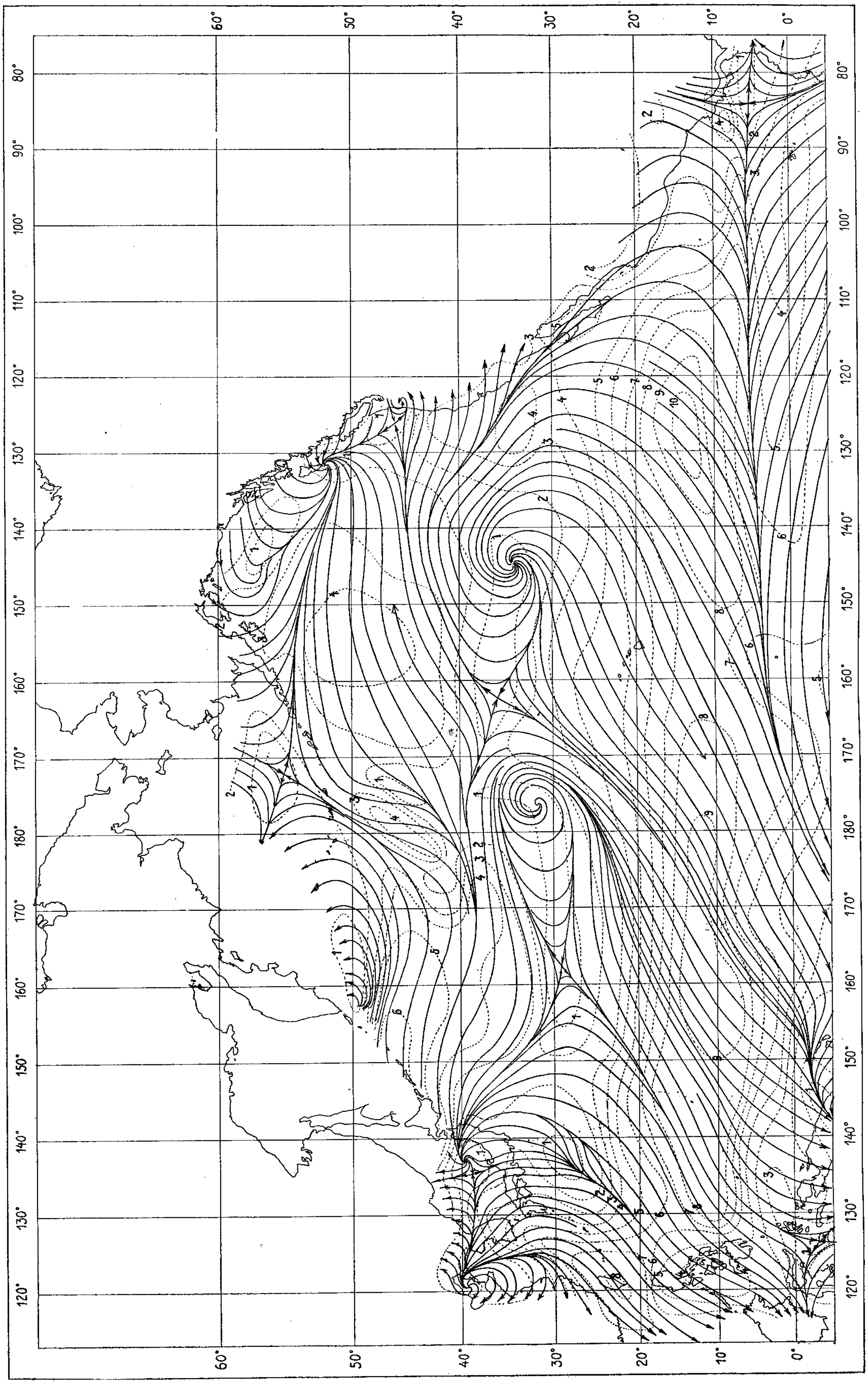

Fig. 28. Mean air transport, March.

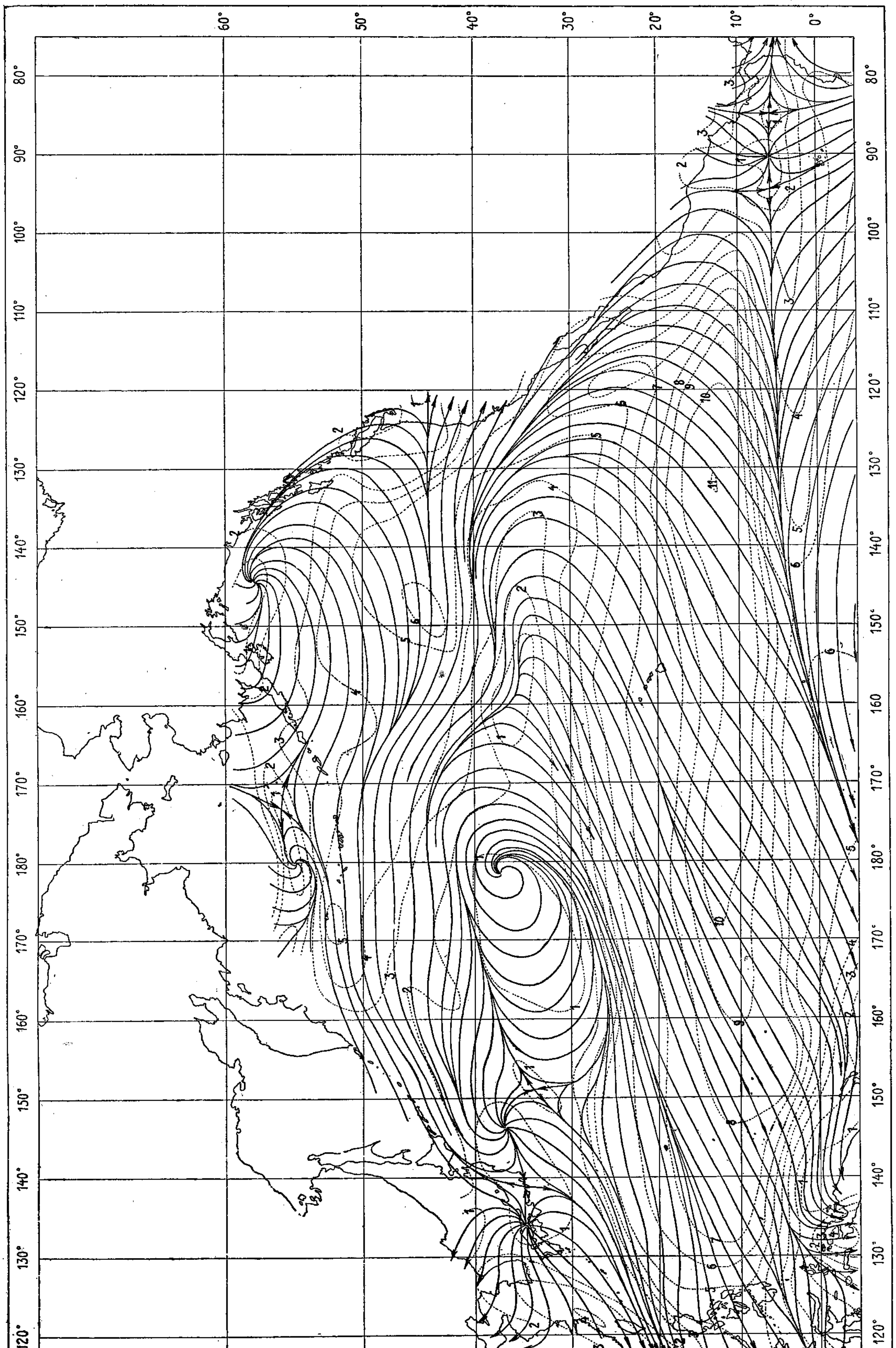

Fig. 29. Mean air transport, April.

Southern Trades are for the most part deflected westwards, and ultimately form part of the Australian NE. Monsoon. East of appr. 90° W. Long., however, the Southern Trades blow towards the north. The Subtropic Anticyclone is situated in 35° N. Lat., 145° W. Long. From here a line of divergence runs to the west, to a neutral point. The western section of the North Trades continue as the Australian Monsoon. The East Asiatic Monsoon is degenerating. An anticyclone is formed in the Gulf of Pechili, and another in the Sea of Japan. From here the monsoon blows towards S and SW; off Celebes, however, the South Monsoon already makes its appearance. The stream lines which start at the coast of Japan south of 40° N. Lat. do not reach farther east than to 160° E. Long.; but the stream lines from the Tsugaru Strait reach as much as 35° farther east, and here join the NE Trades. An eddy is formed in about 175° W. Long. The calms of the Horse Latitudes reach from 155° E. Long. to 140° W. Long., or about 65°. Between Japan and Kamtchatka the wind blows strongly to the east, and turns partly to the north, partly to the east, and joins the north wind coming from the Bering Sea. As usual, a line of convergence runs at some distance parallel to the coast of British Columbia; from Puget Sound and southwards to about 30° N. Lat. the wind is on-shore.

34. *April* (Fig. 29). — The equatorial line of convergence runs in an east westerly direction in about 5° N. Lat. to 140° W. Long., and turns here more to WSW and passes out of the chart at 175° W. Long. — Of the Southern Trades, only the northernmost extension is shown on the chart, the conditions being very much like those of the preceding month. The Subtropic Anticyclone and the neighbouring neutral point have combined to form a divergent »broom«, owing to the great extension of the east branch of the Asiatic Monsoon. The point in question is situated in 155° W. Long., 35° N. Lat. The Northern Trades continue as East winds, from about 160° E. Long. and westwards.

