ON QUASI STATIC WAVEMOTION IN BAROTROPIC FLUID STRATA

BY *V. BJERKNES*

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1. A state of motion is said to be quasi static when the vertical accelerations are small enough to be neglected as compared with the horizontal accelerations. The equation of equilibrium may then be used for the vertical coordinate, while the complete equations of motion are retained only for the horizontal coordinates.

During a motion of this type we have apparent equilibrium distribution of pressure and mass along any vertical at any time. But these apparent equilibrium distributions vary in space from vertical to vertical, and in time for every single vertical. All great atmospheric and oceanic motions are of this kind.

Lagrange gave the first example of a fluid motion of this type, considering waves of great length as compared with the depth of the fluid. This kind of wave has been simply called »Long« waves. Later *Horace Lamb* substituted »tidal« for the somewhat indeterminate word »long«, for the reason that the tides are waves of this kind. But as this terminology would force us to divide the »tidal waves« met with in the atmosphere and the sea into two classes, those which are of tidal origine and those which are not, I have preferred the word »quasi static« which alludes to the dynamical nature of the waves in question, and not to a special although important class among them.

Quasi static wave motion has hithertoo been considered almost exclusively for incompressible fluid systems, stratified or not. The aim of this paper is to extend some of the results to the case of *compressible* systems in view of applications to the discussion of atmospheric motions. The generalisation is easy as long as the *condition of barotropy* is retained for the entire system in case of no stratification, and for each stratum in case of a stratified system.

I.

A single barotropic stratum.

2. Let z be the vertical coordinate, counting positively upwards. At the height $z = z_2$ the fluid is to be limited by a horizontal rigid bottom. In the case of equilibrium the free surface is to form a horizontal plane at the height $z = z_1$. The pressure in case of equilibrium is denoted by $\bar{p}$, and has the value $\bar{p}_2$ at the bottom, and $\bar{p}_1$ at the free surface. The latter is identical with the constant external pressure p_0. The equilibrium

density is $\bar{\varrho}$ with the values $\bar{\varrho}_2$ at the bottom and $\bar{\varrho}_1$ at the free surface. g being the acceleration of gravity we have the following equilibrium conditions: —

$$\text{(a)} \qquad \bar{\varrho}\, g = -\frac{\partial \bar{p}}{\partial z}$$

$$\text{(b)} \qquad \bar{\varrho} = F(\bar{p}) \,,$$

the first being the hydrostatic equation, and the second the barotropic condition, which gives the physical law of compressibility for the fluid. It is supposed that the differences of density originate exclusively from the pressure.

In the following we assume that the statical problem is solved, and that thus $\bar{p}$ and $\bar{\varrho}$ are known functions of the coordinate z.

3. Let p be pressure and ϱ density during motion. Let the displacement of a fluid particle from its equilibrium position be ξ in a horizontal direction along the axis of x, and ζ in a vertical direction along the axis of z.

Every quantity defining the departure from conditions of equilibrium is supposed to be small of the first order. Thus $\varrho - \bar{\varrho}$ is small of the first order as compared with ϱ or $\bar{\varrho}$, and the same with $p - \bar{p}$ compared with p or $\bar{p}$. The horizontal displacement ξ is supposed to be small as compared with the other characteristic horizontal distances of the problem. In the case of regular waves the other principal horizontal distance will be the wave length. Otherwise it may be defined more generally as the horizontal distance between points in which the quantities $\varrho - \bar{\varrho}$, $p - \bar{p}$, ξ, ζ show differences of their own order of magnitude. The condition may also be expressed thus, that the horizontal variations in space of these quantities are slow.

Just as in all problems concerning small oscillations we may then express the acceleration by the local time derivative $\frac{\partial}{\partial t}$ instead of by the individual $\frac{d}{dt}$. We may then write the following equations: (1) the equation of motion for the horizontal axis

$$\varrho \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x}$$

(2) the equation of equilibrium for the vertical axis

$$\varrho\, g = -\frac{\partial p}{\partial z}$$

(3) the equation of continuity, which gives the local decrease of density during the motion

$$-(\varrho - \bar{\varrho}) = \frac{\partial (\varrho \xi)}{\partial x} + \frac{\partial (\varrho \zeta)}{\partial z}$$

(4) the law of compressibility

$$\varrho = F(p)$$

where F is the same function as in the static case.

The equations may be simplified by the omission of second order terms. Where the unknown dynamic density ϱ is the factor of a small quantity we may substitute for it the known static density $\bar{\varrho}$. Thus while we must retain the exact value of ϱ in the static equation where it is the factor of the acceleration of gravity g, we may change it to $\bar{\varrho}$ in the dynamic equation where it is the factor of the small horizontal acceleration.

In the same manner we must retain the exact value of ϱ in the first member of the equation of continuity, but may change it to $\bar{\varrho}$ in the second member. A further simplification then comes, because this $\bar{\varrho}$ is independent of x. We arrive in this way at the following system of equations which is linear with respect to the independent variables p, ϱ, ξ, ζ

(a) $$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\bar{\varrho}} \frac{\partial p}{\partial x}$$

(b) $$g = -\frac{1}{\varrho} \frac{\partial p}{\partial z}$$

(c) $$-(\varrho - \bar{\varrho}) = \bar{\varrho} \frac{\partial \xi}{\partial x} + \frac{\partial (\bar{\varrho}\, \zeta)}{\partial z}$$

(d) $$\varrho = F(p)$$

To these equations, which determine the conditions for the interior of the fluid, we have to add the surface conditions. The kinematical condition expresses the fact that the vertical displacement is zero at the bottom

(e) $$z = z_2\,, \quad \zeta_2 = 0\,.$$

The dynamical condition involves the circumstance that the pressure at the free surface is equal to the exterior pressure p_0. As the free surface during the motion is at the height $z_1 + \zeta_1$, this condition takes the form

(f) $$p\,(z_1 + \zeta_1) = p^0\,.$$

Finally it is important to remember one general property of the motion. By barotropic conditions vortices are conserved, and as in the case before us there are no initial vortices, the motion will be irrotational. The components of velocity being $\frac{\partial \xi}{\partial t}$ and $\frac{\partial \zeta}{\partial t}$, this property of the motion will be expressed

$$\frac{\partial}{\partial z} \frac{\partial \xi}{\partial t} - \frac{\partial}{\partial x} \frac{\partial \zeta}{\partial t} = 0\,.$$

As we can here interchange the order of the differentiations, we can integrate with respect to time. Equating the irrelevant constant to zero, we find

(g) $$\frac{\partial \xi}{\partial z} = \frac{\partial \zeta}{\partial x}$$

This equation has an important consequence. We have introduced no restriction for the variation of the dependent variables as function of z. And indeed, p, ϱ and ζ may vary at finite rate as functions of this variable. But equation (g) shows that the variable ξ varies at the same slow rate as a function of z as ζ does as function of x. *For sufficiently short vertical distances we may therefore consider* $\frac{\partial \xi}{\partial z}$ *as a constant.*

4. From the identity of the equations 3, (b) and (d) with 2, (a) and (b) it follows that the dynamic distributions of pressure and density must be identical with the statical

ones, only with a displacement which must be identical with the elevation ζ_1 of the free surface. I. e. we may write

(a) $$\varrho(z) = \bar{\varrho}(z - \zeta_1)$$

(b) $$p(z) = \bar{p}(z - \zeta_1)$$

As ζ_1 is small we may develop according to Taylor

(c) $$\varrho(z) = \bar{\varrho}(z) - \frac{\partial \bar{\varrho}}{\partial z}\zeta_1$$
$$p(z) = \bar{p}(z) - \frac{\partial \bar{p}}{\partial z}\zeta_1$$

From the last of these equations we may eliminate $\frac{\partial \bar{p}}{\partial z}$ by the hydrostatic equation 2 (a)

(d) $$p(z) = \bar{p}(z) + g\,\bar{\varrho}\,\zeta_1$$

Using (d) in the equation of motion and (c) in the equation of continuity we get

(e $$\frac{\partial^2 \xi}{\partial t^2} = -g\frac{\partial \zeta_1}{\partial x}$$

(f) $$\frac{\partial \bar{\varrho}}{\partial z}\zeta_1 - \frac{\partial(\bar{\varrho}\zeta)}{\partial z} = \bar{\varrho}\frac{\partial \xi}{\partial x}$$

As ζ_1 is independent upon z, the first member of equation (f) can be written as a derivative with respect to z. The second member consists of two factors, of which the one $\bar{\varrho}$ may be written $-\frac{\partial}{\partial z}\left(\frac{\bar{p}}{g}\right)$ in virtue of the hydrostatic equation, while the other $\frac{\partial \xi}{\partial x}$ may, as we have seen, be considered as independent upon z if we limit ourselves to the consideration of sufficiently small vertical distances. Remembering this restriction we may then write the equation of continuity

$$\frac{\partial}{\partial z}(\bar{\varrho}\,\zeta_1 - \bar{\varrho}\,\zeta) = -\frac{\partial}{\partial z}\left(\frac{\bar{p}}{g}\frac{\partial \xi}{\partial x}\right)$$

and we may integrate it from the bottom $z = z_2$ to the free surface $z = z_1$ provided that the distance $z_1 - z_2$ is small as compared with the horizontal wave length. As at the free surface we have $\zeta = \zeta_1$, $\bar{\varrho} = \bar{\varrho}_1$, $\bar{p} = \bar{p}_1$ and at the bottom $\zeta = o$, $\bar{\varrho} = \bar{\varrho}_2$, $\bar{p} = \bar{p}_2$ we get

$$\bar{\varrho}_2\,\zeta_1 = -\frac{1}{g}(\bar{p}_2 - \bar{p}_1)\frac{\partial \xi}{\partial x}$$

Now let us introduce

(g) $$c^2 = \frac{\bar{p}_2 - \bar{p}_1}{\bar{\varrho}_2} = \bar{\alpha}_2(\bar{p}_2 - \bar{p}_1)$$

where $\bar{\alpha}_2$ is the specific volume at the bottom of the stratum. Then the equation of continuity takes the form

(h) $$\zeta_1 = -\frac{c^2}{g}\frac{\partial \xi}{\partial x}$$

and substituting this in the equation of motion (e) it takes the form

(i) $$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}$$

Our problem is now solved: (i) is the well known equation of »vibrating chords«, which has the general integral

(j) $$\xi = A f(x - ct) + B \varphi(x + ct)$$

f and φ being any functions. To the value of the horizontal displacement thus found will by (h) correspond the elevation

(k) $$\zeta_1 = -A \frac{c^2}{g} f'(x - ct) - B \frac{c^2}{g} \varphi'(x + ct)$$

of the free surface. The result then is that in our fluid stratum invariable waves of any waveprofile may propagate in either of the two directions. *The velocity of propagation is by* (g) *equal to the geometric mean of the specific volume of the fluid at the bottom and the difference of pressure between bottom and surface.* It is very remarkable that this result is independent upon the special law of compression, and therefore equally valid for liquids and gases.

