COASTAL CURRENTS

BY

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(Manuscript received 25th January 1935.)

Introduction. In higher latitudes currents of lighter water generally follow the coasts, running with the land to the right on the Northern Hemisphere, to the left on the Southern Hemisphere. The lighter water thus forms a kind of wedge along the coast, and the lines of equal density — isopycnals — or of equal specific volume — isosteres — will be inclined downwards and towards the coast. As a consequence, the isobars, and the actual surface of the sea, will be inclined too, and away from the shore, with a corresponding horizontal pressure gradient, pointing away from the land. The lighter water has a tendency to spread in a sheet over the whole sea, but is kept close to the shore by the deflective force due to the earth's rotation. If now the inclination of the isobars is i, and g the acceleration of gravity, then the horizontal pressure gradient is $i\,g$. The condition for lateral equilibrium is:

$$i\,g = 2\omega v \sin B = \lambda v = 1.46 \times 10^{-4} \sin B . v$$

Here v is the velocity of the current, ω the angular velocity of the earth's rotation, and B the latitude. The friction is disregarded.

This fundamental relation was first used by *Mohn* (1885) in his paper on the Norwegian Sea, and has ever since formed the base for the computation of the velocity of currents due to the unequal distribution of density. The theory has, however, been greatly improved by *Bjerknes*, and the execution of the work has been made easier in the course of time, but the computations are still rather cumbersome. First the density or specific volume must be determined from observations of temperature, salinity and depth, then whole series of summations are necessary in order to determine the exact distribution of pressure, or the slope of the isobaric surfaces. The homogeneous bottom water is supposed to be at rest, with horizontal isobars. Starting from one of these, the pressure at a certain higher level can be found by an integration of the differential equation:

$$dp = - g\varrho dz$$

where ϱ is the density, z the vertical distance. Or the height of an isobaric surface can be determined from the equation:

$$gdz = - \alpha dp$$

where α is the specific volume. Both procedures require a considerable amount of somewhat tedious work.

If however the density has been determined, and a set of isopycnals — or isosteres — has been drawn on a section across the current, it is possible to determine the velocity from the sloppe of these curves.

The velocity of coastal currents. We will consider a vertical section across a coastal current. The lighter water runs along the coast; the isopycnals are inclined towards the coast, the isobars away from the same. The density of the water is ϱ, the inclination of the isopycnals is j, the inclination of the isobars is i. At the boundary between two layers of density ϱ_n and ϱ_{n+1} we have: (Fig. 1.)

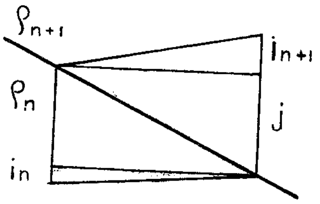

Fig. 1.

$$(i_{n+1} + j)\,\varrho_{n+1} = (i_n + j)\,\varrho_n$$

or:

$$i_{n+1}\,\varrho_{n+1} - i_n\varrho_n + (\varrho_{n+1} - \varrho_n)\,j = 0$$

that is:

$$\triangle\,(i\varrho) + j\triangle\varrho = 0$$

If the density varies continuously, we obtain:

$$d\,(i\varrho) + jd\varrho = 0$$

This is valid for the variations along a vertical line.[1])
By summation we get:

$$i\varrho - i_0\varrho_0 = -\int_{\varrho_0}^{\varrho} jd\varrho$$

or:

$$i\varrho - i_0\varrho_0 = \Sigma j\triangle\varrho$$

The density allways decreases upwards. The bottom water is supposed to be at rest, and $i_0 = 0$. Accordingly:

$$i\varrho = -\int_{\varrho_0}^{\varrho} j\mathrm{d}\varrho$$

or:

$$i\varrho = \Sigma j\triangle\varrho$$

In the general case the inclination j will vary with the depth. On the oceanographic sections the mass distribution is represented by isopycnals for some constant interval δ. We then have:

$$i\varrho = \delta\,\Sigma j$$

The condition for lateral equilibrium is, that the horizontal pressure gradient ig is equal to the Coriolis force:

$$\lambda v = ig$$

If now $u = \varrho v$, the specific momentum, we obtain:

$$u = \frac{g\delta}{\lambda}\,\Sigma j$$

For the bottom water, $v_0 = u_0 = 0$.