The line of convergence at Celebes has passed southwards again, and out of the margin of the chart. The Asiatic Monsoon has turned along the coast, as far northwards as Vladivostok. Anticyclonic centres now lie across Nippon, and from here the air is streaming out in all directions, but turns to the right both north and south of the centres. The eddy in about 180° is prominent. The calms of the Horse Latitudes reach from Japan to about 150° W. Long. To the north of these, the air drift is from W to E. The limit of the Bering North Wind is pushed northwards, to 55° N. Lat. — A cyclonic centre appears at 142° W. Long., 58° N. Lat., off the south coast of Alaska. The coast of British Columbia has southerly winds, and south of 45° N. Lat. the wind is westerly, to the Mexican border. Here the stream lines run parallel to the coast.

35. *May* (Fig. 30). — The equatorial line of convergence has passed 10° N. Lat. in the eastern section of the chart. Farther west it approaches to the equator, but does not pass out of the chart (5° S. Lat.). The line is, on the contrary, again deflected towards the north, and ultimately passes through the South China Sea. The doldrums reach from Panama to 130° W. Long. — The Southern Trades appear more fully developped on the chart, and only the portion west of 105° W. Long is turned westwards. The Subtropic Anticyclone and the neighbouring neutral point are again separated, and are situated in 145° and 160° W. Long., respectively, in about 35° N. Lat. The northern limit of the trades is indicated by a stream line running from the neutral point and towards SW and W; the area of the trades thus terminates in a wedgelike shape near the Philippine Islands. — A complex singular point appears to the east of Celebes; it approaches the »cusp« in form. — Along the Asiatic coast the wind is on-shore at least as far as the island of Sachalin. An anticyclone is situated at the south cape of Corea, and another broom-like point of divergence is placed in 30° N. Lat., 160° E. Long. The Horse Latitude calms reach from 155° E. to 140° W. Long. The great eddy that made its appearance on the chart of the preceding month, is strongly developped. A part of the air drift to the south of this eddy is deflected to the north, bringing southerly

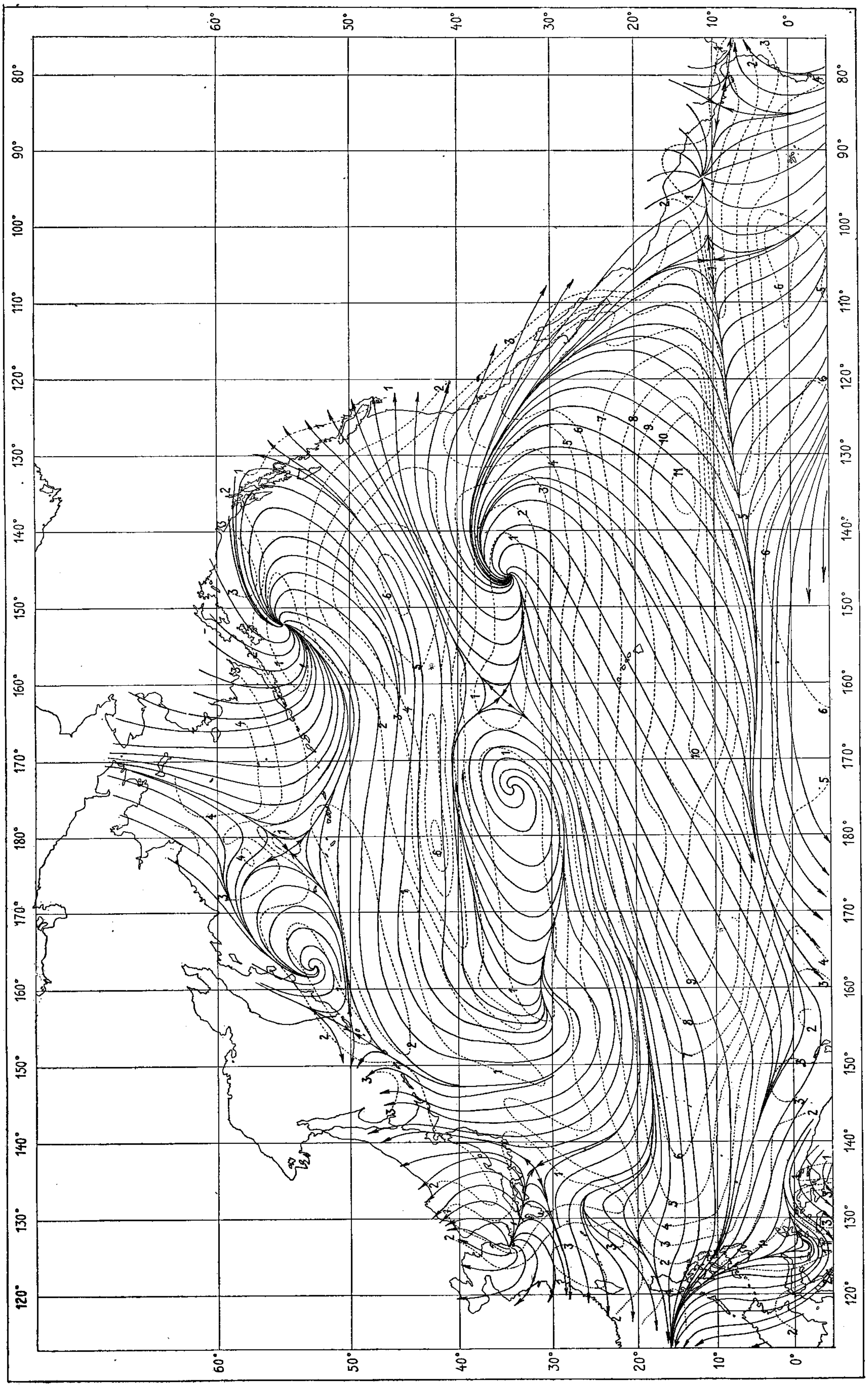

Fig. 30. Mean air transport, May.

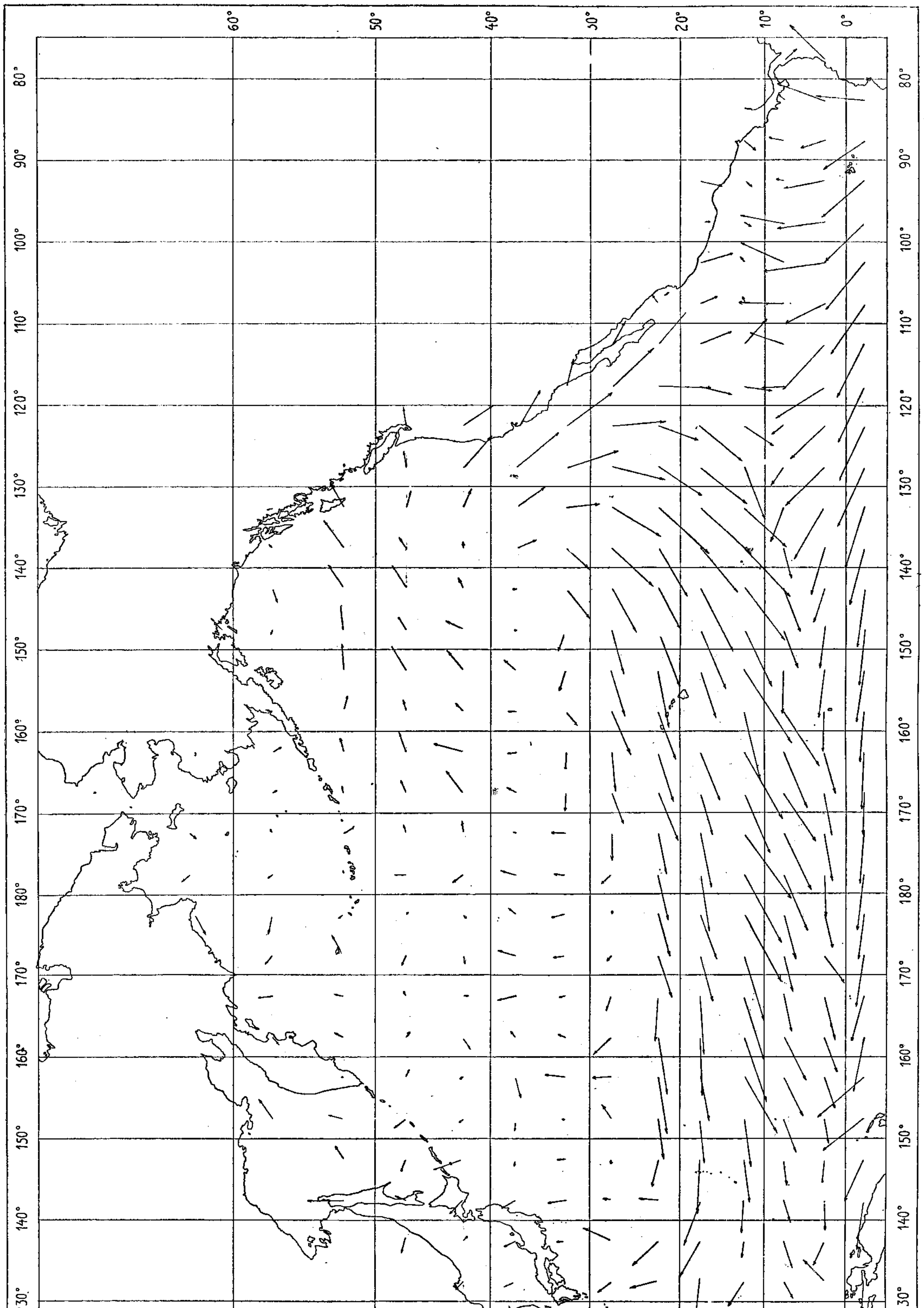

Fig. 31. Mean air transport, June. Each wind rose is reduced to one resultant.