5. After the result is found, we no longer need distinguish between the dynamical values $\bar{\alpha}$, $\bar{p}$ and the practically equal statical values α, p of specific volume and pressure. We may write the formula for the velocity of propagation

(a) $$c = \sqrt{\alpha_2 (p_2 - p_1)}$$

In special cases we may now bring this formula in more special forms.

When the fluid is homogeneous and incompressible α_2 will be the specific volume of the fluid not merely at the bottom but everywhere in the fluid, and the product $\alpha_2 (p_2 - p_1)$ will represent the difference of gravity potential $\Phi_1 - \Phi_2$ between the surface and the bottom. Then (a) takes the simple form

(b) $$c = \sqrt{\Phi_1 - \Phi_2}$$

i. e. *quasistatic waves in a homogeneous and incompressible fluid stratum propagate with a velocity equal to the square root of the difference of potential between surface and bottom.* Writing the formula $\frac{1}{2} c^2 = \frac{1}{2} (\Phi_1 - \Phi_2)$ we see that the velocity c may be interpreted as that acquired by a body falling from the surface to half »the dynamic depth« of the fluid stratum. This is the well known result of *Lagrange,* only slightly generalized: using the »dynamic depth« measured in potential measure instead of the geometric depth measured in length measure, we make the result comprise the case even of so deep layers, that the acceleration of gravity varies sensibly from the surface to the bottom.

To compare the general formula (a) where the compressibility is taken into account, with (b) where it is neglected, we may in the case of oceanic waves use the »Hydrographic Tables« 7 H and 8 H[1]), which give corresponding values of pressure, specific volume and dynamic depth for the »normal« seawater of 0° C and a salinity of 35 per mille. The annexed table gives an extract from these tables, only with transition to MTS units, i. e. from decibar to centibar, and from dynamic meter to dynamic decimeter.

[1]) Dynamic Meteorology and Hydrography by V. Bjerknes and different Collaborators, Hydrographic Tables. Published by the Carnegie Institution of Washington 1910. German edition by Vieweg & Sohn 1912.

Then two columns are added, one for the geometric mean of pressures and sp. volumes, and one for the square root of the differences of potential. These columns then give the velocities of propagation respectively according to (a) and (b) in meters per second.

p Pressure centibar	Φ Depth dyn. decim.	a Specific Volume	$a\,p$	$\sqrt{a\,p}$	$\sqrt{\Phi}$
0	0	0.97264	0	0	0
10 000	9 704.0	0.96819	9 681.9	98.40	98.51
20 000	19 364.2	0.96388	19 277.6	138.85	139.16
30 000	28 982.0	0.95970	28 791.0	169.68	170.24
40 000	38 558.7	0.95566	38 226.4	195.51	196.36
50 000	48 095.6	0.95173	47 586.5	218.15	219.31
60 000	57 593.7	0.94791	56 874.6	238.48	239.99
70 000	67 054.2	0.94421	66 094.7	257.09	258.95
80 000	76 478.2	0.94060	75 248.0	274.32	276.55
90 000	85 866.5	0.93709	84 338.1	290.48	293.03
100 000	95 220.3	0.93368	93 368.0	305.56	308.58

The differences are seen to be practically insignificant: they are one per mille for a depth of the order of magnitude 1 000 meters, and one per cent for the greatest attainable depths of the order of magnitude of 10 000 meters.

In looking at the amplitudes we may consider sinus waves. When in 4 (j) we write $\sin\frac{2\pi}{L}(x-ct)$ for $f(x-ct)$, L being the wave length, we find the following general expression for the ratio between the vertical amplitude A_v and the horizontal A_h

(c)
$$\frac{A_v}{A_h}=\frac{2\pi}{L}\frac{c^2}{g}$$

In the special case of homogeneity and incompressibility we may introduce $c^2=\Phi_1-\Phi_2=$ $=Dg$, D being the depth expressed in length measure. We then get

(d)
$$\frac{A_v}{A_h}=2\pi\frac{D}{L}$$

which gives the well known result that in quasi-static waves in incompressible fluids, the ratio of the vertical elevation of the surface to the horizontal amplitude is practically six times the ratio of the depth to the wave length.

Passing to atmospheric problems, we may write $p_1=0$ in formula (a), not because we know that pressure is zero at the boundary of the atmosphere, — provided that such a boundary exists at all, — but because in sufficient heights p_1 would be insignificant relatively to p_2. Using at the same time the equation for gas, $pa=R\vartheta$ we find for the velocity of propagation

(e)
$$c=\sqrt{a_2\,p_2}=\sqrt{R\,\vartheta_2}$$

α_2 being sp. volume, p_2 pressure, and ϑ_2 absolute temperature at sea level. But these are the well known expressions of the *Newtonian sound velocity,* which for 0° C or $\vartheta_2 = 273$ amounts to 280 meters per second. We therefore get this remarkable result:

If the atmosphere forms a single barotropic stratum, quasistatic waves propagate with the Newtonian sound velocity in the air at sea level.

It is interesting that this result comes out as a general consequence of the barotropy combined with quasi static state, and not as the consequence of any special law of compression. The quasistatic waves would propagate at the same velocity (e) in the unlimited isothermal atmosphere by the isothermal law of compression, in the limited adiabatic atmosphere of about 27 000 meters of height by the adiabatic law of compression, and ultimately also in the homogeneous atmosphere of about 7 800 meters of height and specific volume α_2 at all heights by compressibility zero. And in all cases we obtain the same finite elevation of the free surface: Taking the case of sinus waves, we may apply formula (d) to the homogeneous atmosphere to calculate this elevation. We get

(f) $$\frac{A_v}{A_h} = 2\pi \frac{7800}{L}$$

It may at first sight seem astonishing that the transition from the lower isothermal to the higher adiabatic elasticity, which in acoustics leads from the lower Newtonian to the higher Laplácean sound velocity, is here irrelevant. The reason is that the energy of propagation of quasi static waves is gravitational and not elastic. The greater changes of volume due to the adiabatic heating and cooling are used for producing at the same level a stronger vertical motion of the adiabatic than of the isothermal atmosphere, and thus to produce in the limited adiabatic atmosphere the same gravitational energy of propagation as in the unlimited isothermal atmosphere.

II.

Fundamental equations for a stratified system.

6. When the fluid system consists of any number M of strata, we shall denote by Roman figures I, II, III, ... N, ... M the strata themselves, and by Arabian figures 1, 2, 3, ... n, ... $m+1$ their bounding surfaces, 1 and 2 being those of stratum I, 2 and 3 those of stratum II, and so on. To the symbols representing our variables we add the Roman numbers as high, and the Arabian numbers as low indices. In the case of equilibrium the boundary surfaces are horizontal planes in the heights

(a) $$z_1\ ,\ z_2\ ,\ z_3\ ,\ \ldots\ z_n\ ,\ \ldots\ z_{m+1}$$

which may also be represented by the gravity potentials

(b) $$\Phi_1\ ,\ \Phi_2\ ,\ \Phi_3\ ,\ \ldots\ \Phi_n\ ,\ \ldots\ \Phi_{m+1}$$

In the stratum N pressure and density are represented by p^N and ϱ^N, their values at the upper boundary surface by p_n^N and ϱ_n^N, and at the lower boundary surface by p_{n+1}^N, ϱ_{n+1}^N. The corresponding equilibrium values are $\bar{p}^N$, $\bar{\varrho}^N$, $\bar{p}_n^N$, ... The motion within the stratum is represented by the displacements ξ^N, ζ^N.

