THE FIGURE OF THE EARTH

BY *W. WERENSKIOLD*

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1. Both geodetic measurements and theoretical considerations lead to the result that the figure of the earth is represented by an ellipsoid of revolution, to a close approximation; in fact, if therms of the second order are omitted the correspondence is perfect. But discrepancies occur which are not easily explained. The various geodetic measurements give different values of the dimensions of the meridian ellipse. By including the terms of second order in the calculation, one ought to get a better coincidence. But it will then prove impossible to find one single ellipsoid of revolution that fits in best with all geodetic measurements; on the contrary, the dimensions of the "best" ellipsoid are dependent on the latitude. The ellipsoid which has the same axes a and b as the earth's spheroid, may be called the principal ellipsoid.

2. The ellipsoid is represented by the following equation, which is correct to the fourth order:

$$r = a\ [1 - (\alpha + \tfrac{3}{2}\alpha^2)\ \sin^2 v + \tfrac{3}{2}\alpha^2 \sin^4 v] \qquad (1)$$
$$= a\ [1 - \alpha\ \sin^2\ v - \tfrac{3}{8}\alpha^2\ \sin^2 2v]$$

Here v is the geocentric latitude, α the flattening, r the radius vector, a the equatorial radius, or major semi-axis.

By putting:

$$A = -\alpha - \tfrac{3}{2}\alpha^2 \qquad (2)$$
$$B = \tfrac{3}{2}\alpha^2$$

the equation (1) may be written thus:

$$r = a\ [1 + A\ \sin^2 v + B\ \sin^4 v] \qquad (3)$$

Here, A is of order 10^{-3}, and B of order 10^{-5}—10^{-6}.

3. We shall now consider a more general case, namely, a spheroid of revolution, defined by the equation:

$$r = a\ [1 + A_1\ \sin^2 v + B_1\ \sin^4 v] \qquad (3')$$

where the coefficients A_1 and B_1 are different from those defined by (2), but still of the first and second order respectively. By neglecting small quantities of higher order, expressions for the radii of curvature of this spheroid can be developed in the regular manner.

The equation (3') can be written in the following form:

$$r = a\ [1 + \tfrac{1}{2}A_1 + \tfrac{3}{8}B_1 - \tfrac{1}{2}(A_1 + B_1)\cos 2v + \tfrac{1}{8}B_1 \cos 4v] \qquad (4)$$

By derivation is obtained:

$$r' = a\,[(A_1 + B_1)\sin 2v - \tfrac{1}{2}B_1 \sin 4v]$$

and:

$$r'' = a\,[2\,(A_1 + B_1)\cos 2v - 2\,B_1 \cos 4v] \qquad (5)$$

The geographic latitude φ is introduced instead of the geocentric latitude v; the connection between these quantities is:

$$v = \varphi + u$$

where u is defined by:

$$tg\,u = \frac{r'}{r}A_1 \sin 2v = A_1 \sin 2\varphi$$

disregarding quantities of order 2. Now let R be the radius of curvature of the meridian and ϱ the radius of curvature in the prime vertical, then these quantities are defined by:

$$R = \frac{[r^2 + (r')^2]^{3/2}}{r^2 + 2\,(r')^2 - r\,r''} \qquad \varrho = r\frac{\cos v}{\cos\varphi}$$

By a somewhat lengthy calculation is obtained:

$$R = a\,[1 + \tfrac{1}{2}A_1 + \tfrac{1}{4}A^2{}_1 + \tfrac{3}{8}B_1 + \tfrac{3}{2}(A_1 + B_1)\cos 2\varphi + \tfrac{15}{8}(2A^2{}_1 - B_1)\cos 4\varphi] \qquad (6)$$

$$\varrho = a\,[1 - \tfrac{1}{2}A_1 + A_1{}^2 - \tfrac{9}{8}B_1 + \tfrac{1}{2}(A_1 - 4A_1{}^2 + 3B_1)\cos 4\varphi + (A_1{}^2 - \tfrac{3}{8}B_1)\cos 4\varphi]$$

This calculation is of course also valid in the event of the spheroid in question being a true ellipsoid, and by substitution of the values of A and B from the equations (2) the above expressions are found to represent the radii of curvature of the ellipsoid (1).