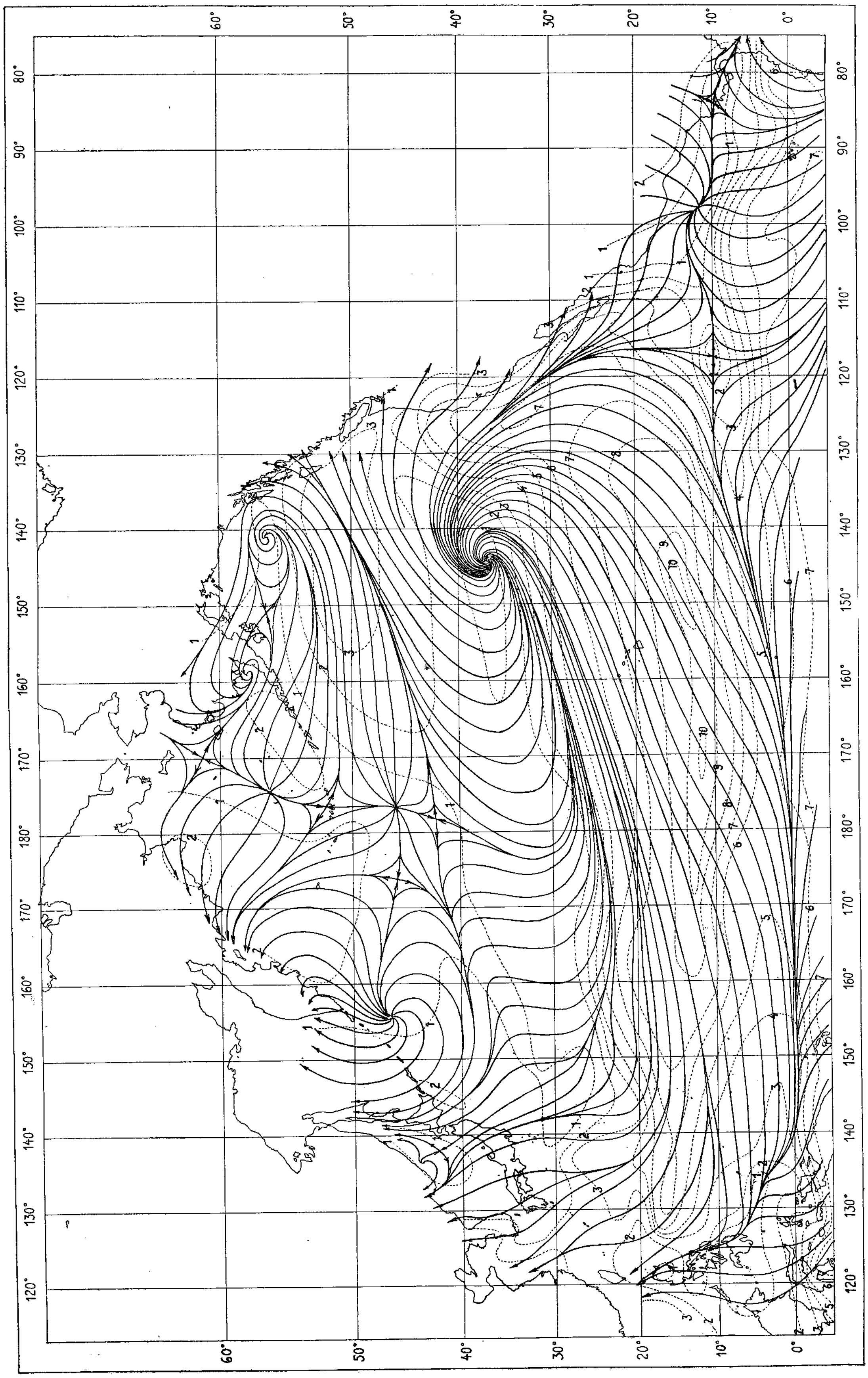

Fig. 32. Mean air transport, June. Stream-lines and curves of equal velocity.

winds to the Curile Islands. — To the north of the Horse Latitudes, the wind is generally westerly. A strong cyclonic centre appears in 55° N. Lat., 150° W. Long., and a branch of the Bering North Wind terminates here. The American coast between 58° and 30° N. Lat. has westerly winds.

36. *June* (Fig. 32). — The equatorial line of convergence runs from Panama and reaches to about 12° N. Lat. at 100° W. Long. Farther west it approaches the equator and runs practically along this from 180° to 140° E. Long. From this point it runs towards NW to Formosa. — The doldrums are greatly extended along the west coast of Mexico, and to the west they reach to about 120° W. Long. — The Southern Trades are strongly developped, and the eastern branch, which does not turn to the west, is the broader one. The neutral point on the equatorial line of convergence, that marks the limit between the two parts of the Southern Trades, is situated in about 118° W. Long. In the Indonesian seas, the wind blows from SE, as a continuation of the trades. — The Subtropic Anticyclone is very conspicuous, and lies at 36° N. Lat., 145° W. Long. Two lines of divergence start from here, one towards the east, turning to SE and S, the other towards the west. The trades continue without interruption into the East Asiatic Summer Monsoon. The great eddy of the spring months, in the western part of the Horse Latitudes, has disappeared. — The anticyclonic points of the Sea of Japan have wandered northwards, to about 45° N. Lat. — In the whole area to the north of the 40th parallel the winds are light and variable, especially along the 180°th meridian. Here not less than five singular points are situated along the same line of divergence. The shape and position of these singular points are of course rather uncertain, as some slight displacement of the curves $A = 0$ or $B = 0$ will make great change in the arrangement. Off the Alaskan coast, two cyclonic centres are found, corresponding to the Aleutian Low. Along the coasts of America, the wind is on-shore as far south as to 40° N. Lat.; farther southwards the wind is more parallel to the coast, from NNW. — The Bering North Wind reaches down to some 60° N. Lat. only. The east coast of Kamtchatka has easterly winds. — The Horse Latitude calms do not seem to exist as a continuous belt.

37. *July* (Fig. 33). — The equatorial line of convergence runs from Panama to 15° N. Lat., 110° W. Long., and from here turns to the west. It does not reach farther south than some 4° N. Lat., and west of 160° E. Long. it turns to NW and N and ultimately passes through Japan. The doldrums are more developped in a westward direction, terminating near a neutral point at 120° W. Long. To the west of this same point, the Southern Trades turn westwards, and west of the 180th meridian they turn to NW and N, and continue as the East Asiatic Summer Monsoon. This monsoon is now fully developped, and the picture of the stream lines is singularly clear and simple, with a single great anticyclone occupying most of the area of the North Pacific. — The Subtropic Anticyclone is situated a little to the north of 40° N. Lat, 150° W. Long. The Northern Trades are limited by a line of divergence running almost due west, in about 20° N. Lat. West of 140° E. Long. the trades are deflected to the north, and form part of the monsoon. — To the north of the above mentioned line of divergence the wind blows towards west, but is strongly deflected northwards along a belt stretching from the Subtropic Anticyclone to the Philippine Islands. North of this belt, the wind is generally southerly and spreads towards the coasts of the continents, especially towards the American shores. The wind is thus on-shore along the whole Asiatic coast and most of the American, too. The line of divergence which runs from the Subtropic Anticyclone towards east and south east, now turns into Southern Mexico. The Bering North Wind does not appear at all on this chart.

38. *August* (Fig. 34). — The equatorial line of convergence starts at Panama as usual, and reaches to about 12° N. Lat. The doldrums are extended towards 122° W. Long.