The kinematic boundary condition, which must be fulfilled at every boundary surface, demands the equality of the normal components of displacement on the two sides of the surface. Now within the degree of approximation we may identify the tangential displacement with the horizontal, and the normal displacement with the vertical. At the boundary surface n we have then two different horizontal displacements ξ_n^{N-1} and ξ_n^N, but only one vertical displacement $\zeta_n^{N-1} = \zeta_n^N$. The fact that the kinematical surface condition is satisfied may therefore be symbolized simply by leaving out the indices $N-1$ and N representing the strata, and retaining only the low indices n representing the surface. Thus we represent the elevations of the boundary surfaces by the symbols

(c) $$\zeta_1 , \zeta_2 , \zeta_3 , \dots \zeta_n , \dots \zeta_m , \zeta_{m+1} .$$

In a quite similar way we may symbolize the fact that the dynamical surface condition is fulfilled. In virtue of the principle of the equal action and reaction the pressures p_n^{N-1} and p_n^N must be identical. For the pressure at any bounding surface we may therefore leave out the index giving reference to the one or the other of the two strata on the two sides of the surface. In the case of equilibrium the known pressures at the boundary surfaces (a) are therefore represented

(d) $$\bar{p}_1 , \bar{p}_2 , \bar{p}_3 , \dots \bar{p}_n , \dots \bar{p}_m , \bar{p}_{m+1} .$$

In the case of motion the boundary surfaces are at the variable heights

(e) $$z_1 + \zeta_1 , z_2 + \zeta_2 , z_3 + \zeta_3 , \dots z_n + \zeta_n , \dots z_m + \zeta_m , z_{m+1}$$

and in these moving surfaces we have the varying pressures

(f) $$p_1 , p_2 , p_3 , \dots p_n , \dots p_m , p_{m+1}$$

Finally it should be remembered that we know the last member of the series (c) and the first of the series (f). As the bottom does not move we have

(g) $$\zeta_{m+1} = 0$$

And as both p_1 and $\bar{p}_1$ must be identical with the given constant exterior pressure we have

(h) $$p_1 = \bar{p}_1$$

7. In any stratum N the internal equilibrium conditions are

(a) $$\bar{\varrho}^N g = -\frac{\partial \bar{p}^N}{\partial z}$$

(b) $$\bar{\varrho}^N = F^N(\bar{p}^N)$$

We assume that the equilibrium values $\bar{p}^N$ and $\bar{\varrho}^N$ of pressure and density, which satisfy these equations and the surface conditions (6, d), are known.

Corresponding to (3, a—d) the equations of motion for the stratum N will be

(c) $$\frac{\partial^2 \xi^N}{\partial t^2} = -\frac{1}{\bar{\varrho}^N}\frac{\partial p^N}{\partial x}$$

(d) $$g = -\frac{1}{\varrho^N}\frac{\partial p^N}{\partial z}$$

(e) $$-(\varrho^N - \bar{\varrho}^N) = \bar{\varrho}^N \frac{\partial \xi^N}{\partial x} + \frac{\partial(\bar{\varrho}^N \zeta^N)}{\partial z}$$

(f) $$\varrho^N = F^N(p^N)$$

The formal identity of the equations (d) and (f) with (a) and (b) enables us to express the unknown dynamical values of pressure and density by the known statical values in the form

(g) $$\varrho^N(z) = \bar{\varrho}^N(z + h^N)$$

(h) $$p^N(z) = \bar{p}^N(z + h^N)$$

The auxiliary quantity h^N, which may be called the »dynamical elevation«, is for the highest stratum oppositely equal to the elevation ζ_1 of the free surface as in the case of a single stratum (4, a and b). But for any other stratum it must be a function of the elevations $\zeta_1, \zeta_2 \ldots \zeta_n$ of all overlying boundary surfaces. We shall use it provisionally as a convenient auxiliary variable, which we shall later eliminate. In developed form equations (g) and (h) give

(i) $$\varrho^N(z) = \bar{\varrho}^N(z) + \frac{\partial \bar{\varrho}^N}{\partial z} h^N$$

$$p^N(z) = \bar{p}^N(z) + \frac{\partial \bar{p}^N}{\partial z} h^N$$

Or, using (a) in the last of these equations

(j) $$p^N(z) = \bar{p}^N(z) - \bar{\varrho}^N g\, h^N$$

If we now introduce (j) in the equation of motion (c), and (i) in the equation of continuity (e), these equations become

(k) $$\frac{\partial^2 \xi^N}{\partial t^2} = g \frac{\partial h^N}{\partial x}$$

(l) $$-\frac{\partial \bar{\varrho}^N}{\partial z} h^N = \bar{\varrho}^N \frac{\partial \xi^N}{\partial x} + \frac{\partial (\bar{\varrho}^N \zeta^N)}{\partial z}$$

where now dynamical pressure p^N and dynamical density ϱ^N are eliminated, and replaced by the auxiliary variable h^N.

To find a first relation connecting this auxiliary variable with the fundamental variables of our problem, we may use the dynamical surface condition at any surface n. According to (j) the pressures in the strata $N - \mathrm{I}$ and N are respectively

$$p^{N-\mathrm{I}}(z) = \bar{p}^{N-\mathrm{I}}(z) - g\, \bar{\varrho}^{N-\mathrm{I}} h^{N-\mathrm{I}}$$
$$p^N(z) = \bar{p}^N(z) - g\, \bar{\varrho}^N h^N$$

Neglecting second order quantities, we find as values of these pressures at the boundary surface n

$$p^{N-\mathrm{I}}(z_n + \zeta_n) = \bar{p}^{N-\mathrm{I}}(z_n) + \frac{\partial \bar{p}^{N-\mathrm{I}}}{\partial z} \zeta_n - g \bar{\varrho}_n^{N-\mathrm{I}} h^{N-\mathrm{I}}$$

$$p^N(z_n + \zeta_n) = \bar{p}^N(z_n) + \frac{\partial \bar{p}^N}{\partial z} \zeta_n - g\, \bar{\varrho}_n^N h^N$$

Or, using the equilibrium equation (a) and the corresponding equation for the stratum $N - \mathrm{I}$

$$p^{N-\mathrm{I}}(z_n + \zeta_n) = \bar{p}^{N-\mathrm{I}}(z_n) - g\, \bar{\varrho}_n^{N-\mathrm{I}} (h^{N-\mathrm{I}} + \zeta_n)$$
$$p^N(z_n + \zeta_n) = \bar{p}^N(z_n) - g\, \bar{\varrho}_n^N (h^N + \zeta_n)$$

When we identify these pressures, we get

(m) $$\bar{\varrho}_n^N (h^N + \zeta_n) - \bar{\varrho}_n^{N-\mathrm{I}} (h^{N-\mathrm{I}} + \zeta_n) = 0$$

For each boundary surface we obtain an equation of this form, connecting the auxiliary quantities h with the elevations of the bounding surface.

A second relation connecting the auxiliary quantity h^N with the fundamental variables is obtained by integrating the equation of continuity (l) through the stratum. As h^N is independent of z we may write the equation

$$-\frac{\partial}{\partial z}(\bar{\varrho}^N h^N + \bar{\varrho}^N \zeta^N) = \bar{\varrho}^N \frac{\partial \xi^N}{\partial x}$$

In the second member $\bar{\varrho}^N$ may expressed as a derivative of $\bar{p}^N$ with respect to z in virtue of the static equation (a). Further is $\frac{\partial \xi^N}{\partial x}$ a quantity which varies slowly as function of z, and may in the first approximation be considered as a constant within a stratum of which the depth is small as compared with the horizontal wave length. Limiting our considerations to strata of this limited depth, we may then write the equation

$$-\frac{\partial}{\partial z}[\bar{\varrho}^N (h^N + \zeta^N)] = -\frac{\partial}{\partial z}\left(\frac{\bar{p}^N}{g}\frac{\partial \xi^N}{\partial x}\right)$$

Then integrating through the stratum from its lower boundary $n + 1$ to its upper boundary n we have the equation of continuity in the form

(n) $$\bar{\varrho}^N_{n+1}(h^N + \zeta^N_{n+1}) - \bar{\varrho}^N_n (h^N + \zeta_n) = \frac{1}{g}(\bar{p}_{n+1} - \bar{p}_n)\frac{\partial \xi^N}{\partial x}$$

One such equation for each stratum gives m equations connecting the auxiliary quantities h^N with ζ_n and ξ^N.

(k), (m) and (n) now give 3 m equations for determining the 3m quantities h^N, ζ_n, ξ^N and thus give the solution of the problem.

8. Dynamical pressure p and dynamical density ϱ have now disappeared from our equations, and have been completely replaced by the statical values $\bar{p}$ and $\bar{\varrho}$. We therefore no longer need two sets of notations to distinguish between two parallel sets of quantities. We may simplify our notations, and write everywhere p and ϱ, remembering that these are now the known statical values.

We first write the complete system of equations (m) and (n) as follows, remembering that the first equation (m) is simplified because above the free surface we have a vacuum, and thus $\varrho_1{}^0 = 0$, while in the last of equations (n) we have a corresponding simplification because $\zeta_{m+1} = 0$

(1) $$\varrho_1{}^{\mathrm{I}}(h^{\mathrm{I}} + \zeta_1) = 0$$

(I) $$\varrho_2{}^{\mathrm{I}}(h^{\mathrm{I}} + \zeta_2) - \varrho_1{}^{\mathrm{I}}(h^{\mathrm{I}} + \zeta_1) = \frac{1}{g}(p_2 - p_1)\frac{\partial \xi^{\mathrm{I}}}{\partial x}$$

(2) $$\varrho_2{}^{\mathrm{II}}(h^{\mathrm{II}} + \zeta_2) - \varrho_2{}^{\mathrm{I}}(h^{\mathrm{I}} + \zeta_2) = 0$$

(II) $$\varrho_3{}^{\mathrm{II}}(h^{\mathrm{II}} + \zeta_3) - \varrho_2{}^{\mathrm{II}}(h^{\mathrm{II}} + \zeta_2) = \frac{1}{g}(p_3 - p_2)\frac{\partial \xi^{\mathrm{II}}}{\partial x}$$

(3) $$\varrho_3{}^{\mathrm{III}}(h^{\mathrm{III}} + \zeta_3) - \varrho_3{}^{\mathrm{II}}(h^{\mathrm{II}} + \zeta_3) = 0$$

(III) $$\varrho_4{}^{\mathrm{III}}(h^{\mathrm{III}} + \zeta_4) - \varrho_3{}^{\mathrm{III}}(h^{\mathrm{III}} + \zeta_3) = \frac{1}{g}(p_4 - p_3)\frac{\partial \xi^{\mathrm{III}}}{\partial x}$$

. . . .