4. We now return to the spheroid:

$$r = a\,[1 + A_1 \sin^2 v + B_1 \sin^4 v]$$

If the coefficients A_1 and B_1 are slightly altered, the radii of curvature will be correspondingly altered. We may introduce:

$$A_2 = A_1 + \Delta A_1$$
$$B_2 = B_1 + \Delta B_1$$

By this substitution, the major axis a is unaltered; if the condition is introduced that the minor axis b also is to retain its value, then obviously:

$$\Delta A_1 = -\Delta B_1 = c \qquad (7)$$

a quantity of the second order. The resulting variations of the radii are:

$$\Delta R = \tfrac{1}{8}c\,a\,(1 + 15\cos 4\varphi) \qquad (8)$$
$$\Delta\varrho = \tfrac{1}{8}c\,a\,(5 - 8\cos 2\varphi + 3\cos 4\varphi)$$

5. This result can be interpreted in the following way:
The equation of the principal ellipsoid is:

$$r = a\,[1 + A\sin^2 v + B\sin^4 v]$$

and the equation of a spheroid, slightly different, is:

$$r = a\,[1 + (A + c)\sin^2 v + (B - c)\sin^4 v] \qquad (9)$$

Then the difference ΔR, $\Delta \varrho$ between the radii of curvature, R_1, ϱ_1 of the spheroid, and those of the ellipsoid, R, ϱ, is expressed by:

$$R_1 - R = \Delta R$$
$$\varrho_1 - \varrho = \Delta \varrho$$

where ΔR and $\Delta \varrho$ have the values stated above. (8).

Then character of the two trigonometric functions is seen from figure 1. It may be remarked that:

$$5 - 8 \cos 2\varphi + 3 \cos 4\varphi = 2\,(1 - 3 \cos 2\varphi)\,(1 - \cos 2\varphi)$$

6. If the coefficient c has a value different from 0, it is clearly impossible to determine a "standard ellipsoid" for the figure of the earth. But, for each latitude φ it is possible to determine an osculating ellipsoid, having the same radii of curvature as the spheroid — in fact, the "ellipsoid of reference".

This osculating ellipsoid will be somewhat different from the principal ellipsoid (1), that is, the major axis and the flattening will be a little different from the values a and α belonging to the principal ellipsoid.

If we introduce:

$$a_1 = a + \Delta a \quad \text{and}$$
$$\alpha_1 = \alpha + \Delta \alpha$$

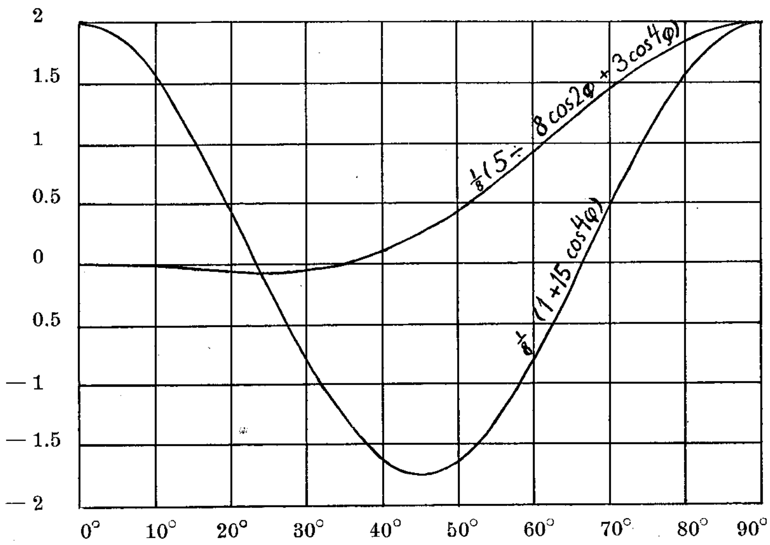

Figure 1. Graphical representation of the functions $\frac{1}{8}$ (1 + 15 cos 3 φ) and $\frac{1}{8}$ (5 — 8 cos 2 φ + 3 cos 4 φ)