There is no corresponding expression for the velocity. But, as a matter of fact, the density will not vary more than about 1/1000 in the open sea, and with sufficient approximation we can put: $u = \varrho_0 v$, and:

$$v = \frac{g\delta}{\lambda\varrho_0}\,\Sigma j$$

On many oceanographic sections, however, the mass distribution is represented by isosteres, curves of equal specific volume, with interval δ. From the formula:

$$\triangle\,(i\varrho) + j\triangle\,\varrho = 0$$

we obtain:

$$a\triangle i = (i + j)\,\triangle a$$

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[1]) Applied to the atmosphere, this relations leads to well known formulas, for the stability of motion at a surface of discontinuity *(Margules)* and for the change of horizontal pressure gradient with height *(Sir Napier Shaw)*.

Now, the values of the inclination i are very small compared with the values j; also, the variation of the specific volume α is of order 1/1000 in the open sea; accordingly we put:

$$\triangle i = \frac{1}{\alpha_0} j \triangle \alpha$$

and by summation:

$$i = \frac{\delta}{\alpha_0} \Sigma j$$

and further:

$$v = \frac{g\delta}{\lambda \alpha_0} \Sigma j$$

On the oceanographic sections the isopycnals and isosteres are often arranged in a strikingly parallel pattern, so that the vertical distance between two and the same curves is nearly constant. If this is exactly fulfilled, the inclination j is the same at all points along a vertical line. In this case the relation:

$$d\,(i\varrho) + j d\varrho = 0$$

can be integrated directly:

$$i\varrho = j\,(\varrho_0 - \varrho)$$

or in another form:

$$(i + j)\,\varrho = j\,\varrho_0$$

which can be found immediately from a figure. (Fig. 2). If the specific volume is introduced, we get:

$$i = \frac{\alpha - \alpha_0}{\alpha_0} j$$

In this case, the following expressions are exactly valid:

$$u = \frac{g}{\lambda} j\,(\varrho_0 - \varrho)$$

$$v = \frac{g}{\lambda \alpha_0} j\,(\alpha - \alpha_0)$$

These expressions will be of use later on.

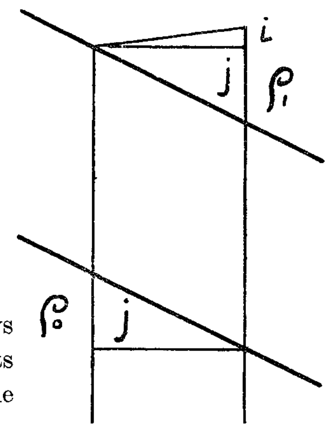


Fig. 2.

On the oceanographical sections, the vertical scale is always greatly exaggerated, but as the inclination j is measured by its tangent, all values j are multiplied by the same factor. If the proportion vertical scale/horizontal scale is n, then we have:

$$j = J/n$$

where J is the tangent measured on the diagram. We then ultimately obtain:

$$v = \frac{g\delta}{n\lambda\varrho_0} \Sigma J \qquad \text{(isopycnals)}$$

or:

$$v = \frac{g\delta}{n\lambda\alpha_0} \Sigma J \qquad \text{(isosteres)}$$

We will consider two examples.

First, let the density distribution be shown by isopycnals with interval 10^{-4}. The vertical scale is 1:5000, the horizontal scale is 1 : 3 millions. The proportion n is 600. Moreover, the latitude is 63°. — This corresponds to the Stad Sections in: *Helland Hansen & Nansen,* The Norwegian Sea. — We then have:

$$v = \frac{g\delta}{n\lambda\varrho_0} \Sigma J = \frac{9.82 \times 10^{-4}}{600 \times 1.46 \times 10^{-4} \times 1.028 \times \sin B} \Sigma J \text{ (m/sec)}$$

$$= \frac{1.09/\Sigma J}{\sin B} \text{ (cm/sec)} = 1.22\ \Sigma J \text{ (cm/sec)}$$

Secondly, the mass distribution is shown by isosteres for intervals of 5×10^{-5}. The vertical scale is 1 : 2500, the horizontal scale is 1 : 185 200; the number n accordingly will be 74.08. The latitude B is 68° 20′. This corresponds to a section prepared at the Bureau of Fisheries, Bergen. — We obtain:

$$v = \frac{g\delta}{n\lambda a_0} \Sigma J = \frac{9.82 \times 5 \times 10^{-5} \times 1.028}{74.08 \times 1.56 \times 10^{-4} \times \sin B} \Sigma J \text{ (m/sec)}$$

$$= \frac{4.67}{\sin B} \Sigma J \text{ (cm/sec)} = \underline{5.02\ \Sigma J} \text{ (cm/sec)}$$

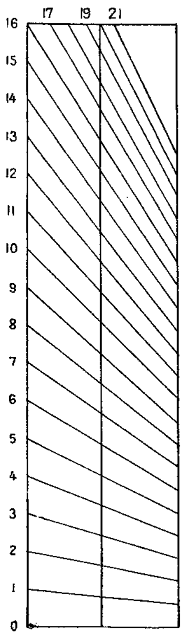


Fig. 3.
Scale of tangents for the determination of slope.