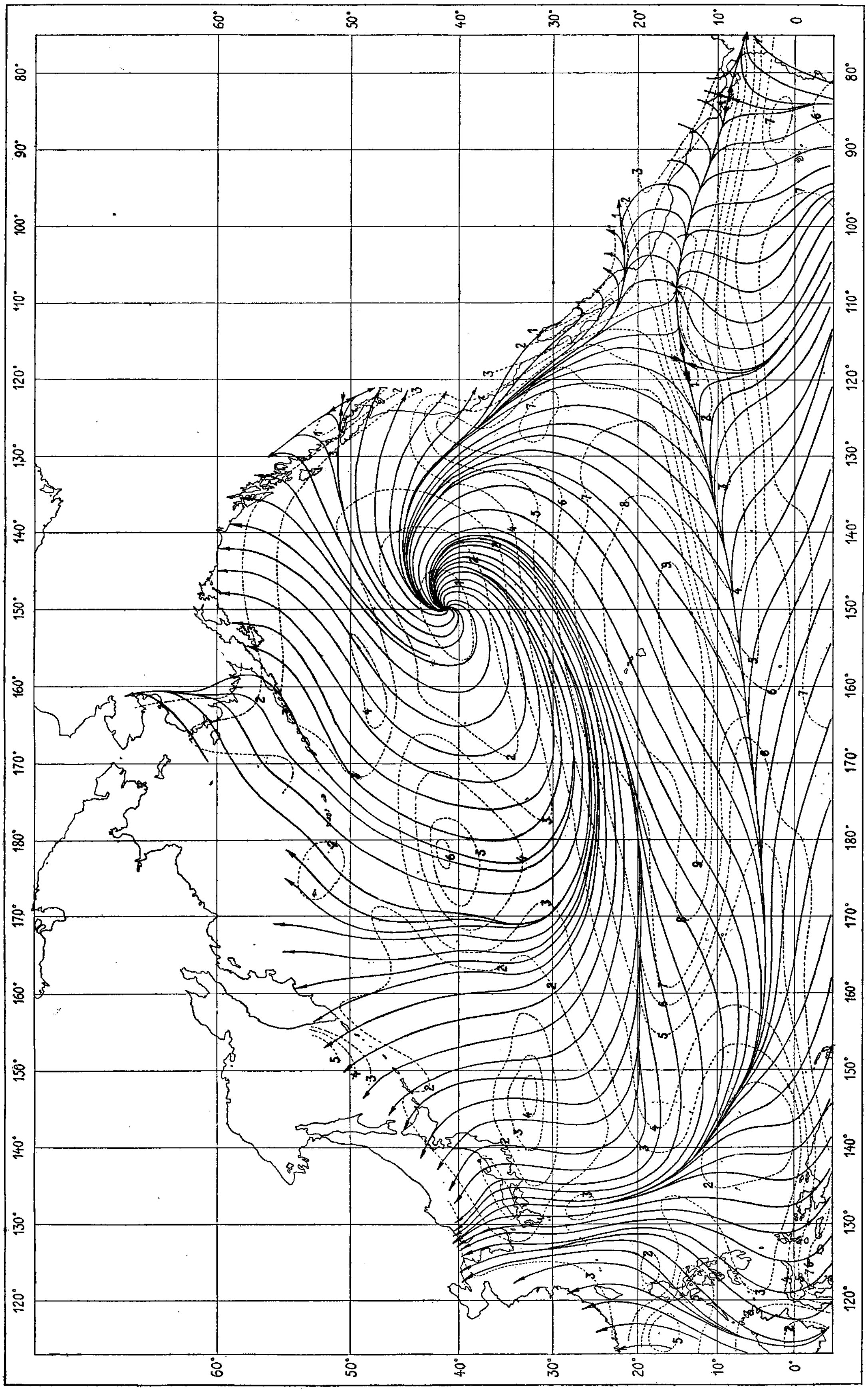

Fig. 33. Mean air transport, July.

The Southern Trades are turned westwards, and then to NW, as usual in the summer months. The Subtropic Anticyclone now lies in about 43° N. Lat., 145° W. Long. The divergence lines marking the boundaries of the Northern Trades are conspicuous, and the western one is split in two parts. West of the 180th meridian the Southern and Northern Trades join in a west drift, which continues towards the Asiatic shores. The Summer Monsoon blows along the whole coast of East Asia, as far north as to the base of the peninsula of Kamchatka. A cyclonic centre appears near the south end of this peninsula, but its existence is doubtful, owing to the scarcity of observations. — North of the Horse Latitudes the same big bend of the stream lines is shown, as on the chart for July, but it is now placed some more to the north. — The whole north-eastern area of the Pacific has south-westerly winds, but along the coast south of Vancouver Island the wind is deflected southwards, and from here and to southern Mexico the wind is NNW. The Bering North Wind again makes its appearance, and reaches down to some 60° N. Lat., and here turns to the east, towards a cyclonic centre to the east of Bering Strait.

39. *September* (Fig. 35). — The equatorial line of convergence starts at Panama, and runs in a westerly direction, but is lost west of the 180th meridian. The doldrums have split into two areas, one forming a belt roughly parallel to the Central American coast, another round the neutral point at 122° W. Long., 11° N. Lat. This same point indicates, as usual, the limit between the two parts of the Southern Trades, one part being deflected eastwards, and the other to the west. The Subtropic Anticyclone lies at 142° W. Long., 38° N. Lat. The line of divergence along the northern limit of the trades, is very prominent, and runs almost due west. West of 170° W. Long., the two trade winds join to form an east wind. There is some disturbance about 145° E. Long., 15° N. Lat., where a cyclonic centre appears, with its conjugated neutral point a little to the west. The stream lines from the Southern Trades terminate here, together with a branch of the Northern Trades, but the bulk of the latter continues to the west and south west, past the south cape of Formosa. In Indonesia, the wind is now blowing from the south, and the stream lines terminate along a line of convergence which runs in an east-west direction through the above mentioned singular points. — An anticyclone has been formed in the Bay of Pechili, and in the China Sea the wind is NNE to NE. A line of convergence runs along the west coasts of Japan and Sachalin. To the west of this line, in the Sea of Japan, the wind comes from the north and west; east of the line, however, the wind is easterly. An anticyclonic centre is situated in the sea off Nippon, in 155° E. Long., 40° N. Lat. The air, that streams out of this centre towards the east and north, meets two other currents, viz. the Bering North Wind, and the SE wind of the north side of the Horse Latitudes. The Bering North Wind reaches some 62 N. Lat., and here a centre of convergence is situated, corresponding to the Aleutian Low. Most of the south coast of Alaska is exposed to southwesterly winds; south of Puget Sound, the wind is more northerly.

40. *October* (Fig. 36). — The equatorial line of convergence passes through Nicaragua and reaches to about 100° W. Long., 15° N. Lat., where it turns a little to the south, and becomes more indistinct. The doldrums extend to about 118° W. Long. The west branch of the Southern Trades continues as an east wind. — The Subtropic Anticyclone lies in 145° W. Long., 35° N. Lat. The west-going line of divergence is very prominent. The Northern Trades turn to the west, and join the extension of the Southern Trades, but the stream lines end in a point of convergence, in about 10° N. Lat., 135° E. Long. In the Indonesian Seas, the wind is southerly up to a line of couvergence, that runs from the above mentioned centre (knot) towards the Sulu Islands. — An anticyclone lies over Peking, and from here the air streams out along curved paths, that form the NE monsoon of the China Sea. The Japanese Islands and the East Siberian coast have

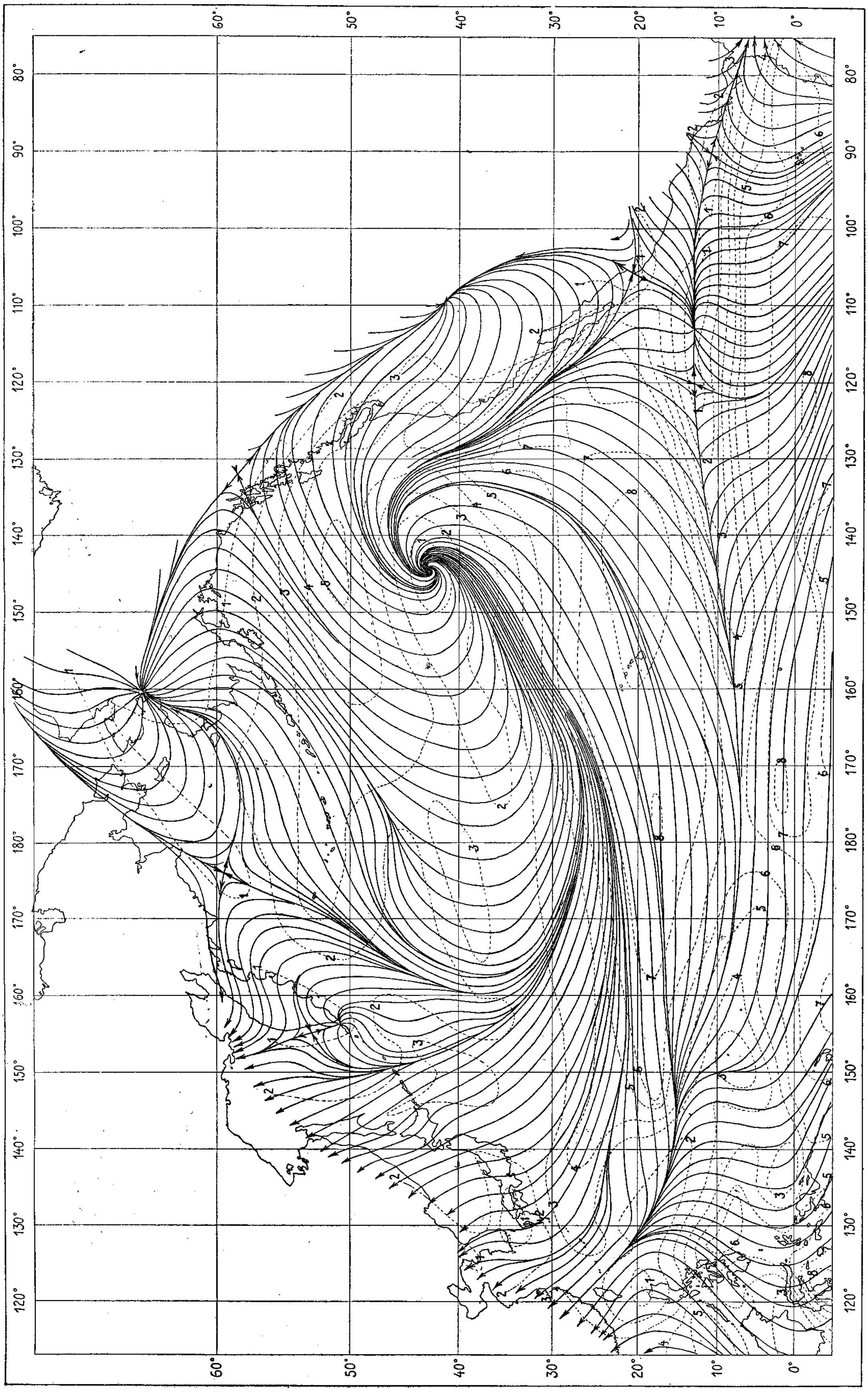

Fig. 34. Mean air transport, August.