the corresponding variation of the radii of curvature of the ellipsoid are found:

$$\Delta R = \Delta a - \tfrac{1}{2}\, a \,\Delta \alpha\, (1 + 3 \cos 2\varphi)$$
$$\Delta \varrho = \Delta a + \tfrac{1}{2}\, a \,\Delta \alpha\, (1 - \cos 2\varphi) \qquad (10)$$

Now the increments Δa and $\Delta \alpha$ which are sufficient to determine the osculating ellipsoid, when the principal ellipsoid is known, are obviously obtained by equalizing the expressions (8) and (10):

$$\Delta a - \tfrac{1}{2}\, a \,\Delta \alpha\, (1 + 3 \cos 2\varphi) = \tfrac{1}{8}\, a\, c\, (1 + 15 \cos 4\varphi) \qquad (11)$$
$$\Delta a + \tfrac{1}{2}\, a \,\Delta \alpha\, (1 - \cos 2\varphi) = \tfrac{1}{8}\, a\, c\, (5 - 8 \cos 2\varphi + 3 \cos 4\varphi)$$

From these equations we get by elimination:

$$2\,(1 + \cos 2\varphi)\,\Delta a = \tfrac{3}{8}\, a\, c\, (-2 + \cos 2\varphi + 2 \cos 4\varphi - \cos 6\varphi)$$
$$2\,(1 + \cos 2\varphi)\,\Delta \alpha = \quad c\, (1 - 2 \cos 2\varphi - 3 \cos 4\varphi)$$

But the expressions can be simplified, because:

$$\cos 6\varphi - 2 \cos 4\varphi - \cos 2\varphi + 2 = 4\,(1 - \cos 2\varphi)^2\,(1 + \cos 2\varphi)$$
$$- 3 \cos 4\varphi - 2 \cos 2\varphi + 1 = 2\,(2 - 3 \cos 2\varphi)\,(1 + \cos 2\varphi)$$

The value $\cos 2\varphi = -1$ makes the problem indefinite. Putting this a side, we obtain:

$$\Delta a = -\tfrac{3}{4}\, a\, c\, (1 - \cos 2\varphi)^2 \qquad (12)$$
$$\Delta \alpha = c\, (2 - 3 \cos 2\varphi)$$

These expressions can be transformed; by putting:

$$a = 1/N$$

is obtained:

$$\Delta N = -\frac{c}{a^2}(2 - 3 \cos 2\varphi) = -c N^2 (2 - 3 \cos 2\varphi) \qquad (13)$$

Also:

$$\Delta a = -3\, a\, c\, \sin^4 \varphi$$

The character of these trigonometric functions is seen from figure 2.

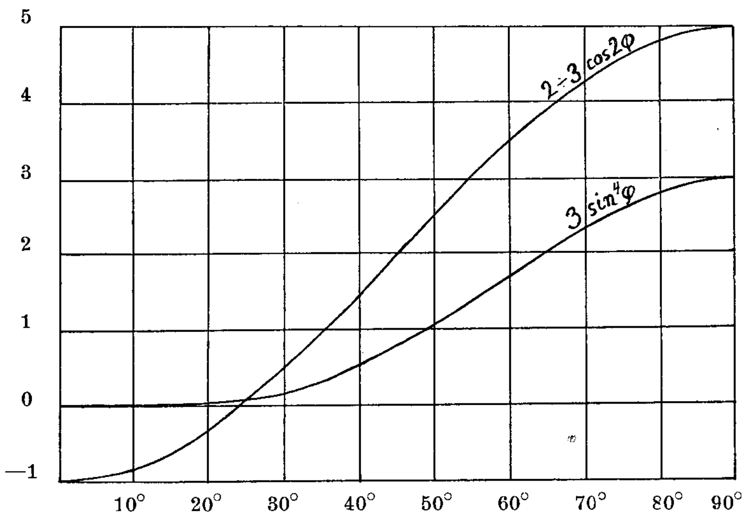

Figure 2. Graphical representation of the functions $(2 - 3\cos 2\varphi)$ and $3 \sin^4 \varphi$