The inclination J of the isopycnals (or isosteres) is measured with a scale on a slip of transparent paper (Fig. 3), along a vertical line, and the numbers are written in place. The velocity differences are found by means of a slide-rule. Then the velocity differences are added together from below, and the sums written in the spaces between the isopycnals. These numbers are the velocities. When first the factor before the summation sign has been determined, the rest is quickly done. (Fig. 7—10 and 11—14, pp. 8—9.)

Some remarks on the sections. The Norwegian Atlantic current might be supposed to run northwards along the coast, with minor irregularities. Such is also the case on the 1903 sections; the 1904 sections however show quite different conditions: the current runs northwards and southwards in stripes. This fact has been noticed by Helland-Hansen and Nansen (Norwegian Sea, pp. 118—128, 152—170). For the Stad and Lofoten sections, the velocities have been calculated according to the method based upon Bjerknes' circulation theorem. From data given by the authors I have drawn a figure showing the distribution of velocity on the Lofoten section 1904. (Fig. 4). There is a close correspondence between the results of the classical method and that presented in this paper. The cause of the peculiar arrangement of the currents, in this sections as well as in others, has been discussed at some length by the said authors. The strongly marked divide some 350 km outwards from the Lofoten coast, between southwards and northwards currents, coincides with the edge of the Helgeland ridge, which here slopes down from a dept of 2000 to a depth of 3000 metres.

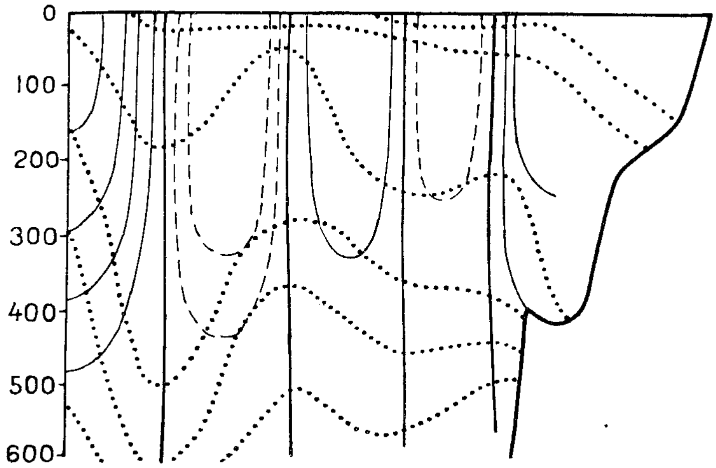


Fig. 4. Lofoten 1904
Compare fig. 14.

The water transport of a coastal current. When the velocity has been found for each point of the section, the total transport can be found by numerical summation or graphical integration. This is rather irksome work, too but the result can be found by an easier method.

The current is now supposed to run along a vertical wall which shall represent the coast. A coordinate system is introduced, with the Z-axis along the wall, and positive downwards. The X-axis is horizontal, along the surface, and positive outwards. The bottom water of density ϱ_0 is at rest.

Firstly, let us consider an ideal case. The whole mass of water of the current shall be homogeneous, and the boundary against the underlying bottom water is a straight line on the section. (Fig. 5). The density of the lighter water is ϱ_1, and the inclination of the boundary line, j. Then we have:

$$u = -\frac{g}{\lambda} j \triangle \varrho = \frac{g}{\lambda}(\varrho_0 - \varrho_1) j$$

The specific momentum u is constant over the whole area of the section of the current. The total mass transport is,

$$U = \tfrac{1}{2} XZu = \frac{g}{2\lambda}(\varrho_0 - \varrho_1) jXZ$$

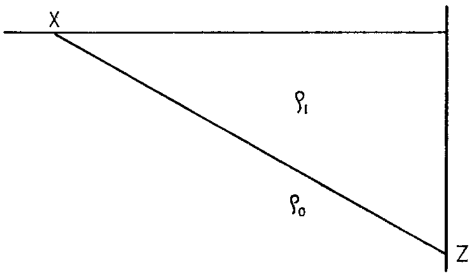


Fig. 5.

But now, $jX = Z$, and ultimately we get:

$$U = \frac{g}{2\lambda}(\varrho_0 - \varrho_1) Z^2$$

The transport is accordingly only dependent upon the depth of the current, the lateral dimensions have no influence.