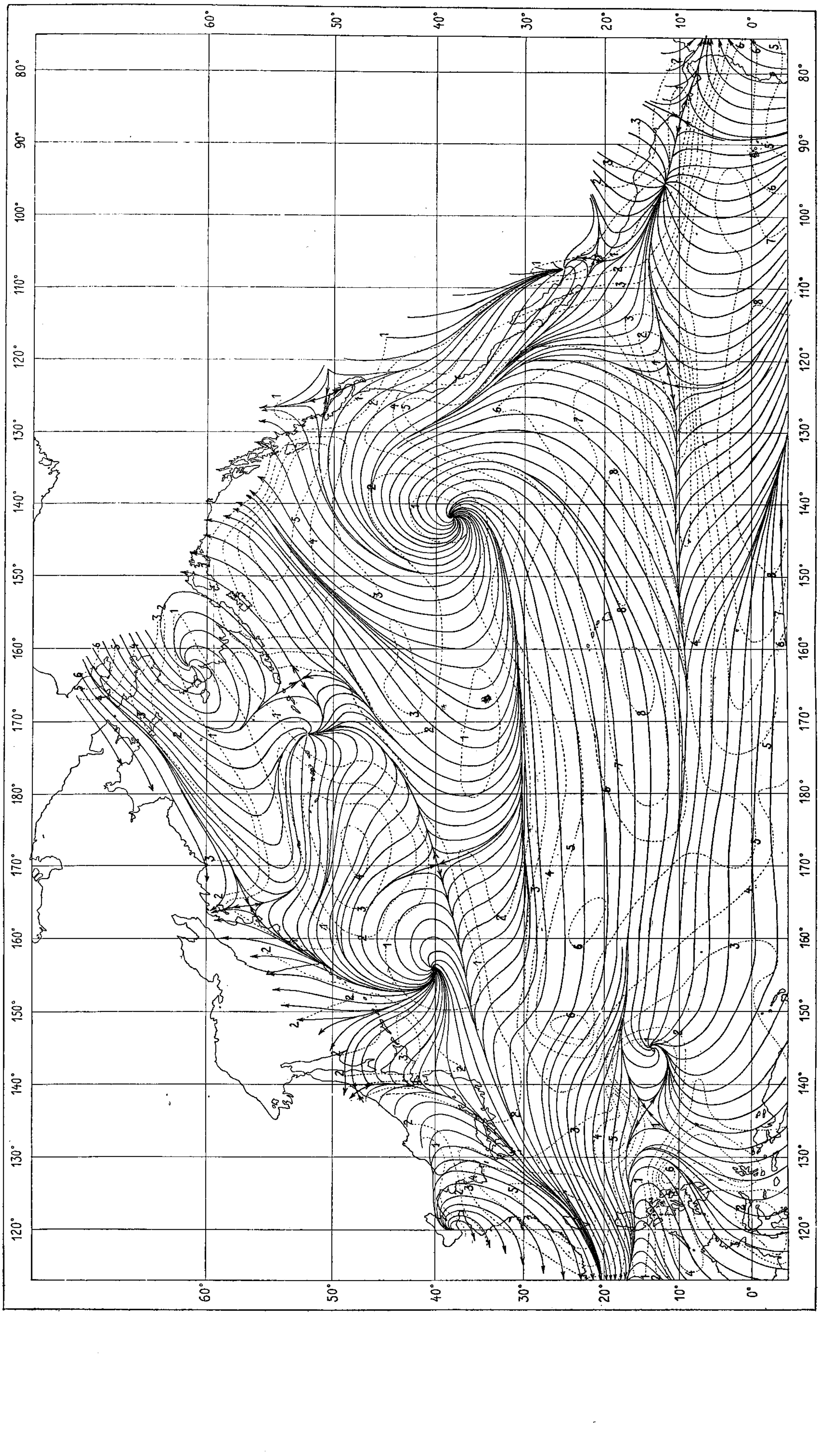

Fig. 35. Mean air transport, September.

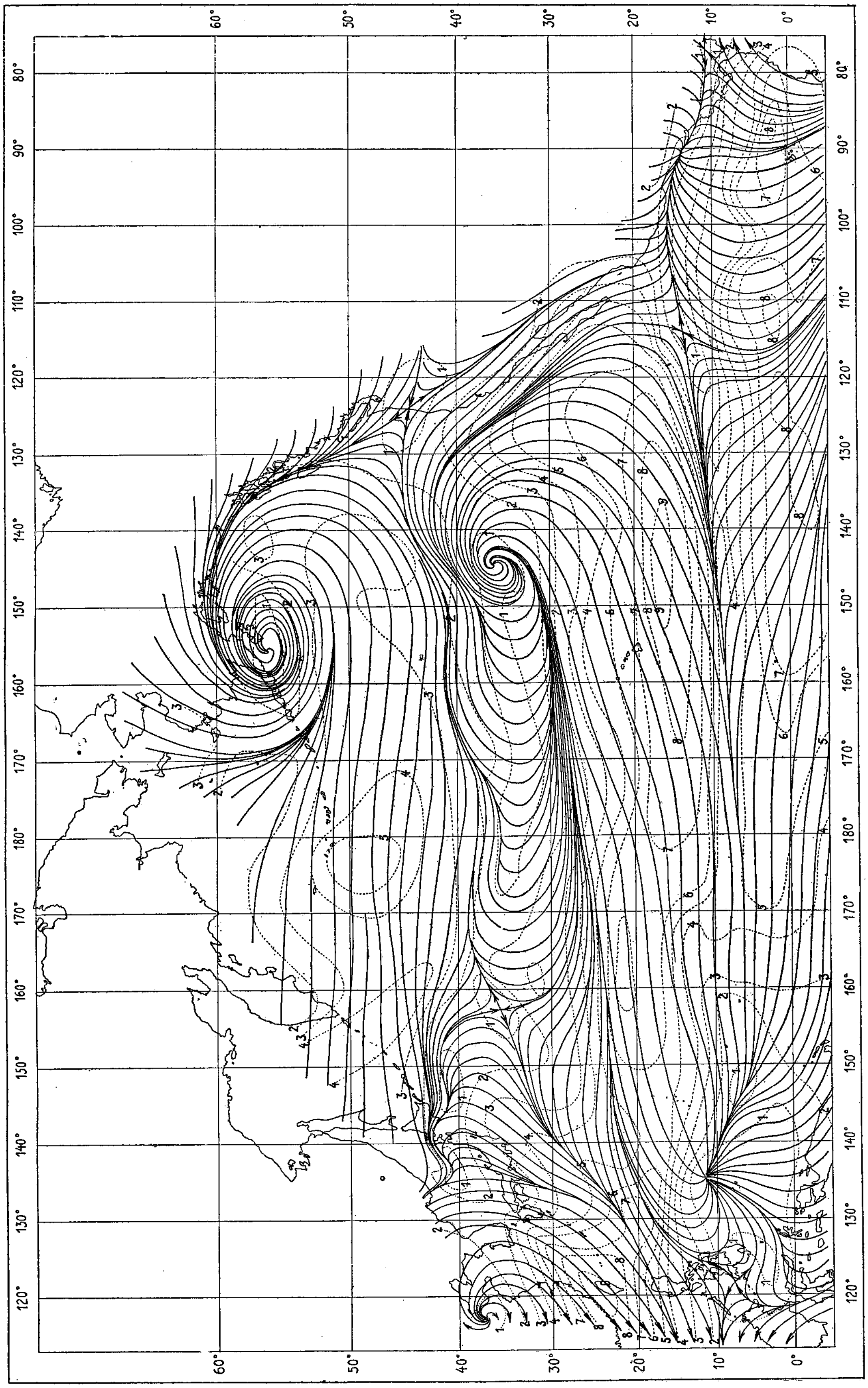

Fig. 36. Mean air transport, October

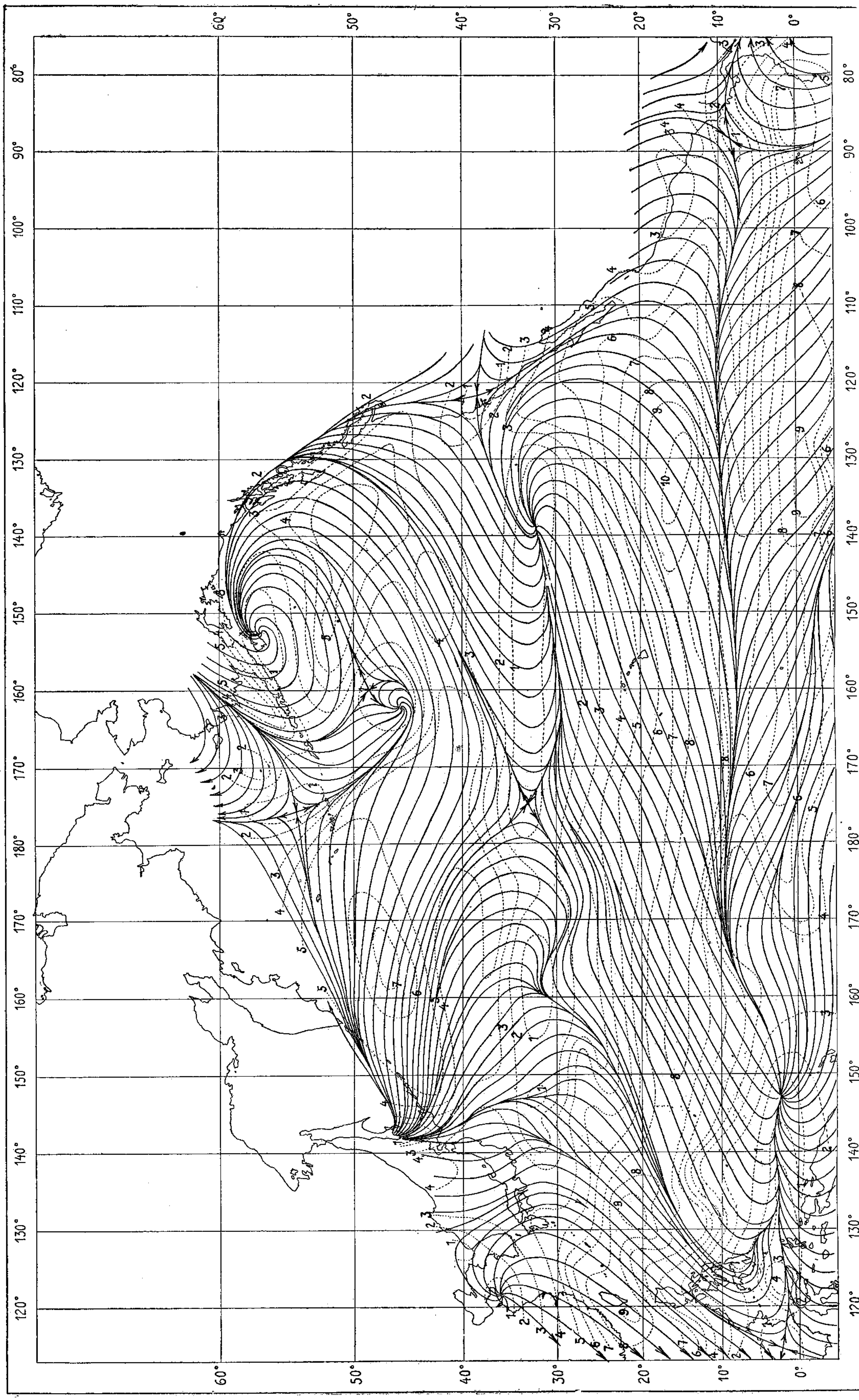

Fig. 37. Mean air transport, November.