7. We have now found the corrections which must be added to the constants of the principal ellipsoid in order to get the osculating ellipsoid (or ellipsoid of reference). The significance of this correction will depend upon the magnitude of the coefficient c.

According to Helmert, figures of the earth different from an ellipsoid have been computed by Paucker, Bowditch and Clarke. Paucker found an expression for the radius vector of the meridian, dependent upon the second, fourth, and sixth powers of the geocentric latitude. The deviation from the ellipsoid at 45° Lat. was found to be 107 m outwards. Clarke and Bowditch computed the radius of curvature of the meridian section, as a function of the sine of the geographical latitude: $R = A + B s^2 + C s^4$ and $s = \sin \varphi$. The deviation from the ellipsoid at 45° Lat. was found to be 18 m inwards by Bowditch, and 54 m outwards by Clarke.

Helmert (1884, pp. 80—82) has deduced the following equation of the spheroid of the earth:

$$r = a\,[1 - [\alpha(1 + \beta - \alpha) + \delta] \sin^2 v + [\alpha(\beta - \alpha) + \delta] \sin^4 v]$$

where: $\delta = \tfrac{1}{3}(7\alpha^2 - 4\alpha\beta + \beta_2)$

or: $r = a\,[1 - (\alpha + \tfrac{4}{3}\alpha^2 - \tfrac{1}{3}\alpha\beta + \tfrac{1}{3}\beta_2) \sin^2 v + (\tfrac{4}{3}\alpha^2 - \tfrac{1}{3}\alpha\beta + \tfrac{1}{3}\beta_2) \sin^4 v]$

Comparing with the equation of the ellipsoid, the difference c between the coefficients is found:

$$c = \tfrac{1}{6}(\alpha^2 + 2\alpha\beta - 2\beta_2)$$

Here α is the flattening, the coefficients β and β_2 are determined from measurements of gravity.

According to Trabert (1911, p. 315) the best value of the coefficient β_2 is — 0,000007, while $\beta = 0{,}00530$, and $\alpha = 0{,}00336$

From these numbers is found:

$$c = 10{,}15 \cdot 10^{-6} = 0{,}9\,\alpha^2, \text{ and } ac = 64{,}7 \text{ m} \qquad \text{(I)}$$

Jordan (1923, p. 632) computes the following formula for the radius vector of the meridian:

$$r = a\,[1 - \alpha\ (1 - \alpha + \beta)\ \sin^2 v + \alpha\ (\beta - \alpha)\ \sin^4 v)]$$

where:

$$\alpha = 1/296{,}7, \text{ and } \beta = 0{,}00528$$

Then: $c = \frac{5}{2}\,\alpha^2 - \alpha\beta = 10{,}4 \cdot 10^{-6} = 0{,}94\,\alpha^2$, and $ac = 69$ m

Buchholz (1916) puts the equation of the spheroid into the following form:

$$r = a\ [1 - (\alpha + \tfrac{1}{2}\,\alpha^2)\ \sin^2 v + \tfrac{1}{2}\,\alpha^2\ \sin^4 v]$$

I have not seen the book myself, it is here cited from Jankowski (p. 10). This expression leads to the value of:

$$c = \alpha^2 = 11{,}3 \cdot 10^{-6}, \text{ and } ac = 72 \text{ m.}$$

This is very close to the value found from Trabert's numbers.