Now, let the boundary be curved; the inclination j is a function of the coordinate x. The water of the current is still homogeneous, with density ϱ_1 (Fig. 6). At a point at the boundary we have:

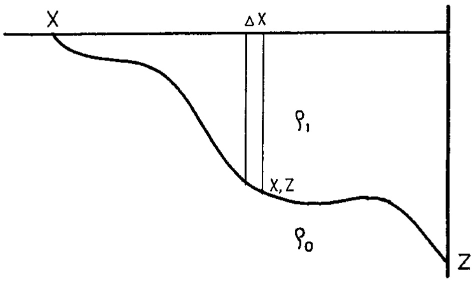


Fig. 6.

$$u = \frac{g}{\lambda}(\varrho_0 - \varrho_1) j$$

and the specific momentum is the same at all points along the same ordinate, above the boundary curve. The transport through a stripe with height z and breadth $\triangle x$ is:

$$\triangle U = uz \triangle x = \frac{g}{\lambda}(\varrho_0 - \varrho_1) jz \triangle x$$

But along the boundary curve, we have:

$$j \triangle x = \triangle z$$

and then:

$$\triangle U = \frac{g}{\lambda}(\varrho_0 - \varrho_1) z \triangle z$$

or in differential form:

$$dU = \frac{g}{\lambda}(\varrho_0 - \varrho_1) zdz$$

By integration we obtain:

$$U = \frac{g}{2\lambda}(\varrho_0 - \varrho_1) Z^2$$

Now, let the water of the current be built up of layers of density $\varrho_0, \varrho_1 \ldots\ldots\ldots$ etc. A vertical column of the section, from the surface to the bottom water, will then be divided into smaller elements by the various boundaries between the water layers, crossing at depths $z_0, z_1, \ldots\ldots$ etc. with inclinations $j_0, j_1, \ldots\ldots$ etc.