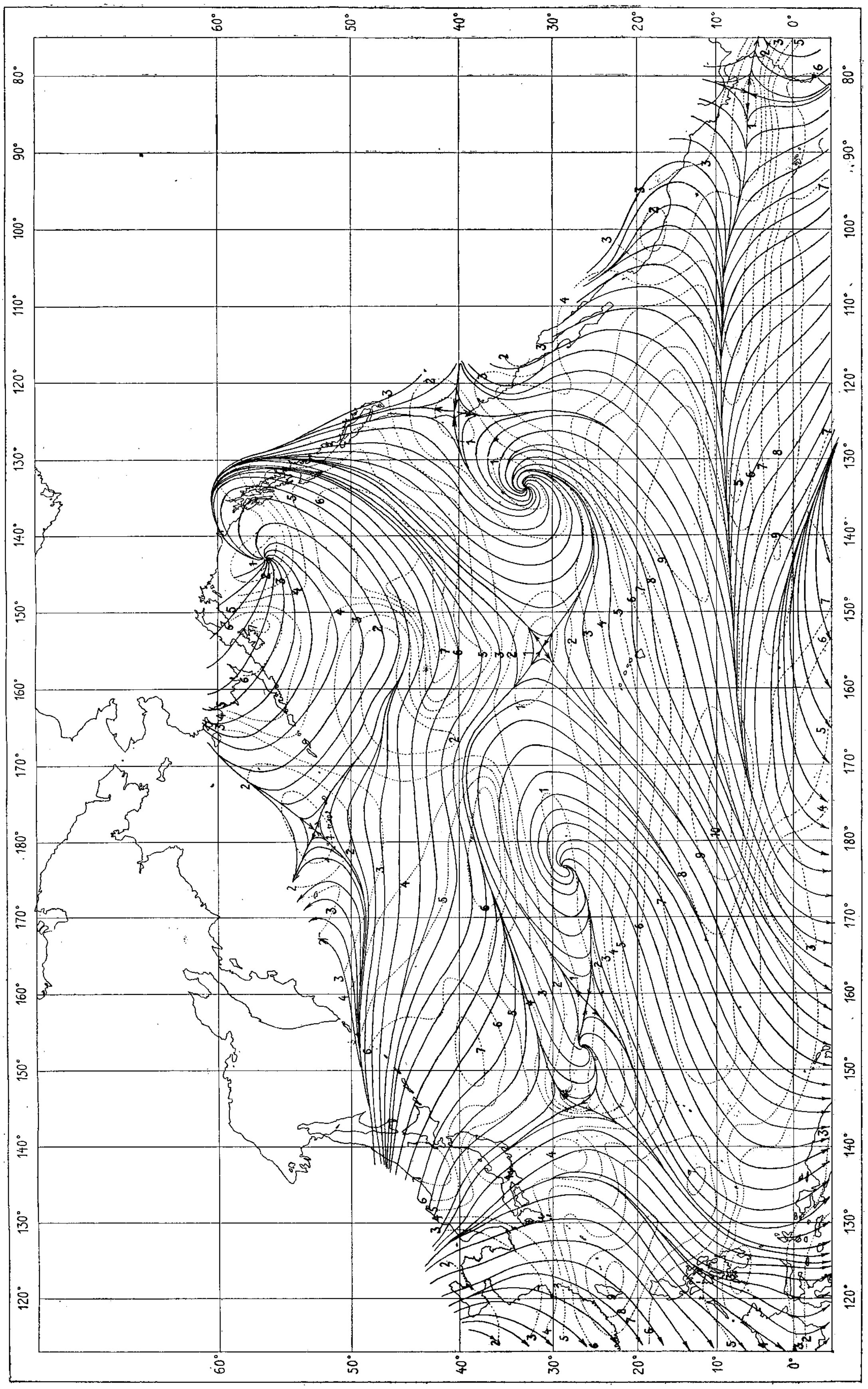

Fig. 38. Mean air transport, December.

westerly winds. East of Nippon this air current meets the east winds of the Subtropic, and is turned southwards (at 156° E. Long., 36° N. Lat.). The stream lines that pass out of the Tsugaru Strait, however, continue eastwards, and join the SW wind north of the Horse Latitudes, forming a broad belt of westerly winds in the north part of the Pacific, ultimately ending in a big cyclone off the SW coast of Alaska. In the formation of this spiral the Bering North Wind also has a considerable share. This wind reaches down to about 52° N. Lat. and turns to the east and north. A line of convergence runs along the coast of S. Alaska and British Columbia, and enters the continent near the mouth of Columbia River. From here and southwards along the coast, the wind is NNW, as usual.

41. *November* (Fig. 37). — The equatorial line of convergence passes through Panama, but does not reach higher latitudes than some 10° N. Towards the west the line ends in a point of convergence. The doldrums are limited to a small area round the neutral point in about 90° W. Long. The eastern branch of the Southern Trades is narrow; the western branch turns westwards, as usual, and terminates in the point of convergence, mentioned above, in 2° N. Lat., 148° E. Long. The Subtropic Anticyclone lies at 140° W. Long., 32° N. Lat. The line of divergence, that marks the northern limit of the Trades, runs towards the west to a neutral point, in 175° W. Long., 23° N. Lat. Here the Trades join the deflected monsoon, and continue towards SW, ending against a line of convergence, which runs E—W a little north of equator. The Indonesian Seas have south winds, which also terminate in the same line of convergence. — An anticyclonic centre is placed off Shantung, and from here the wind blows towards S and E, turning towards SSW over the China Sea. The Sea of Japan has northerly winds, as far as to the south cape of Sachalin. The monsoon wind reaches as far east as to the neutral point in about 175° W. Long. To the north of this point, the stream lines of the monsoon continue as westerly winds, which turn northwards off the Alaskan coast, and end in a cyclonic centre at Kadiag. — The Bering North Wind reaches as far south as to 45° N. Lat. — The west wind to the north of the Horse Latitudes turn northwards off the American coast, from San Francisco and north. In Southern California the wind runs parallel to the coast, from NNW.

42. *December* (Fig. 38). — The equatorial line of convergence starts south of Panama, and approaches 10° N. Lat. At the 180th meridian it turns to SW and S. The doldrums are limited to a small area, reaching only to appr. 87° W. Long. The east branch of the Southern Trades forms a narrow belt along the coast. The west branch turns to the west and south west, in about 150° W. Long. The Subtropic Anticyclone is situated at 134° W. Long., 33° N. Lat. The west going line of divergence is short and curved, and ends in the neutral point at 155° W. Long., 30° N. Lat. The united trades continue as the Australian NE Monsoon. A secondary anticyclonic centre is placed in 177° E. Long., 28° N. Lat. The stream lines that diverge from this point turn all ultimately to SW, and form part of the Indonesian North Monsoon. The East Asiatic Monsoon blows outwards from the continent. One part turns to the south and joins the Indonesian Monsoon; another part, north of Tokio, continues as a west wind across the Ocean, and ends in the usual cyclonic centre off the south coast of Alaska. The Bering North Wind turns to the east and reaches as far south as 45° N. Lat.; farther on it turns to NE and N, and forms part of the Alaskan Cyclone. — The west wind of the north side of the Horse Latitudes turns to SE, S and SW, and forms part of the trades. East of the neutral point in about 155° W. Long. the wind turns to NE, and blows from the south along the coast of British Columbia and Alaska. The coast of California has, however, northerly winds.

Wind and Pressure.

43. As the wind is caused by differences in barometric pressure, the picture of the stream lines must be expected to be in close agreement with the field of isobars. The Pilot Charts have small inset charts, which show the mean barometric pressure for each month. The correspondence is found to be very good, except at some places, where discrepancies occur, but that is in areas of light and variable winds, and the deviations from the rule may partly depend upon some deficiency of materials or methods employed. — It has to be remarked, however, that the pressure gradient is connected with the acceleration of the velocity by mathematical laws — namely, the hydrodynamical equations; and in these also enters the frictional force, which depends much upon the movement of the upper layers of the air. And, as the mean of the velocities does not correspond to the mean of the accelerations, the agreement between the mean velocity and the mean pressure gradient cannot be expected to be perfect. The deviations will be most prominent at places, where the resulting mean of the velocities is very near zero. Moreover, the direction of the velocity is difficult to determine with exactness in these areas. — In general, the wind blows from the highs towards the lows, and areas of divergence correspond to areas of high pressure, convergences to areas of low pressure. The equatorial line of convergence depends upon the belt of low pressure in these regions, and the much discussed »Permanent Ocean Highs« give rise to equally permanent anticyclonic centres. The »Aleutian Low« is generally present on the isobaric charts, and so is the corresponding area of strong convergence, on the wind charts. One ore two cyclonic centres will appear in these parts in all months, except June and July. As a matter of fact, the cyclones wander towards the east, one following the other, in an endless succession, but the net result, as obtained by our method, is represented by one or two apparently stationary cyclones near the centre of the »low«. The cold heavy air from the Polar Regions streams out through the Bering Strait, in all months except July. This north wind is strongest in the winter months. It is caused by the constant arctic »high«.