(m) $$\varrho^M_m (h^M + \zeta_m) - \varrho^{M-\mathrm{I}}_m (h^{M-\mathrm{I}} + \zeta_m) = 0$$

(M) $$\varrho^M_{m+1} h^M - \varrho^M_m (h^M + \zeta_m) = \frac{1}{g}(p_{m+1} - p_m)\frac{\partial \xi^M}{\partial x}$$

The equations originating from (7, m), and which refer to boundary surfaces, have been numbered with Arabian figures, and those originating from (7, n) and which refer to strata, have been numbered with Latin figures.

Fortunately this system can be solved with respect to h and ζ without the calculation of determinants. We first derive a new system in the following way: The first equation of the new system is equation (1) of the old system, divided by $\varrho_1{}^{I}$; the second equation of the new system is the sum of equations (1) and (I) of the old one, divided by $\varrho_2{}^{I}$; the third of the new system is the sum of the equations (1), (I), (2) of the old one, and divided by $\varrho_2{}^{II}$, and so on. When in the second members we write the specific volumes $\alpha_1{}^{I}$, $\alpha_2{}^{I}$, ... instead of the corresponding reciprocal densities, we get

$$\text{(1)}\quad h^{I} + \zeta_1 = 0$$

$$\text{(I)}\quad h^{I} + \zeta_2 = \frac{\alpha_2{}^{I}}{g}(p_2 - p_1)\frac{\partial \xi^{I}}{\partial x}$$

$$\text{(2)}\quad h^{II} + \zeta_2 = \frac{\alpha_2{}^{II}}{g}(p_2 - p_1)\frac{\partial \xi^{I}}{\partial x}$$

$$\text{(II)}\quad h^{II} + \zeta_3 = \frac{\alpha_3{}^{II}}{g}(p_2 - p_1)\frac{\partial \xi^{I}}{\partial x} + \frac{\alpha_3{}^{II}}{g}(p_3 - p_2)\frac{\partial \xi^{II}}{\partial x}$$

$$\text{(3)}\quad h^{III} + \zeta_3 = \frac{\alpha_3{}^{III}}{g}(p_2 - p_1)\frac{\partial \xi^{I}}{\partial x} + \frac{\alpha_3{}^{III}}{g}(p_3 - p_2)\frac{\partial \xi^{II}}{\partial x}$$

$$\text{(III)}\quad h^{III} + \zeta_4 = \frac{\alpha_4{}^{III}}{g}(p_2 - p_1)\frac{\partial \xi^{I}}{\partial x} + \frac{\alpha_4{}^{III}}{g}(p_3 - p_2)\frac{\partial \xi^{II}}{\partial x} + \frac{\alpha_4{}^{III}}{g}(p_4 - p_3)\frac{\partial \xi^{III}}{\partial x}$$

. . . .

$$\text{(m)}\quad h^{M} + \zeta_m = \frac{\alpha_m^M}{g}(p_2 - p_1)\frac{\partial \xi^{I}}{\partial x} + \frac{\alpha_m^M}{g}(p_3 - p_2)\frac{\partial \xi^{II}}{\partial x} + \frac{\alpha_m^M}{g}(p_4 - p_3)\frac{\partial \xi^{III}}{\partial x} + \dots + \frac{\alpha_m^M}{g}(p_m - p_{m-1})\frac{\partial \xi^{M-I}}{\partial x}$$

$$\text{(M)}\quad h^{M} = \frac{\alpha_{m+1}^M}{g}(p_2 - p_1)\frac{\partial \xi^{I}}{\partial x} + \frac{\alpha_{m+1}^M}{g}(p_3 - p_2)\frac{\partial \xi^{II}}{\partial x} + \frac{\alpha_{m+1}^M}{g}(p_4 - p_3)\frac{\partial \xi^{III}}{\partial x} + \dots + \frac{\alpha_{m+1}^M}{g}(p_m - p_{m-1})\frac{\partial \xi^{M-I}}{\partial x} + \frac{\alpha_{m+1}^M}{g}(p_{m+1} - p_m)\frac{\partial \xi^{M}}{\partial x}$$

The solution of this system is immediate. Beginning with (1), subtracting (I), adding (2), subtracting (II), adding (3) and so on we find ζ_1; beginning with (I), subtracting (2), adding (II), subtracting (3) and so on, we find h^{I}; then beginning with (2) and alternately subtracting and adding the subsequent equations, we find ζ_2, and so on.

The expressions for the quantities ζ and h, which we find in this way, may be written very concisely if we introduce some abbreviated notations which naturally suggest themselves.

We first write

$$\text{(a)}\quad \begin{aligned}
a_1 &= \alpha_2{}^{I} - \alpha_2{}^{II} + \alpha_3{}^{II} - \alpha_3{}^{III} + \alpha_4{}^{III} \dots\dots - \alpha_m^M + \alpha_{m+1}^M \\
b_2 &= \quad - \alpha_2{}^{II} + \alpha_3{}^{II} - \alpha_3{}^{III} + \alpha_4{}^{III} \dots\dots - \alpha_m^M + \alpha_{m+1}^M \\
a_2 &= \quad \alpha_3{}^{II} - \alpha_3{}^{III} + \alpha_4{}^{III} \dots\dots - \alpha_m^M + \alpha_{m+1}^M \\
b_3 &= \quad - \alpha_3{}^{III} + \alpha_4{}^{III} \dots\dots - \alpha_m^M + \alpha_{m+1}^M \\
a_3 &= \quad \alpha_4{}^{III} \dots\dots - \alpha_m^M + \alpha_{m+1}^M \\
&\dots\dots\dots\dots \\
b_m &= \quad - \alpha_m^M + \alpha_{m+1}^M \\
a_m &= \quad \alpha_{m+1}^M
\end{aligned}$$

It is useful to examine at once the properties of the auxiliary quantities a and b thus introduced. We first remark that stability requires that the sp. volumes always decrease from the surface and downwards. Thus we have in the general case

(b) $$\alpha_1^{I} > \alpha_2^{I} > \alpha_2^{II} > \alpha_3^{II} > \alpha_3^{III} > \ldots \qquad > \alpha_{m+1}^{M}$$

Equality among quantities α which have the same upper index

(c) $$\alpha_1^{I} = \alpha_2^{I}, \qquad \alpha_2^{II} = \alpha_3^{II} \qquad \ldots.$$

indicates the special case of homogeneity and incompressibility within the different strata. On the other hand, equality among quantities α which have the same lower index

(d) $$\alpha_2^{I} = \alpha_2^{II}, \qquad \alpha_3^{II} = \alpha_3^{III}, \ldots$$

indicates that the different discontinuities disappear, so that we come back to the case of a single barotropic stratum.

Now $a_1, a_2, a_3, \ldots$ and $-b_2, -b_3, -b_4, \ldots.$ evidently are positive quantities which may be defined as specific volumes, we shall call them »virtual« specific volumes to distinguish them from the real ones $\alpha_1^{I}, \alpha_2^{I} \ldots$ For obvious reasons we may say that there exists correspondence between a_1 and α_2^{I}, between a_2 and α_3^{II}, between a_3 and α_4^{III}, and so on. Remembering the relations (b), (c), (d) we easily find:

(1) Between the corresponding virtual and real sp. volumes we have the relations

(e) $$a_1 \geqq \alpha_2^{I}, \quad a_2 \geqq \alpha_3^{II}, \quad a_3 \geqq \alpha_4^{III}, \ldots$$

the equalities corresponding to the special case of homogeneity and incompressibility.

(2) The differences between the virtual sp. volumes a are identical with the differences between the corresponding real sp. volumes α

(f) $$a_1 - a_2 = \alpha_2^{I} - \alpha_2^{II}, \quad a_2 - a_3 = \alpha_2^{II} - \alpha_3^{II}, \quad \ldots.$$

(3) The virtual sp. volumes a form a decreasing series

(g) $$a_1 > a_2 > a_3 > \ldots \qquad > a_m > 0$$

(4) The virtual sp. volumes $-b$ form a decreasing series

(h) $$-b_2 > -b_3 > -b_4 > \ldots \qquad > -b_m > 0$$

(5) In the case of homogeneity and incompressibility we have identity between the corresponding virtual and real sp. volumes, a and α, while the virtual sp. volumes b disappear

(i) $$a_1 = \alpha_2^{I}, \quad a_2 = \alpha_3^{II}, \quad a_3 = \alpha_4^{III}$$

(j) $$b_2 = b_3 = b_4 = \ldots = b_m = 0$$

(6) In the case of complete barotropy, i. e. when all discontinuities disappear, the virtual sp. volumes a become all identical with the real sp. volume at the bottom