On the other hand, Jeffreys (1924, p. 192—193) states that the spherical harmonic in $(\frac{1}{3} - \cos^2 v)$ is the only one present in the figure of the earth on the theory of hydrostatic equilibrium. In this case, the equation of the spheroid is simply:

$$r = a\ (1 - \alpha \sin^2 v)$$

and accordingly:

$$c = 1{,}5\,\alpha^2 = 16{,}93 \cdot 10^{-6}, \text{ and } ac = 108 \text{ m.} \qquad \text{(II)}$$

8. The correction to the length of the meridian quadrant is found to be:

$$s = \frac{\pi}{16}\,ac$$

and the greatest vertical distance between the surface of the spheroid and the ellipsoid of equal flattening is:

$$h = \tfrac{1}{4}\,ac$$

With the values (I) is obtained: $s = 12{,}7$ m, and $h = 16{,}2$ m
and with valus (II) $s = 21{,}2$ m, $h = 27$ m

These differences are quite insignificant, but the case is somewhat different when the radii of curvature are considered. In the following table the corrections to the values computed from the ellipsoid are given, for the two hypotheses (I) and (II).

φ	ΔR m I	ΔR m II	$\Delta \varrho$ m I	$\Delta \varrho$ m II	$\Delta R/R$ mm/km I	$\Delta R/R$ mm/km II	$\Delta \varrho/\varrho$ mm/km I	$\Delta \varrho/\varrho$ mm/km II	Δa m I	Δa m II	ΔN I	ΔN II	Δs m I	Δs m II
0°	129	216	0	0	20.3	33.9	0	0	0	0	0.9	1.5	30	30
10°	101	169	— 2	— 3	15.8	26.4	— 0.3	— 0.5	0.2	0.3	0.7	1.2	20.9	34.8
20°	29	49	— 5	— 8	4.6	7.6	— 0.8	— 1.3	2.6	4.5	0.3	0.45	32.7	54.4
30°	— 52	— 88	— 4	— 7	— 8.2	—13.8	— 0.6	— 1.1	12	20	—0.45	—0.75	30.5	50 9
40°	—106	—177	6	11	—16.6	—27.7	1.0	1.7	33	55	—1.3	—2.2	16.0	26.7
50°	—106	—177	29	48	—16.6	—27.7	4.5	7.6	67	112	—2.3	—3.8	— 3.3	— 5.5
60°	— 52	— 88	61	101	— 8.2	—13.8	9.5	15.9	109	182	—3.2	—5.3	—17.8	—29.7
70°	29	49	94	157	4.6	7.6	14.8	24.7	151	253	—3.9	—6.4	—20.0	—33.3
80°	101	169	120	200	15.8	26.4	18.8	13.4	183	305	—4.3	—7.2	— 8.2	—13.6
90°	129	216	129	216	20.3	33.9	20.3	33.9	194	324	—4.5	—7.5	12.7	21.2

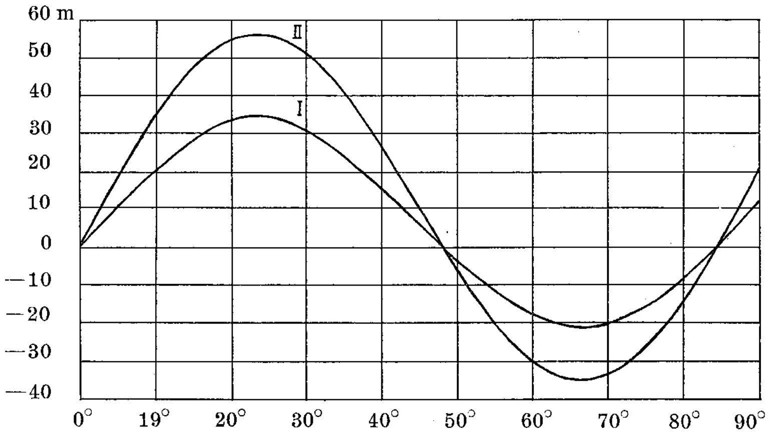

Figure 3. Corrections to the length of an arc of meridian measured from equator.