The distribution of the barometric pressure in the eastern parts of the North Pacific does not change very much from one season to another, and is moreover much like the conditions at corresponding latitudes on the eastern parts of the South Pacific, as well as the North and South Atlantic. The whole system is translated somewhat to the north in summer, to the south in winter months. — It is commonly accepted as a matter of fact, that the air circulation on a globe entirely covered with water — or any other uniform stuff — would show the same characteristics: a line of convergence along the (thermal) equator, belts of trade winds rising in belts of high pressure, at about 30° Lat., and outside these, southwesterly winds, up to some 50—60°. Here the encounter with polar winds would cause irregular, turbulent motions, appearing as a series of cyclones, passing eastwards in a continuous succession. This may be, but it is more difficult to find a clear and comprehensible explanation of this arrangement. The equatorial low, and the polar highs, are easily understood, but the permanent highs at some 30° N. and S. Latitude, are still somewhat mysterious[1]. The existence of the great continents makes a profound change in this so-called »planetary« circulation. As the general drift of the air is westerly in the temperate zones, the influence of the continents is most prominent on their eastern sides, while the oceanic conditions are prevailing on the western shores. The most conspicuous feature of the East Asiatic Summer Monsoon is the total disappearance of the equatorial low, or, more correctly, its displacement into the inner parts of the continent. This fact is easily explained, as well as the reverse conditions of the

[1] See, however, *V. Bjerknes*, On the dynamics of the circular vortex. etc. Geofysiske Publ. Vol. II. No. 4, 1922.

winter circulation. It is interesting to note the progressive decay of the winter monsoon, that begins in the south, and proceeds northwards along the coast of China and Japan. The monsoon, that is still blowing over the sea, is by degrees cut off from the inland source by areas of convergence. These are obviously due to the circumstance, that the low lying coast lands are more easily and early heated in the spring months, than the interior highlands on the one side, and the sea off shore on the other hand. In March, the monsoon begins to shift in the Gulf of Pe-chi-li, and in April, the change has proceeded as far to the north as to Vladivostok. In the next month, the wind has changed at least as far northwards as the island of Sakhalin. — The summer monsoon now blows regularly along the whole coast of East Asia for two months, but in September, the wind is again off-shore along the coast near Vladivostok. — In the autumn months, the interior plateaus and mountains will be cooled at a swifter rate than the coastal plains, and the monsoon is cut off by an area of convergence in China. —

The continent of Australia, which is outside the margin of the charts, makes its influence felt on the course of the stream lines in the western equatorial seas; in January, for instance, the hot interior acts like a pump, and the Trades are drawn towards this land. In July, on the contrary, Australia acts as an area of divergence, and the air is streaming out from the continent. This is in agreement with the temperature changes. —

46. *Correlation between pressure gradient and wind in the Northern Trades.* — The monthly means of pressure and temperature are shown on inset charts on the pilot charts. In the area of the Trades the conditions may be assumed to be rather uniform, and we have thus a means to calculate the acceleration of the pressure gradient for each month. In order to compare the gradient with the force of the wind, we will consider the point where the trades attain their maximal velocity, which is tolerably stationary, throughout the whole year. The acceleration of the velocity can be neglected, as the tangential projection is $= 0$, and the curvature of the stream lines is so slight, that the normal projection becomes insignificant too. (Except for March.) Assuming that balance exists between the three vectors: acceleration of gradient $= \alpha G$, deflective acceleration due to the rotation of the earth, $2\,\omega\,V \sin\varphi = L$, and a vector αR representing the friction, we could find the relation between gradient and wind for each month. — By means of some formulas developped by *H. U. Sverdrup*[1] and the author[2] a measure of the correlation between the two vectors is obtained. Let one vector be s, and the other t, and let ψ denote the angle between the two vectors. Further:

$$\Sigma s^2 = S \qquad \Sigma t^2 = T \qquad \Sigma s\,t \cos\psi = P \qquad \Sigma s\,t \sin\psi = Q$$

then the factor of correlation r is determined by:

$$r^2 = \frac{P^2 + Q^2}{S\,T}$$

and the ratio

$$\frac{s}{t} = \sqrt{\frac{S}{T}}$$

By putting $s = V$, velocity of the wind, and $t = \alpha\,G$, acceleration of gradient, the following values are obtained (see Table p. 52):

[1] *H. U. Sverdrup,* Über die Korrelation zwischen Vektoren. Meteorol. Zeitschr. Heft 8/9, p. 285, 1917.
[2] *W. Werenskiold,* Bemerkungen über Korrelation. Ibid. Heft 7, p. 194, 1920.

Coefficient of correlation $r = 0.983$

Mean ratio $\frac{\alpha G}{V} = 4.41 \cdot 10^{-5}$

Mean angle $\psi_m = 62° \; 10'$

The latitude is assumed to be 15°. Then $\frac{L}{V} = 3.78 \cdot 10^{-5}$ and $\frac{\alpha R}{V} = 2.06 \cdot 10^{-5}$. The mean angle between the inverse direction of velocity and the vector of friction is 3°, to the right. These values are obtained from a triangle, $A\,B\,C$, where $A\,B = \frac{L}{V}$, $B\,C = \frac{\alpha R}{V}$, $C\,A = \frac{\alpha G}{V}$, and the angle $A = 90° - \psi_m$. Equilibrum is supposed to exist between the mean values of the vectors, as determined by the relations above.

For the sake of comparison, an example from *Mohn's* Meteorology may be cited:[1]

He starts from the following values: latitude 19°.3, temperature 21°, pressure 760 mm., gradient 0.61 mm. pr. meridian degree, mean force of wind, 2.8. (land scale.) In the meter-ton-sec. system these values are:

$$\alpha G = 5.15 \cdot 10^{-4}$$

$$V = 11.5$$

and accordingly: $$\frac{\alpha G}{V} = 4.48 \cdot 10^{-5}$$

The ratio between acceleration of friction and velocity is:

$$\frac{\alpha R}{V} = 2.24 \cdot 10^{-5}$$

and the vector of friction points backwards and to the right, about 15° 40′ to the (inverse) direction of velocity.

Mohn does not take this course; he starts with the assumption that the vector of friction is parallel to the velocity, and computes the magnitude of the velocity from the other data.

These results agree well with those found from the Pacific. But compared with values obtained in other parts of the world, the friction seems to be too little, in comparison with the wind velocity. The ratio R/V is of course dependent on the condition of the surface, but on the high sea the friction might be assumed to be like everywhere. According to *Sverdrup*[2] the ratio between the same vector of friction and the velocity is $= 7.8 \cdot 10^{-5}$, on the North Atlantic, in appr. 45° N. Lat. This value is more than three times as great as that found above, for the Trades. But as a matter of fact, the expression: $\frac{\alpha R}{2\,\omega\,V \sin\varphi}$ is roughly constant:

N. Pacific,	15°	0.55
N. Atlantic,	19.°3	0.48
N. Atlantic,	45°	0.75

or in mean about $$\frac{\alpha R}{L} = 0.6.$$

[1] *H. Mohn*, Meteorologi. Kristiania 1903. § 350, p. 250.

[2] *H. U. Sverdrup*. Druckgradient, Wind und Reibung an der Erdoberfläche. Annalen der Hydrographie &c. 1916, Heft VIII, p. 413.

The material is not, however, in any way sufficient for a more detailed discussion of the relation of friction to velocity etc.; the chief difficulty is, that the velocity, that enters into the equations of the »surface conditions«, is not determined at all. The wind velocity is measured at a height of several metres above the surface of the sea, and the values obtained in this way are much greater than the values of the actual surface velocities. Furthermore, the wind is much stronger at the crest of the waves than in the depressions; here even eddies may appear, as every yachtsman knows from experience. It is then well-nigh impossible to determine the velocity at the surface.

We shall not therefore enter upon any discussion of the theory of friction, but only refer to the papers by *Hesselberg and Sverdrup*, and by *G. I. Taylor*[1].

Tables for computation of correlation between wind and pressure, Northern Trades.