(k) $$a_1 = a_2 = a_3 = \ldots = a_m = \alpha_{m+1}^{M}$$

If then we form the quantities ζ and h according to the rules given above, we get as coefficients products of the virtual sp. volumes a and $-b$ into differences of pressure for successive boundary surfaces. But according to (5, a) each such product represents the squared velocity of propagation of quasi static waves in a single barotropic stratum having just that difference of pressure between surface and bottom and this sp. volume at the bottom. We agree to introduce the following symbols for these squared velocities of propagation

$$
(\mathrm{l})\quad
\begin{array}{lll}
\gamma^{\mathrm{I}}_1 = a_1(p_2 - p_1) & \gamma^{\mathrm{I}}_2 = a_2(p_2 - p_1) & \gamma^{\mathrm{I}}_3 = a_3(p_2 - p_1)\ldots \\
 & \gamma^{\mathrm{I}}_{m-1} = a_{m-1}(p_2 - p_1) & \gamma^{\mathrm{I}}_m = a_m(p_2 - p_1) \\
\delta^{\mathrm{I}}_2 = -b_2(p_2 - p_1) & \gamma^{\mathrm{II}}_2 = a_2(p_3 - p_2) & \gamma^{\mathrm{II}}_3 = a_3(p_3 - p_2)\ldots \\
 & \gamma^{\mathrm{II}}_{m-1} = a_{m-1}(p_3 - p_2) & \gamma^{\mathrm{II}}_m = a_m(p_3 - p_2) \\
\delta^{\mathrm{I}}_3 = -b_3(p_2 - p_1) & \delta^{\mathrm{II}}_3 = -b_3(p_3 - p_2) & \gamma^{\mathrm{III}}_3 = a_3(p_4 - p_3)\ldots \\
 & \gamma^{\mathrm{III}}_{m-1} = a_{m-1}(p_4 - p_3) & \gamma^{\mathrm{III}}_m = a_m(p_4 - p_3) \\
\ldots\ldots & \ldots\ldots & \ldots\ldots \\
\delta^{\mathrm{I}}_{m-1} = -b_{m-1}(p_2 - p_1) & \delta^{\mathrm{II}}_{m-1} = -b_{m-1}(p_3 - p_2) & \delta^{\mathrm{III}}_{m-1} = -b_{m-1}(p_4 - p_4)\ldots \\
 & \gamma^{M-1}_{m-1} = a_{m-1}(p_m - p_{m-1}) & \gamma^{M-1}_m = a_m(p_m - p_{m-1}) \\
\delta^{\mathrm{I}}_m = -b_m(p_2 - p_1) & \delta^{\mathrm{II}}_m = -b_m(p_3 - p_2) & \delta^{\mathrm{III}}_m = -b_m(p_4 - p_3)\ldots \\
 & \delta^{M-1}_m = -b_m(p_m - p_{m-1}) & \gamma^{M}_m = a_m(p_{m+1} - p_m)
\end{array}
$$

From the inequalities (g) and (h) existing between the virtual specific volumes we immediately derive corresponding inequalities between these squared velocities of propagation. Separately for the quantities γ and the quantities δ we find:

Quantities γ which have the same upper index and decreasing low indices form a decreasing series.

Precisely the same is the case with the quantities δ.

If then we calculate the quantities h according to the rule given above, and introduce first the abbreviated notations a and then the abbreviated notations γ we get

$$
(\mathrm{m})\quad
\begin{aligned}
h^{\mathrm{I}} &= \frac{1}{g}\left(\gamma_1{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \gamma_2{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} + \gamma_3{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} + \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} + \cdots + \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
h^{\mathrm{II}} &= \frac{1}{g}\left(\gamma_2{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \gamma_2{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} + \gamma_3{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} + \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} + \cdots + \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
h^{\mathrm{III}} &= \frac{1}{g}\left(\gamma_3{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \gamma_3{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} + \gamma_3{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} + \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} + \cdots + \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
h^{\mathrm{IV}} &= \frac{1}{g}\left(\gamma_4{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \gamma_4{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} + \gamma_4{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} + \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} + \cdots + \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
&\ldots\ldots\ldots\ldots\\
h^{M} &= \frac{1}{g}\left(\gamma_m{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \gamma_m{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} + \gamma_m{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} + \gamma_m{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} + \cdots + \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)
\end{aligned}
$$

And when in the same manner we calculate the quantities ζ and introduce first the abbreviated notations a and b, and then the abbreviated notations γ and δ we find

$$
(\mathrm{n})\quad
\begin{aligned}
\zeta_1 &= \frac{1}{g}\left(-\gamma_1{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} - \gamma_2{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} - \gamma_3{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} - \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} - \cdots - \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
\zeta_2 &= \frac{1}{g}\left(\delta_2{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} - \gamma_2{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} - \gamma_3{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} - \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} - \cdots - \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
\zeta_3 &= \frac{1}{g}\left(\delta_3{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \delta_3{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} - \gamma_3{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} - \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} - \cdots - \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
\zeta_4 &= \frac{1}{g}\left(\delta_4{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \delta_4{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} + \delta_4{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} - \gamma_4{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} - \cdots - \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)\\
&\ldots\ldots\ldots\ldots\\
\zeta_m &= \frac{1}{g}\left(\delta_m{}^{\mathrm{I}}\frac{\partial\xi^{\mathrm{I}}}{\partial x} + \delta_m{}^{\mathrm{II}}\frac{\partial\xi^{\mathrm{II}}}{\partial x} + \delta_m{}^{\mathrm{III}}\frac{\partial\xi^{\mathrm{III}}}{\partial x} + \delta_m{}^{\mathrm{IV}}\frac{\partial\xi^{\mathrm{IV}}}{\partial x} + \cdots - \gamma^{M}_{m}\frac{\partial\xi^{M}}{\partial x}\right)
\end{aligned}
$$

Equations (m) now allow us to eliminate the auxiliary quantities h from the differential equations (7, k). The integration then gives the horizontal displacements ξ, and then equations (n) give the vertical elevations ζ of the boundary surfaces.

9. Substituting the values (8, m) of the auxiliary quantities h into the equation of motion (7, k) we arrive at the following system of differential equations for the determination of the horizontal displacements

$$
\text{(a)}\qquad
\begin{aligned}
\frac{\partial^2\xi^{I}}{\partial t^2} &= \gamma_1^{I}\frac{\partial^2\xi^{I}}{\partial x^2} + \gamma_2^{II}\frac{\partial^2\xi^{II}}{\partial x^2} + \gamma_3^{III}\frac{\partial^2\xi^{III}}{\partial x^2} + \gamma_4^{IV}\frac{\partial^2\xi^{IV}}{\partial x^2} + \dots + \gamma_m^{M}\frac{\partial^2\xi^{M}}{\partial x^2}\\
\frac{\partial^2\xi^{II}}{\partial t^2} &= \gamma_2^{I}\frac{\partial^2\xi^{I}}{\partial x^2} + \gamma_2^{II}\frac{\partial^2\xi^{II}}{\partial x^2} + \gamma_3^{III}\frac{\partial^2\xi^{III}}{\partial x^2} + \gamma_4^{IV}\frac{\partial^2\xi^{IV}}{\partial x^2} + \dots + \gamma_m^{M}\frac{\partial^2\xi^{M}}{\partial x^2}\\
\frac{\partial^2\xi^{III}}{\partial t^2} &= \gamma_3^{I}\frac{\partial^2\xi^{I}}{\partial x^2} + \gamma_3^{II}\frac{\partial^2\xi^{II}}{\partial x^2} + \gamma_3^{III}\frac{\partial^2\xi^{III}}{\partial x^2} + \gamma_4^{IV}\frac{\partial^2\xi^{IV}}{\partial x^2} + \dots + \gamma_m^{M}\frac{\partial^2\xi^{M}}{\partial x^2}\\
\frac{\partial^2\xi^{IV}}{\partial t^2} &= \gamma_4^{I}\frac{\partial^2\xi^{I}}{\partial x^2} + \gamma_4^{II}\frac{\partial^2\xi^{II}}{\partial x^2} + \gamma_4^{III}\frac{\partial^2\xi^{III}}{\partial x^2} + \gamma_4^{IV}\frac{\partial^2\xi^{IV}}{\partial x^2} + \dots + \gamma_m^{M}\frac{\partial^2\xi^{M}}{\partial x^2}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{\partial^2\xi^{M}}{\partial t^2} &= \gamma_m^{I}\frac{\partial^2\xi^{I}}{\partial x^2} + \gamma_m^{II}\frac{\partial^2\xi^{II}}{\partial x^2} + \gamma_m^{III}\frac{\partial^2\xi^{III}}{\partial x^2} + \gamma_m^{IV}\frac{\partial^2\xi^{IV}}{\partial x^2} + \dots + \gamma_m^{M}\frac{\partial^2\xi^{M}}{\partial x^2}
\end{aligned}
$$

These equations will be satisfied by the following values of ξ

$$
\text{(b)}\qquad \xi^{I} = A^{I} f(x-ct)\,,\qquad \xi^{II} = A^{II} f(x-ct)\,,\dots\qquad \xi^{M} = A^{M} f(x-ct)
$$

provided that the amplitudes A^{I}, A^{II}, A^{III}, ... satisfy the linear and homogeneous equations

$$
\text{(c)}\qquad
\begin{aligned}
(\gamma_1^{I} - c^2)A^{I} + \gamma_2^{II}A^{II} + \gamma_3^{III}A^{III} + \gamma_4^{IV}A^{IV} + \dots + \gamma_m^{M}A^{M} &= 0\\
\gamma_2^{I}A^{I} + (\gamma_2^{II} - c^2)A^{II} + \gamma_3^{III}A^{III} + \gamma_4^{IV}A^{IV} + \dots + \gamma_m^{M}A^{M} &= 0\\
\gamma_3^{I}A^{I} + \gamma_3^{II}A^{II} + (\gamma_3^{III} - c^2)A^{III} + \gamma_4^{IV}A^{IV} + \dots + \gamma_m^{M}A^{M} &= 0\\
\gamma_4^{I}A^{I} + \gamma_4^{II}A^{II} + \gamma_4^{III}A^{III} + (\gamma_4^{IV} - c^2)A^{IV} + \dots + \gamma_m^{M}A^{M} &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
\gamma_m^{I}A^{I} + \gamma_m^{II}A^{II} + \gamma_m^{III}A^{III} + \gamma_m^{IV}A^{IV} + \dots + (\gamma_m^{M} - c^2)A^{M} &= 0
\end{aligned}
$$