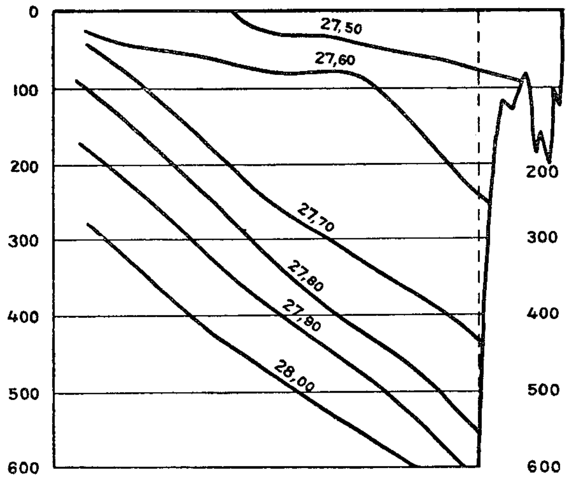


Fig. 7. Stad 1903

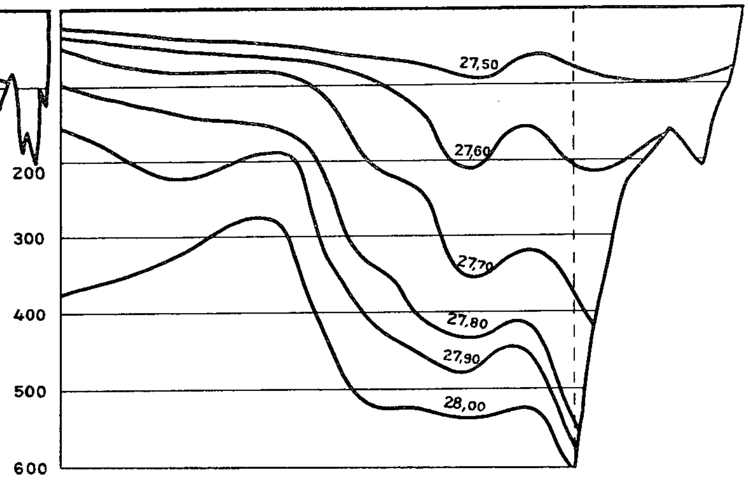


Fig. 8. Stad 1904

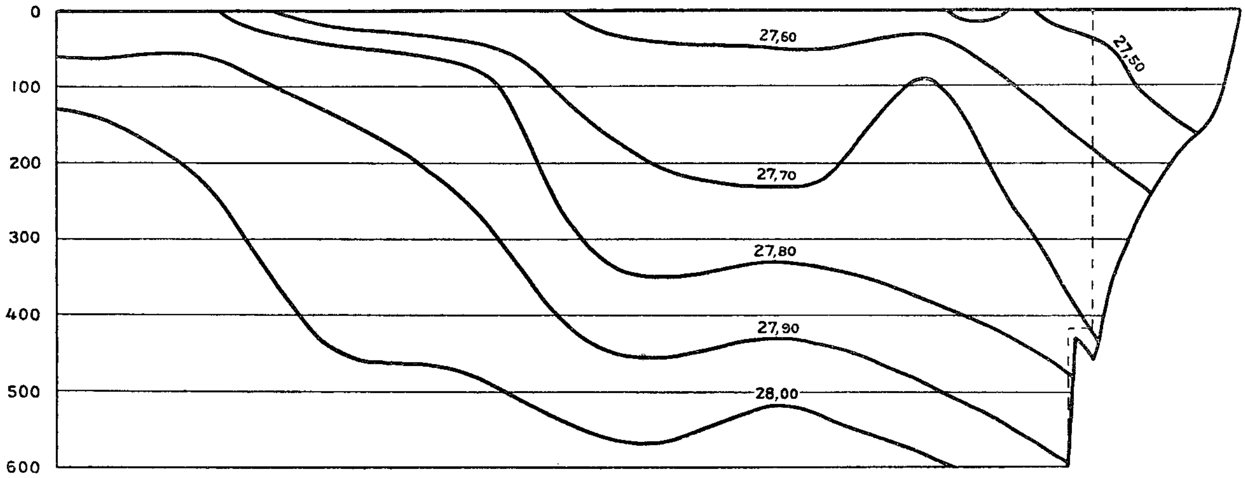


Fig. 9. Lofoten 1903

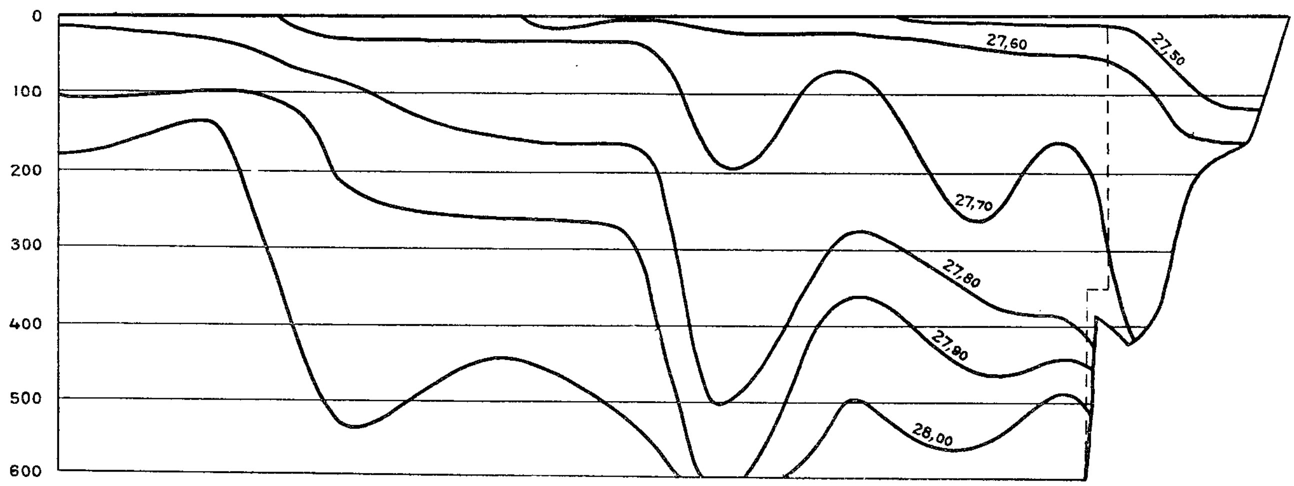


Fig. 10. Lofoten 1904

Figures 7, 8, 9, 10. Sections across the Norwegian coastal current. Horizontal scale, 1: 6000000; vertical scale, 1: 10000. Redrawn from Helland Hansen & Nansen, The Norwegian Sea, Pls. XXI A, and XXIV A.