The first pair of columns gives the corrections to the radius of curvature of meridian section, due to the spheroid's deviation from the principal ellipsoid. The second pair of columns gives the corresponding corrections to the radius of in the prime vertical.

The corrections to the radius of curvature of the meridian are greatest at the equator and the poles, and zero at the arctic and tropic circles, curiously enough; at Lat. 45° the correction attains its greatest negative value. The radius of curvature of section perpendicular to meridian is little different from that belonging to the ellipsoid, from equator to the middle latitudes, but at the poles the correction is of course equal for both radii of curvature.

I have also tabulated the percentage deviations, in order to get an impression as to the resulting disagreement in the geodetic measurements; in fact, the numbers in the third and fourth pairs of columns give the difference in millimetres per kilometre of distances, computed on the surface of the spheroid and the principal ellipsoid. In a north-south direction, these differences are not insignificant, amounting to more than two centimetres per kilometre at equator and the poles, and 1,7 at 45° Lat.; at the arctic and tropic circles a measurement of an arc of meridian will give the correct values of the ellipsoid having the same axes as the spheroid.

In the fifth and sixth pairs of columns are given the corrections which should be added to the axis a and the inverse number of the flattening, $N = {}^1/\alpha$ of the principal ellipsoid, in order to obtain the axis and flattening of the ellipsoid of reference for the latitude in question. The corrections are insignificant for the tropics.

The correction to an arc of meridian from latitude φ_1 to latitude φ_2 ist:

$$\Delta s = \tfrac{1}{8}\, a\, c \left[\frac{\pi}{180}(\varphi_2 - \varphi_1) + \frac{15}{4}(\sin 4\,\varphi_2 - \sin 4\,\varphi_1)\right]$$

$$= \tfrac{1}{8}\, a\, c \left[\frac{\pi}{180}(\varphi_2 - \varphi_1) + \frac{15}{2}\sin 2\,(\varphi_2 - \varphi_1)\cos 2\,(\varphi_2 + \varphi_1)\right]$$

The numbers are given in the last pair of columns, for $\varphi_1 = 0$.

The correction is greatest for $23^1/_3°$ Lat., and amounts to 33,5 or 56 m respectively. This may be of some interest in connection with the tables of lengths of arcs of meridian prepared by Mr. Hinks. See also figure 3.

We may perhaps regard Haydords values as the best ones: $a = 6378388$ and $N = 297$; but these numbers are obviously only valid for the mean latitude of his arcs, 38°. The true major semi-axis of the earth should be: $6378388 - 3\,a\,c\,\sin^4 38°$. According to (I) or (II) the correction is 28 m or 46 m and the true value of a should be: $a =$ (I) 6378360, or (II) 6378342; likewise the inverse of the flattening should be: $N =$ (I) 298 or (II) 299, instead of 297.

Consequently, the ellipsoid fitting best for the latitude of Oslo, 60°, should have the dimensions:

$$a = \text{(I) } 6378469, \text{ or (II) } 6378524, \text{ and } N = \text{(I) } 295, \text{ or (II) } 294.$$

10. The above calculations are correct, but there may be grave doubts as to the true value of the coefficient c[1]. The problem can be solved by a careful comparison between the results of the various measurements of arcs of different latitudes. A difficulty arises from the fact that some of the arcs have been computed from an accepted value of the flattening, namely that determined by Helmert from the measurements of gravity.

I have neither time nor opportunity to make this investigation, but I think it ought to be done. Then perhaps some of the difficulties pointed out by Mc Caw and Hinks, concerning the adoption of a standard figure of the earth, might be reduced to regular consequences of a law of nature.

I am indebted to Jankowski for the suggestion that the flattening is dependent on the latitude. Otherwise, the treatment is quite different from that above, and leads also to different results.

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[1] Jeffreys has simply omitted the second order terms, and Trabert has apparently made a mistake in his computation.