The amplitudes A will be different from zero only with the condition that the following determinant is equal to zero

$$
\text{(d)}\qquad
\begin{vmatrix}
\gamma_1^{I} - c^2 & \gamma_2^{II} & \gamma_3^{III} & \gamma_4^{IV} & \dots & \gamma_m^{M}\\
\gamma_2^{I} & \gamma_2^{II} - c^2 & \gamma_3^{III} & \gamma_4^{IV} & \dots & \gamma_m^{M}\\
\gamma_3^{I} & \gamma_3^{II} & \gamma_3^{III} - c^2 & \gamma_4^{IV} & \dots & \gamma_m^{M}\\
\gamma_4^{I} & \gamma_4^{II} & \gamma_4^{III} & \gamma_4^{IV} - c^2 & \dots & \gamma_m^{M}\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\gamma_m^{I} & \gamma_m^{II} & \gamma_m^{III} & \gamma_m^{IV} & \dots & \gamma_m^{M} - c^2
\end{vmatrix} = 0
$$

The characteristic symmetry of this determinant should be observed. The high indices of the quantities γ increase along the horizontal lines from left to right, while the low indices increase along diagonal lines. Physically, all the quantities γ have just as c^2 the significance of squares of velocities of propagation. The determinant is an equation of the m^{th} degree in c^2, and determines m values of this quantity as function of the known quantities of the same nature γ.

Substituting any of these values of c_n^2 in the linear equations (c) these may be solved in the form:

$$
\text{(e)}\qquad \frac{A^{I}}{\Delta_1} = \frac{A^{II}}{\Delta_2} = \frac{A^{III}}{\Delta_3} = \dots = \frac{A^{M}}{\Delta_m}
$$

$\triangle_1 \cdots \triangle_m$ being the underdeterminants to any horizontal line of the determinant (d) after the substitution $c^2 = c_n^2$. This system of values for velocity of propagation and amplitudes makes (b) a solution of the differential equations (a). Then (b) represents possible horizontal displacements within the different strata as functions of x and t. Equations (8, c) give the corresponding elevations of the boundary surfaces. These horizontal displacements and vertical elevations will retain their values relatively to a system of coordinates which moves with the velocity c_n. Thus we have a wave motion propagating with this velocity in all strata, and with unvariable wave profiles in each of them. Superposing m motions of this kind corresponding to the m different values of c^2 and propagating in the positive direction, and m other motions of the same nature propagating with the same velocities in the negative direction, we get the most general motion of the system.

The problem is thus formally solved. The explicit solution in every special case will depend entirely upon the determination of the roots of the algebraic equation (d), including the examination of whether they are always real and positive. We will give a detailed discussion of the simple case of two strata.

III.

A system of two strata.

10. When the number of strata is reduced to two, the horizontal displacements ξ and ξ^{II} within these two strata will, by wave motion propagating in the positive direction, be

$$\text{(a)} \qquad \xi^{\mathrm{I}} = A^{\mathrm{I}} f(x - ct), \qquad \xi^{\mathrm{II}} = A^{\mathrm{II}} f(x - ct)$$

The corresponding elevations ζ_1 and ζ_2 of the boundary surfaces may be written

$$\text{(b)} \qquad \zeta_1^{\mathrm{I}} = B^{\mathrm{I}} f'(x - ct), \qquad \zeta_2 = B^{\mathrm{II}} f'(x - ct)$$

The velocity of propagation will be found as a root of the equation

$$\text{(c)} \qquad \begin{vmatrix} \gamma_1^{\mathrm{I}} - c^2 & \gamma_2^{\mathrm{II}} \\ \gamma_2^{\mathrm{I}} & \gamma_2^{\mathrm{II}} - c^2 \end{vmatrix} = 0$$

This is a quadric in c^2, which gives two roots c_1^2 and c_2^2. Each defines a definite wave motion having a definite ratio between the amplitudes A^{I} and A^{II}. This ratio may be found with equal right by any of the two equations

$$\text{(d)} \qquad \begin{aligned} \frac{A^{\mathrm{I}}}{\gamma_2^{\mathrm{I}}} &= \frac{A^{\mathrm{II}}}{c^2 - \gamma_2^{\mathrm{II}}} \\ \frac{A^{\mathrm{I}}}{c^2 - \gamma_1^{\mathrm{I}}} &= \frac{A^{\mathrm{II}}}{\gamma_2^{\mathrm{II}}} \end{aligned}$$

when in it we substitute the value c_1^2 or c_2^2. According to (8, n) the corresponding values of B^{I} and B^{II} will be

$$\text{(e)} \qquad \begin{aligned} B^{\mathrm{I}} &= -\frac{1}{g}(\gamma_1^{\mathrm{I}} A^{\mathrm{I}} + \gamma_2^{\mathrm{II}} A^{\mathrm{II}}) \\ B^{\mathrm{II}} &= -\frac{1}{g}(-\delta_2^{\mathrm{I}} A^{\mathrm{I}} + \gamma_2^{\mathrm{II}} A^{\mathrm{II}}) \end{aligned}$$

All the fundamental data of the problem are thus expressed by the four symbols γ and δ which represent known quantities. Physically each of these represents the velocity of propagation in a certain single barotropic stratum. Mathematically we come back to the primary data of the problem by the relations

(f)
$$\begin{array}{ll} \gamma_1^{I} = a_1 (p_2 - p_1) & a_1 = \alpha_2^{I} - \alpha_2^{II} + \alpha_3^{II} \\ \gamma_2^{I} = a_2 (p_2 - p_1) & b_2 = \quad - \alpha_2^{II} + \alpha_3^{II} \\ \gamma_2^{II} = a_2 (p_3 - p_2) & a_2 = \qquad \alpha_3^{II} \\ \delta_2^{I} = - b_2 (p_2 - p_1) & \end{array}$$

Besides the quantities γ themselves, we shall in the following developments meet with certain combinations of them. The sum of γ_1^{I} and γ_2^{II} may be expressed

(g)
$$\gamma_1^{I} + \gamma_2^{II} = (\gamma_1^{I} - \gamma_2^{I}) + (\gamma_2^{I} + \gamma_2^{II}) = \Gamma_1 + \Gamma_2$$

where

(h)
$$\Gamma_1 = \gamma_1^{I} - \gamma_2^{I} = (a_1 - a_2)(p_2 - p_1) = (\alpha_2^{I} - \alpha_2^{II})(p_2 - p_1)$$

(i)
$$\Gamma_2 = \gamma_2^{I} + \gamma_2^{II} = a_2 (p_3 - p_1) = \alpha_3^{II} (p_3 - p_1)$$

Even these quantities Γ_1 and Γ_2 may be interpreted as squared velocities of propagation. The following three definitions should be born in mind in connection with the following developments:

$\sqrt{\gamma_2^{II}}$ — velocity of propagation in the lower stratum alone

$\sqrt{\Gamma_1}$ — . . . . in the upper stratum if at its lower boundary it had the sp. volume $\alpha_2^{I} - \alpha_2^{II}$.

$\sqrt{\Gamma_2}$ — . . . . when both strata are joined into one, retaining the total difference of pressure $p_3 - p_1$ and the sp. volume at the bottom α_3^{II}.

11. Developing the determinant (10, c) we have the quadric

(a)
$$c^4 - (\gamma_1^{I} + \gamma_2^{II}) c^2 + (\gamma_1^{I} - \gamma_2^{I}) \gamma_2^{II} = 0$$

which has the two roots

(b)
$$c^2 = \tfrac{1}{2} (\gamma_1^{I} + \gamma_2^{II}) \pm \tfrac{1}{2} \sqrt{(\gamma_1^{I} + \gamma_2^{II})^2 - 4 (\gamma_1^{I} - \gamma_2^{I}) \gamma_2^{II}}$$

which may also be written

(c)
$$c^2 = \tfrac{1}{2} (\gamma_1^{I} + \gamma_2^{II}) \pm \tfrac{1}{2} \sqrt{(\gamma_1^{I} - \gamma_2^{II})^2 + 4 \gamma_2^{I} \gamma_2^{II}}$$

The last form shows that the expression under the radical sign is always positive. Then the first form shows that the rational term is always greater than the irrational term. We conclude that c^2 is always real and positive, and thus gives two real and oppositely directed velocities of propagation c and $-c$.

When to these results we add the circumstance that the sum of the two positive roots c_1^2 and c_2^2 must be the coefficient $\gamma_1^{I} + \gamma_2^{II}$ in the equation, we see that we have

(d)
$$\gamma_1^{I} + \gamma_2^{II} > c_1^2 > \tfrac{1}{2} (\gamma_1^{I} + \gamma_2^{II}) > c_2^2 > 0 .$$

We can therefore speak without ambiguity of a greater velocity of propagation c_1 and a smaller c_2.

That these two different velocities of propagation correspond to essentially different motions follows from equations (10, d). The one shows that the amplitudes A^{I} and A^{II} will have the same or the opposite signs according as $c^2 \gtrless \gamma_2^{II}$, the other that the same will be the case according as $c^2 \gtrless \gamma_1^{\rm I}$. And as these conditions cannot contradict each other, they can be joined into one $c^2 \gtrless \frac{1}{2}(\gamma_1^{\rm I} + \gamma_2^{\rm II})$. This inequality combined with (d) shows that the greater velocity of propagation corresponds to horizontal displacements of the same direction in the two strata, and the smaller velocity of propagation to horizontal displacements of the opposite direction in the two strata.