	Temp. F	Pressure Inches	Sp. Vol.	Grad. 10^{-7}	α G 10^{-4}	Velocity m	Deflection angle
January . .	72	30	840	4.7	3.95	10.4	73
February . .	72	29.95	840	5.9	4.95	10.2	73
March . . .	77	29.9	850	5.7	4.85	10.3	46
April. . . .	72	30	845	5.9	4.98	11.0	56
May	75	29.95	845	5.6	4.73	11.2	54
June. . . .	77	30	850	6.1	5.18	10.3	60
July	77	30	850	5.1	4.33	9.2	62
August. . .	78	29.95	855	4.8	4.10	8.8	59
September .	77	29.95	850	3.9	3.32	8.1	63
October . .	77	29.95	850	4.0	3.40	9.2	69
November .	76	29.97	850	5.3	4.50	10.2	68
December .	74	30	845	4.4	3.72	9.8	70

$x = \alpha G . 10^4$ y = velocity m/sec. ψ = angle of deflection between wind and gradient:

	x	y	ψ	x^2	y^2	xy	xycosψ	xysinψ
1	3.95	10.4	73	15.60	108.16	41.08	12.0	39.2
2	4.95	10.2	73	24.50	104.04	50.49	14.8	48.2
3	4.85	10.3	46	23.52	106.09	49.96	34.6	35.8
4	4.98	11.0	56	24.80	121.00	54.78	30.5	45.2
5	4.73	11.2	54	22.37	125.44	52.98	31.0	42.8
6	5.18	10.3	60	26.83	106.09	53.35	26.6	46.1
7	4.33	9.2	62	18.75	84.64	39.84	18.7	35.1
8	4.10	8.8	59	16.81	77.44	36.08	18.5	30.8
9	3.32	8.1	63	11.02	65.61	26.89	12.2	23.9
10	3.40	9.2	69	11.56	84.64	31.28	11.2	29.2
11	4.50	10.2	68	20.25	104.04	45.90	17.2	42.5
12	3.72	9.8	70	13.84	96.04	36.46	12.5	34.3
				229.85 S	1183.23 T		239.8 P	453.1 Q

Co-efficient of correlation: $r = \sqrt{\frac{P^2 + Q^2}{S\,T}} = 0.983.$

[1] *G. I. Taylor*, Eddy Motion in the Atmosphere. Phil. Trans. Roy. Soc. London. Ser. A, Vol 215, pp. 1—26. 1915. — Skin Friction of the Wind on the Earth's Surface. Proc. Roy. Soc. London, Ser. A, Vol. XCII, pp. 196—199. (No. A 637, Jan. 1, 1916). — Phenomena Connected with Turbulence in the Lower Atmosphere. Proc. Roy. Soc. London, Vol. XCIV, pp. 137—155. (No. A 658, Jan. 1, 1918.)

Wind and Precipitation.

44. *Precipitation.* — As a general rule, the on-shore winds are accompanied by precipitation, while the off-shore winds are dry. Also, polar winds are dry, and winds running from lower latitudes into higher, carry much moisture. In regions with permanent or strictly periodical winds the distribution of rainfall ought to be in close accordance with the character of the mean air transport, as presented on the wind charts. The equatorial areas of convergence — the doldrums — have abundant rainfall, and the belts of high pressure, only a scanty precipitation. The region of the Trade Winds is also dry. — The seasonal shifting of these zones in the eastern parts of the Pacific is accompanied by a corresponding migration of the wet and dry belts. The pacific coast of the State of Columbia has, however, rain at all seasons, or, more correctly, two rainy seasons, one from March to July, and one from October to December. Farther northwards, along the coast from Panama to Guatemala, about 90 % of the annual precipitation falls during the 6 months from May to October. — Near the Mexican northern frontier there is practically no rainfall throughout the whole year, and on the north side of this dry region, the winter is the rainy season. — This type of climate prevails at least as far north as to Vancouver Island. The coast of British Columbia and Southern Alaska have much rain during the whole year, but most in the winter; this is obviously due to the greater frequency of cyclonic storms in the cold season. — The shores of the Bering Sea have off-shore winds in the winter, but southern winds in the summer, and correspondingly, the summer is also the wetter season. The climate of East Asia is totally dependent upon the monsoon. The two sides of one little but mountainous island may have rainy season at opposite seasons. As a rule, the coast of the mainland and the east and south coasts of the islands have great rainfall in the summer, but very slight precipitation in the winter. The west- and north-facing parts of the islands get their rainfall — or snow — in the winter. This is very conspicuous in Japan and on the island of Formosa. The western part of the island of Luzon is wet in summer, dry in winter; to the east of the divide, the conditions are opposite. — A rather curious fact is the diminu-

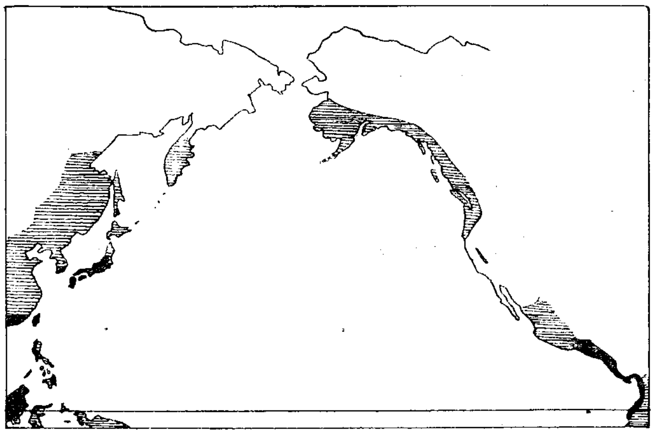

Fig. 39. Rainfall in summer, May—October.

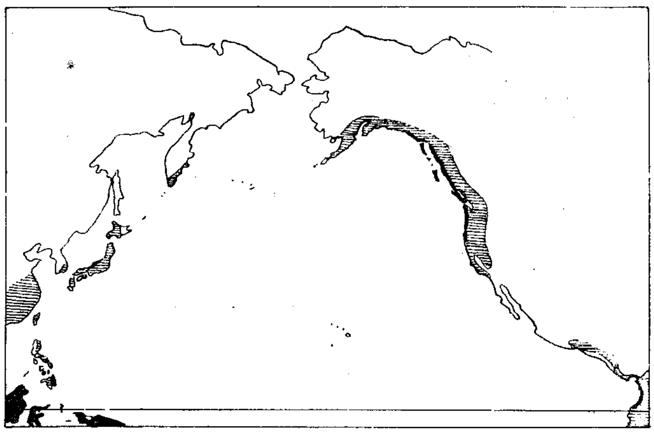

Fig. 40. Rainfall in winter, November—April.
Shaded area: above 250 m.m. Black: above 1000 m.m.

tion of rainfall along the south coast of Borneo and adjacent islands, in the (Southern) winter months, caused by the dry wind from the arid Australian continent. — It is impossible to enter upon the vast subject, the climatology of the Pacific; for particulars and references, see *Hann's* Klimatologie [1] and *Köppen's* charts of the Pacific [2]. A remarkable fact is the great rainfall on the Marshall Islands, especially the low island Jaluit, compared with the arid climate of the so-called Guano Islands, which are situated near equator, at 155° to 160° W. Long.

As to the distribution of rainfall over the sea, we cannot infer much from the wind conditions as represented on the charts. As a rule, warm air blowing into cool areas will bring precipitation, while cold winds entering hot regions, are dry. But much the greater part of the rainfall is caused by ascending masses of air, and in order to get an idea of these conditions, an investigation ought to be carried out as to the distribution of the vertical component of velocity, near the sea level. As a matter of fact, the manuscript charts of the south and west components of the mean velocity of the wind furnish the materials sufficient for such an investigation. It is possible, by methods set forth by *Bjerknes,* to carry out the graphical differentiations necessary for the formation of the (two-dimensional) divergence of the velocity, and by means of the equation of continuity, the vertical component of velocity can be found, at some arbitrary height, not too great. (e. g., 100 m.). This work must be postponed to some later date. —

45. *Climatic changes.* — The question of climatic changes has always been discussed, and in the later years renewed interest has been taken in these problems, as the material provided by direct measurement of the solar energy has been brought into the discussion.[3] A short summary of the most important facts is rendered by *H. U. Sverdrup* [4]. He compares the system of atmospheric circulation with a caloric machine, that runs faster, when the energy supply is greater than usual, and more slowly, when the supply is diminished. The effect is that the great wind systems are strengthened in periods of increased solar activity, and in this way again the climate is influenced in accordance with the direction of the mean air transport. — The area occupied by the Trades will thus have a tendency to be cooled at times of greater energy supply from the sun. — It seems likely, that the transformation of heat into kinetic energy is to a great extent performed in the upper parts of the atmosphere, and the chief result of an augmented solar activity would be to increase the so called planetary circulation, without affecting the monsoon winds to the same degree. This would involve a shifting of wind divides, lines of convergence and divergence, displacement of centres and so on. In this way it would be possible to account for the remarkable fact, that the temperature of some regions change in a regular way together with the sun-spots for some time, and then suddenly begins to vary in the opposite way.[5] For instance, the temperature may have a maximum corresponding to a maximum of sun-spots, for several periods, but then the character of the coincidence changes, and now a maximum of temperature corresponds to a minimum of sun-spots. This must be due to a shifting of the prevalent wind system of the region in question. Even if the direction and force of the wind is seemingly

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[1] *Dr. Julius Hann,* Handbuch der Klimatologie, Stuttgart 1908—1911, spec. B. II & III.

[2] Atlas des Stillen Ozeans der Deutschen Seewarte, Hamburg.

[3] For references, see: *Helland-Hansen, B.,* und *Fridtjof Nansen,* Temperaturschwankungen des Nordatlantischen Ozeans und in der Atmosphäre Videnskapsselskapets Skr. I, Mat.-Naturv. Klasse, 1916, Nr. 9. Kristiania 1917.

[4] *H. U. Sverdrup:* Die Beziehung der elfjährigen Klimaschwankungen zur Sonnentätigkeit. Meteorologische Zeitschrift, 1918.

[5] The maximum of sun spots seems to coincide with a maximum of solar energy.

the same, the stream lines may have shifted in such a way, that the origin of the air-current has been entirely changed. This is a field for future investigations.

I beg to express my sincerest thanks to the board of directors of the »Universitetets Jubilæumsfond«, and of the »Nansen Fond« for pecuniary support, that enabled me to carry out the long and often complicated work, connected with the preparation of the wind charts.

Mr. *Olaf Aasen* has carried out most of the numerical computations for the determination of the resulting means of the wind roses.