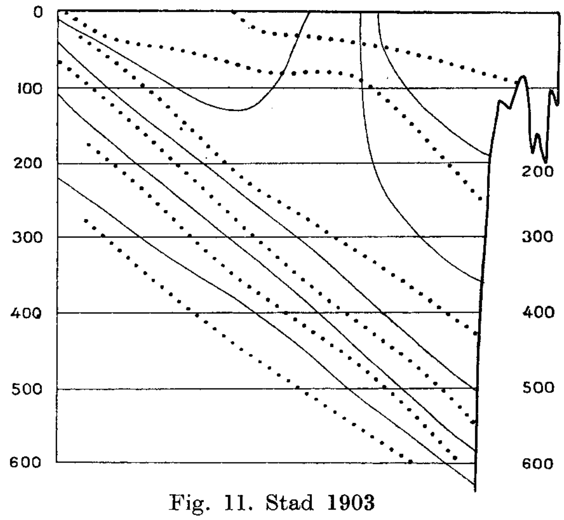


Fig. 11. Stad 1903

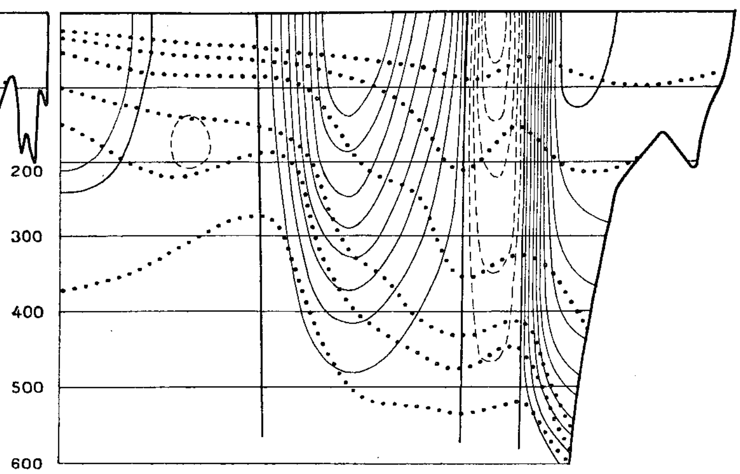


Fig. 12. Stad 1904

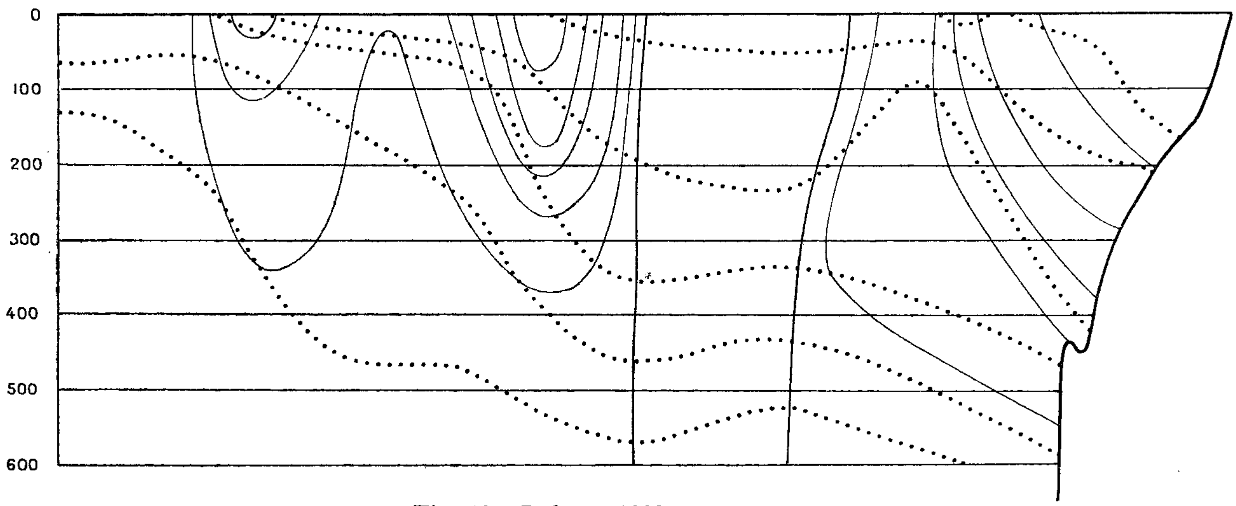


Fig. 13. Lofoten 1903

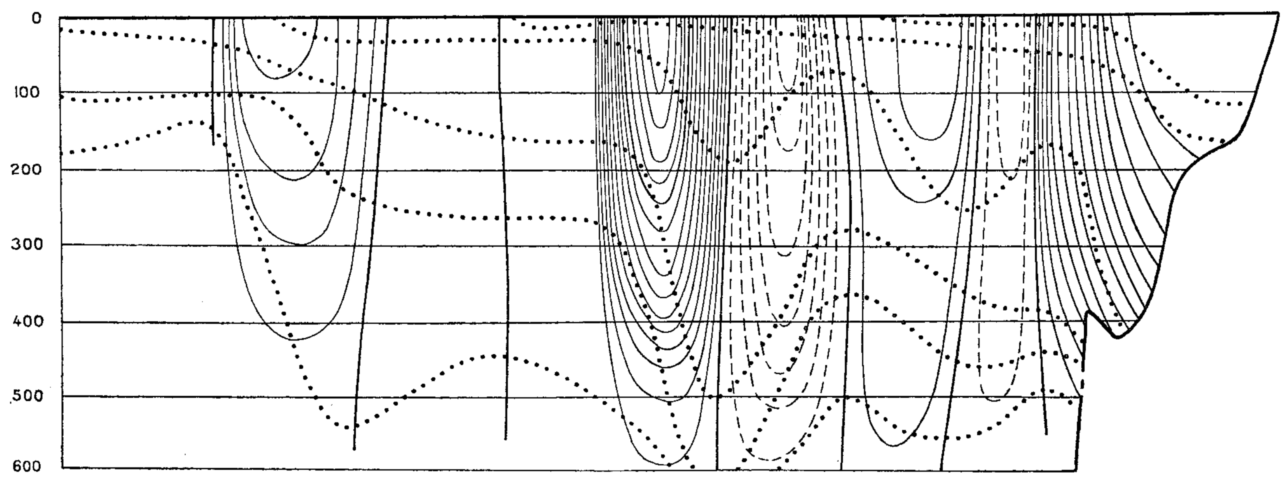


Fig. 14. Lofoten 1904.

Figs. 11, 12, 13, 14. Distribution of velocity. Contour lines with interval 1 cm pr. sec. Full lines, current running northwards. Broken lines, current running southwards. Isopycnals shown by dotted curves.

(Fig. 15). The specific momentum of the water in each layer is $u_0, u_1, \ldots\ldots$ Then the mass transport through the column is:

$$\begin{aligned}\triangle U &= \triangle x\,[u_1(z_0 - z_1) + u_2(z_1 - z_2) + \ldots\ldots] \\ &= \triangle x\,[z_0(u_1 - u_0) + z_1(u_2 - u_1) + \ldots\ldots] \\ &= \triangle x\,[z_0 \triangle u_0 + z_1 \triangle u_1 + \ldots\ldots]\end{aligned}$$

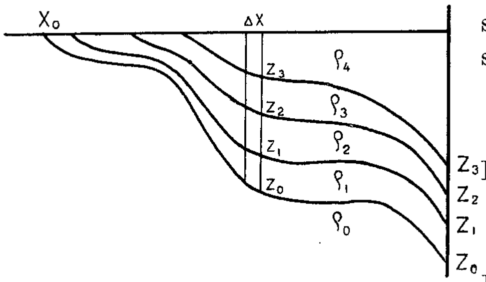


Fig. 15.

For the bottom water $u_0 = 0$. The difference in specific momentum between two layers in the same vertical is:

$$\triangle u = \frac{g}{\lambda} j \triangle \varrho$$

Introducing this, we have:

$$\triangle U = \frac{g \triangle x}{\lambda} [z_0 j_0 \triangle \varrho_0 + z_1 j_1 \triangle \varrho_1 + \ldots\ldots]$$