The corresponding vertical elevations of the boundary surfaces are found from equations (10, e). The general result is seen intuitively. By equal direction of the horizontal motion in the two strata, the places of conflux and of efflux will be the same in both of them. Above an elevation in the internal boundary surface will then be formed a still higher elevation in the free surface. This rapidly propagating wave is therefore strikingly seen from without, and may be called the *external wave*. But by oppositely directed horizontal motions in the two strata, the conflux in the upper stratum is mainly used to fill the cavity formed in the internal surface by the corresponding efflux in the lower stratum. The main vertical displacements will therefore be those in the internal boundary surface. At the free surface the vertical elevations will only give a small negative image of those in the internal surface. This more slowly propagating wave is therefore not so perspicuously seen from without, and may be called the *internal wave*.

12. To find how the velocities of propagation vary according to the situation of the internal boundary surface, we introduce the quantities Γ_1 and Γ_2 in the quadric (11, a)

(a) $$c^4 - (\Gamma_1 + \Gamma_2)\,c^2 + \Gamma_1\,\gamma_2^{\rm II} = 0$$

and in the root (11, b)

(b) $$c^2 = \tfrac{1}{2}(\Gamma_1 + \Gamma_2) \pm \tfrac{1}{2}\sqrt{(\Gamma_1 + \Gamma_2)^2 - 4\,\Gamma_1\,\gamma_2^{\rm II}}$$

where we have to remember that

(c) $$\Gamma_1 = (\alpha_2^{\rm I} - \alpha_2^{\rm II})(p_2 - p_1)\,, \qquad \Gamma_2 = \alpha_3^{\rm II}(p_3 - p_1)\,, \qquad \gamma_2^{\rm II} = \alpha_3^{\rm II}(p_3 - p_2)$$

First let the internal boundary surface coincide with the free surface $p_2 = p_1$. The velocity of propagation for the external wave takes the well known value for a single barotropic stratum, while that for the internal wave becomes zero, corresponding to the fact that this wave disappears:

(d) $$c_1^2 = \Gamma_2 = \alpha_3^{\rm II}(p_3 - p_1)\,, \qquad c_2^2 = 0$$

As now the internal boundary surface moves downwards, we get a stratum of less dense masses above it. As the bottom pressure p_3 is constant and thus the total mass of the system does not vary, the free surface must move upwards according as the internal boundary surface goes downwards, and thus always more mass is transformed to the attenuated form. Thus the total depth of the system increases. During this change Γ_2 remains constant, while Γ_1 increases from its initial value 0, and $\gamma_2^{\rm II}$ decreases from its initial value Γ_2. It follows that both the rational and the irrational part of c^2 increase, but the rational part at the strongest rate, as it must always remain greater than the irrational part. Thus the two velocities of propagation increase from the values (d), taking values comprised between the limits

(e) $$\Gamma_1 + \Gamma_2 > c_1^2 > \Gamma_2\,, \qquad \Gamma_1 > c_2^2 > 0$$

As the internal boundary surface continues its motion downwards and the free surface mounts, the value of c_1^2 always goes on increasing, while c_2^2 passes a maximum and then decreases again. When the internal surface has reached the bottom, $p_2 = p_3$, we again have only a single barotropic stratum, having at its bottom the specific volume

$a_3^{II} + (a_2^{I} - a_2^{II})$. But as a_2^{II} and a_3^{II} are the specific volumes at the upper and the lower boundary of an infinitely thin stratum, they must be like each other so that the sp. volume at the bottom will be a_2^{I} i. e. that sp. volume which the fluid of the upper stratum takes under the pressure p_3 in virtue af the equation $a = f(p)$. The velocity of propagation of the external wave has then reached its maximum value, and that of the internal wave its minimal value zero

(f) $$c_1^2 = (\Gamma_1 + \Gamma_2)_{p_2 = p_3} = a_2^{I}(p_3 - p_1); \qquad c_2^2 = 0$$

These results are equally valid for liquid and for gaseous systems, for the upper stratum may be a gas or a vapour, and the lower a liquid. The only necessary condition is that we must have a barotropic state within each stratum.

If in case of two gaseous strata we wish to introduce absolute temperature ϑ instead of sp. volume in the gas equation $pa = R\vartheta$ we simply have for the quantities Γ_1, Γ_2 and γ_2^{II}

(g) $$\Gamma_1 = (\vartheta_2^{I} - \vartheta_2^{II})\left(1 - \frac{p_1}{p_2}\right), \quad \Gamma_2 = \vartheta_2^{II}\left(1 - \frac{p_1}{p_3}\right), \quad \gamma_2^{II} = \vartheta_3^{II}\left(1 - \frac{p_2}{p_3}\right)$$

ϑ_2^{I} and ϑ_2^{I} being the temperatures above and below the internal boundary surface, and ϑ_3^{II} the temperature at the bottom.

13. In the case of homogeneity and incompressibility, when a_1 and a_2 are simply the sp. volumes of the two strata, the quantities γ defined in (10, b) may be interpreted as differences of potential:

(a) $$\gamma_1^{I} = \Phi_1 - \Phi_2 \qquad \gamma_2^{I} = \frac{a_2}{a_1}(\Phi_1 - \Phi_2) = \frac{\varrho_1}{\varrho_2}(\Phi_1 - \Phi_2)$$
$$\gamma_2^{II} = \Phi_2 - \Phi_3 \qquad \gamma_1^{I} + \gamma_2^{II} = \Phi_1 - \Phi_3$$

The quadric (11, a) then becomes

(b) $$c^4 - (\Phi_1 - \Phi_3)c^2 + \left(1 - \frac{\varrho_1}{\varrho_2}\right)(\Phi_1 - \Phi_2)(\Phi_2 - \Phi_3) = 0$$

and its roots

(c) $$c^2 = \tfrac{1}{2}(\Phi_1 - \Phi_3) \pm \tfrac{1}{2}\sqrt{(\Phi_1 - \Phi_3)^2 - 4\left(1 - \frac{\varrho_1}{\varrho_2}\right)(\Phi_1 - \Phi_2)(\Phi_2 - \Phi_3)}$$

which are well known formulae, though they are expressed generally in terms of geometric height instead of in terms of dynamic height or potential, which give the simplest form. The strata being in these formulae defined by differences of potential instead of by differences of pressure, the discussion leads to results differing in a characteristic way from those given in the preceding section. Keeping the pressures p_1 and p_3 at the free surface and at the bottom constant, we have a system of invariable mass, but variable total depth. But keeping constant the potentials Φ_1 and Φ_3, which define free surface and bottom, we have a system with invariable total depth and variable total mass, this mass decreasing as the internal surface of separation moves downwards, and thus a greater part of the given space is filled with the less dense masses and a smaller part with the denser ones.

As the variable potential Φ_2 does not occur in the rational part of the root, only the irrational part of it will vary with Φ_2. In the initial state when $\Phi_2 = \Phi_1$ we have a single barotropic stratum with the density ϱ_2. Only the external wave will exist, and have the velocities of propagation

(d) $$c_1^2 = \Phi_1 - \Phi_3, \qquad c_2^2 = 0$$

As Φ_2 decreases and the internal boundary surface moves downwards, the internal wave motion becomes possible, and gets a velocity of propagation different from zero. Simultaneously the velocity of propagation of the external wave does not increase, as when the system had invariable mass, but decrease. When $\Phi_2 = \frac{1}{2}(\Phi_1 + \Phi_3)$ the velocity of propagation of the external wave passes a maximum, and that of the internal wave a minimum, namely

(e) $$c_1^2 = \tfrac{1}{2}(\Phi_1 - \Phi_3)\left(1 + \sqrt{\frac{\varrho_1}{\varrho_2}}\right) \qquad c_2^2 = \tfrac{1}{2}(\Phi_1 - \Phi_2)\left(1 - \sqrt{\frac{\varrho_1}{\varrho_2}}\right)$$

which in the ultimate case of $\frac{\varrho_1}{\varrho_2} = 0$ become like each other. When then the internal boundary surface continues its motion downwards, c_1^2 increases and c_2^2 decreases again, both returning to the values (d) as soon as $\Phi_2 = \Phi_3$ and thus the system is again a single barotropic stratum, now with the density ϱ_1.

14. At all atmospheric or oceanic surfaces of discontinuity the difference of specific volume $\alpha_2^{\mathrm{I}} - \alpha_2^{\mathrm{II}}$ is small compared to the values α_2^{I} and α_2^{II} of these volumes themselves. This makes Γ_1 small compared to Γ_2, and using this in the formula (12, b) we easily derive simple approximation formulae for the two roots c_1^2 and c_2^2.