By a summation with respect to x we obtain the total transport. But along each boundary curve, we have:

$j \triangle x = \triangle z$

and from this we get:

$$\triangle U = \frac{g}{\lambda} [z_0 \triangle z_0 \triangle \varrho_0 + z_1 \triangle z_1 \triangle \varrho_1 + \ldots\ldots]$$

Under the summation, all the density differences are constant. Moreover,

$$\Sigma z \triangle z = \tfrac{1}{2} Z^2$$

where Z is the depth of the boundary line in question at the wall. Then:

$$U = \frac{g}{2\lambda} [Z_0^2 \triangle \varrho_0 + Z_1^2 \triangle \varrho_1 + \ldots\ldots]$$

or:

$$U = \frac{g}{2\lambda} \Sigma Z^2 \triangle \varrho$$

If now the variation of the density is continous, we obtain;

$$U = \frac{g}{2\lambda} \int Z^2 d\varrho$$

This expression can be transformed. as:

$$\int Z^2 d\varrho = Z_0^2 \varrho_0 - \int \varrho \, dZ^2$$

and we ultimately get:

$$\underline{U = \frac{g}{2\lambda} \int (\varrho_0 - \varrho) \, dZ^2}$$

The integral represents the moment of a vertical rod with respect to its upper end at the surface, and with a mass distribution as given by the values of the density differences in the section along the wall.