These useful approximation formulae may also be derived directly from the equation (12, a), a circumstance which is of some interest in connection with analogous properties of the higher equations which determine the velocities of propagation in case of a greater number of strata. Setting first $\Gamma_1 = 0$ in the equation (12, a) we find Γ_2 as approximate value of the greather root. But as then the last term of the equation is the product of the two roots, we find $\frac{\Gamma_1 \gamma_2^{\mathrm{II}}}{\Gamma_2}$ as corresponding approximate value of the smaller root. Thus for the greater root c_1^2 and the smaller root c_2^2 we find respectively

(a) $$c_1^2 = \Gamma_2, \qquad c_2^2 = \frac{\Gamma_1 \gamma_2^{\mathrm{II}}}{\Gamma_2}$$

If in these formulae we introduce the expressions of the quantities Γ and γ in terms of pressures and sp. volumes we find

(b) $$c_1 = \sqrt{\alpha_3^{\mathrm{II}}(p_3 - p_1)}, \qquad c_2 = \sqrt{\frac{(\alpha_2^{\mathrm{I}} - \alpha_2^{\mathrm{II}})(p_3 - p_2)(p_2 - p_1)}{p_3 - p_1}}$$

Introducing the values of the same quantities in terms of pressures and temperatures for a gaseous system we find

(c) $$c_1 = \sqrt{R\,\vartheta_3^{\mathrm{II}}\left(1 - \frac{p_1}{p_3}\right)}, \qquad c_2 = \sqrt{R\,\frac{(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}})\left(1 - \frac{p_1}{p_2}\right)\left(1 - \frac{p_2}{p_3}\right)}{1 - \frac{p_1}{p_3}}}$$

The corresponding formulae for an incompressible liquid system in terms of potentials of gravity and densities are

(d) $$c_1 = \sqrt{\Phi_1 - \Phi_3}, \qquad c_2 = \sqrt{\frac{\left(1 - \frac{\varrho_1}{\varrho_2}\right)(\Phi_1 - \Phi_2)(\Phi_2 - \Phi_3)}{\Phi_1 - \Phi_3}}$$

From the first approximation formulae we can easily pass to second approximations. A second approximation of the great root is found by subtracting the small root from the coefficient of c^2 in the quadric (12, a)

(d')
$$c_1^2 = \Gamma_1 + \Gamma_2 - \frac{\Gamma_1 \gamma_2^{\mathrm{II}}}{\Gamma_2} = \Gamma_2 \left(1 - \Gamma_1 \frac{\Gamma_2 - \gamma_2^{\mathrm{II}}}{\Gamma_2^2}\right)$$

Thus we find as explicit expressions of the greater root

(e)
$$c_1 = \sqrt{a_3^{\mathrm{II}}(p_3 - p_1)} \left[1 + \tfrac{1}{2} \frac{a_2^{\mathrm{I}} - a_2^{\mathrm{II}}}{a_3^{\mathrm{II}}} \left(\frac{p_2 - p_1}{p_3 - p_1}\right)^2\right]$$

when we use the variables a and p, and

(f)
$$c_1 = \sqrt{R\vartheta_3^{\mathrm{II}}\left(1 - \frac{p_1}{p_3}\right)} \left[1 + \tfrac{1}{2} \frac{\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}}}{\vartheta_3^{\mathrm{II}}} \frac{p_2}{p_3} \left(\frac{1 - \frac{p_1}{p_2}}{1 - \frac{p_1}{p_3}}\right)^2\right]$$

if we use ϑ and p. The corresponding second approximation formula for c for the incompressible liquid systems in terms of potentials of gravity will be

(g)
$$c_1 = \sqrt{\Phi_1 - \Phi_3} \left[1 - \tfrac{1}{2}\left(1 - \frac{\varrho_1}{\varrho_2}\right) \frac{(\Phi_1 - \Phi_2)(\Phi_2 - \Phi_3)}{\Phi_1 - \Phi_3}\right]$$

The corresponding second approximation formulae for the smaller velocity of propagation call for less interest. Evidently they are obtained by multiplying the first approximation value of c_2 by the factor

(h)
$$1 - \tfrac{1}{2} \frac{a_2^{\mathrm{I}} - a_2^{\mathrm{II}}}{a_3^{\mathrm{II}}} \left(\frac{p_2 - p_1}{p_3 - p_1}\right)^2$$

or by the corresponding factor expressed in the other variables.

When we apply the formulae (e) to calculate numerically the velocity of propagation of atmospheric waves, we can introduce $p_1 = 0$, not because we know that pressure is zero at the limit of the atmosphere — if such a limit exists — but because a small value of p_1 or the value zero will make no sensible difference numerically. We then get the simpler formulae

(i)
$$c_1 = \sqrt{R\vartheta_3^{\mathrm{II}}} \qquad c_2 = \sqrt{R(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}})\left(1 - \frac{p_2}{p_3}\right)}$$

For the external wave we get then the Newtonian velocity of sound for the temperature $\vartheta_3^{\mathrm{II}}$ existing at sea level, as if the entire atmosphere formed a single barotropic stratum with this potential temperature $\vartheta_3^{\mathrm{II}}$ at all levels. The second approximation formula (f) for c_1 will for $p_1 = 0$ be

(j)
$$c_1 = \sqrt{R\,\vartheta_3^{\mathrm{II}}} \left(1 + \tfrac{1}{2} \frac{\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}}}{\vartheta_3^{\mathrm{II}}} \frac{p_2}{p_3}\right)$$

The formula (i) for velocity of propagation c_2 of the internal wave has been tabulated in a previous paper[1]). While it may be applied with safety for finding concrete numerical data, it is adviseable to return to the complete formula, where p_1 has not yet disappeared, in all general discussions. It is seen for instance that according to (i) we find for $p_2 = 0$

(k)
$$c_2 = \sqrt{R(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}})}$$

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[1]) V. Bjerknes: Dynamics of the circular vortex, Geof. Publ. Vol. II, No. 4, p. 28.

while from (c) we get $c_2 = 0$ when we first introduce $p_1 = p_2$ and then let p_1 and p_2 converge to zero conjointly. Or while according to (i) c decreases for decreasing values of p, reaching the maximum value (k) when $p_2 = 0$, it will according to (c) reach this maximal value when $p_2 = \sqrt{p_1 p_3}$. And it will take the value 0 when the pressures $p_2 = p_1$ conjointly converge to zero. Thus if questions connected with the existence or non-existence of an atmospheric boundary surface of pressure zero should be discussed, the general formulae containing the pressure p should be used.

No difficulties of this kind arise for the discussion of the internal wave motions in the sea. To tabulate their velocity of propagation according to (d) we may first tabulate the quantity

$$\text{(l)} \qquad D = \sqrt{\frac{(\Phi_1 - \Phi_2)(\Phi_2 - \Phi_3)}{\Phi_1 - \Phi_3}}$$

as function of the total dynamic depth $\Phi_1 - \Phi_3$ and the dynamic depth $\Phi_1 - \Phi_2$ of the surface of discontinuity below the surface. Then the expression for c_2 may be written

$$c_2 = D\left(1 - \tfrac{1}{2}\frac{\varrho_1}{\varrho_2}\right)$$

and this quantity may be tabulated as function of the auxiliary quantity D and the ratio of densities $\frac{\varrho_1}{\varrho_2}$ at the two sides of the surface.

15. The developed formulae for the case of two strata allow us to make some general remarks on atmospheric and oceanic wave motions, and simultaneously to direct attention to certain open questions which may be solved by the examination of wave motions in case of three or more strata.

When the velocity of propagation c_1 of an external quasi static wave in the sea, such as an earthquake wave, be calculated simply from the total depth $\Phi_1 - \Phi_2$ according to the formula (14, d) we obtain a value which is slightly too high. For the sea is always stratified, and considering it in the first approximation as made up of two strata we find according to formula (14, g) the corrected value in multiplying by the factor

$$1 - \tfrac{1}{2}\left(1 - \frac{\varrho_1}{\varrho_2}\right)\frac{(\Phi_1 - \Phi_2)(\Phi_2 - \Phi_3)}{\Phi_1 - \Phi_3}$$

As the ratio $\frac{\varrho_1}{\varrho_2}$ is always very near unity, the correction is exceedingly small. By the same value of the difference of density the correction will be maximum when the surface of discontinuity is just in the middle between the surface and the bottom, and it will converge to zero as the surface of discontinuity approaches the surface or the bottom.

In the same manner the velocity of propagation c_1 of the external atmospheric waves is underestimated if, according to the first approximation formula (14, i), we identify it with the Newtonian sound velocity at sea level. The stratification of the atmosphere will increase this velocity of propagation, and if as a first approximation we consider the atmosphere as made up of two barotropic strata, we get the corrected velocity of propagation by multiplying by the factor

$$1 + \tfrac{1}{2}\frac{\vartheta_2^{I} - \vartheta_2^{II}}{\vartheta_3^{II}}\frac{p_2}{p_3}$$

It is seen that in case of equal values of the temperature inversion $\vartheta_2^{I} - \vartheta_2^{II}$ the influence upon the velocity of propagation is greater the nearer the surface of discontinuity is to the ground. But even then the influence of the stratification is remarkably moderate, thus only 5 per cent when we have $\vartheta_2^{I} - \vartheta_2^{II} = 27°$ and $\vartheta_3^{II} = 273°$. And if the two barotropic strata were compared with troposphere and stratosphere, we should have $\frac{p_2}{p_3} = \frac{1}{3}$ and could go to a temperature inversion of 80° in order to get an influence of 5 per cent upon the velocity of propagation.

This will be sufficient to show that the influence of the stratification of the atmosphere upon the velocity of propagation of the external wave will be very moderate. But for a more accurate estimate the comparison of the atmosphere with a system of only two barotropic strata would be too rough. It would especially be difficult to say what temperature we should choose at the lower boundary of the stratosphere when this isothermal stratum is to be replaced by an adiabatic one, giving the wave motion in the entire system the same velocity of propagation. But already the formulae for the case of three barotropic strata would here give good information.

We meet with difficulties of the same kind when we apply the formula for the velocity of propagation of the internal wave to the atmosphere and the sea. For the two strata, which we must consider as barotropic, are in reality always more or less stratified. Evidently we may compensate for this stratification by an artificial increase of the discontinuity. How great this artificial exaggeration of the discontinuity should be, however, it is difficult to say beforehand. But already the solution of the problem of three strata would give useful hints for this question.

Another difficulty lies in the circumstance that the discontinuity is never sharp. In reality we always have a transitional layer instead of a surface of discontinuity. This may occasion doubt as to how we should extrapolate from the data of the transitional layer to the equivalent surface of discontinuity. Even for the examination of this question the formulae for the case of three barotropic strata will be useful.