No corresponding, exact expression can be found for the volume transport V, but, as before, we can with a very close approximation put $U = \varrho_0 V$, and:

$$V = \frac{g}{2\lambda\varrho_0} \int (\varrho_0 - \varrho) \, dZ^2$$

If the mass distribution is shown by isosteres, we can put:

$$\triangle v = \frac{g}{\lambda \alpha_0} \triangle i$$

and by reasoning along the same lines as above, we obtain the expression:

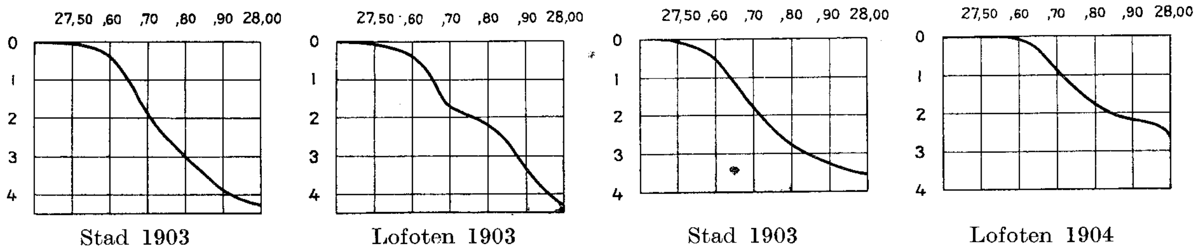
$$V = \frac{g}{2\lambda\alpha_0} \int (\alpha - \alpha_0)\, dZ^2$$

which is exact to an approximation of one or two tenths of one per cent.

In the special case mentioned before, i. e when the isopycnals and isosteres are arranged in a parallel pattern, this last expression is also strictly valid.

Thus it should be possible to compute the total quantity of water carried by a coastal current from measurements and determinations of the density at one station alone, near the shore. This seems rather startling, and the theory must be tested on actual cases, where the transport has been calculated in the regular way. A necessary theoretical condition is, that the curve $\varrho = \varrho_0$ (or $\alpha = \alpha_0$) rises to the surface somewhere in the section.

As to the practical execution of the work, some remarks may here be made. For a given distribution of density along a vertical, the moment can be easily found by a graphical method. On a millimeter paper the density ϱ is put against the square of the corresponding depth Z. On our diagrams, the vertical scale is so chosen, that one unit corresponds to 10^5 square metres; on the horizontal scale, 1 division corresponds to a difference in density of 10^{-4}. (Fig.s 16—19).



Figures 16, 17, 18, 19. Diagrams for the determination of moments. Vertical scale, one unit equals 10^5 square metres. Horizontal scale, $\sigma = 1000\ (\varrho - 1.)$

The value of one square in the diagram thus represents $10^5 \times 10^{-4} = 10$ units. Now, $g/2\lambda\varrho_0 = 3.26 \times 10^4$, and accordingly we get:

$$V = \frac{3.26 \times 10^5 \times A}{\sin B} \text{ cub. m.}$$

where A is the area measured on the diagram.

We shall test the theory upon concrete examples, viz. four oceanographic sections given by Helland-Hansen & Nansen (Norwegian Sea, 1909, Pls. XXI A, May—June 1903, and XXIV A, May—June 1904.) Two of the sections are from Stad and seawards, at Lat. 63°, the other two from the Lofoten Islands toward the west, at 67° N. Lat. The amount of water with salinity above 35 ‰ carried through these sections has been computed by the authors according to the common method, and also, by the present writer, according to the method described above, from one vertical column alone. Of course, the coast and the continental slope do not form a vertical wall, and the isohaline for 35 ‰ does not strictly follow any isopycnal, but corresponds roughly to the line for $\varrho = 1.028$. I have drawn a vertical line near the coast, as it seemed best before beginning the measurements. The coast water with low salinity is practically excluded. The densities with corresponding squares of the depths are shown in the table, and also in figs. 16—19, which should be compared with the sections, figs. 4—7.

	Stad 1903		Stad 1904		Lofoten 1903		Lofoten 1904	
Density	Depth	Square	Depth	Square	Depth	Square	Depth	Square
1.0275	80	6 400	80	6 400	100	10 000	20	400
6	200	40 000	220	48 400	200	40 000	60	3 600
7	440	193 600	420	176 400	420	176 400	300	90 000
8	550	302 500	520	270 400	475	235 625	420	176 400
9	620	384 400	570	324 900	580	336 400	460	213 600
1.0280	660	435 600	600	360 000	660	435 600	510	260 100
Summation		11.4		10.0				
Area		11.25		10.2		8.3		6.3
V mill. cubm.		4.1		3.7		2.9		2.2
H. H. & N.		4.0		3.8		2.7		2.3

The agreement is almost too close, and might arouse suspicion. The water transport in the Lofoten sections has been computed by Helland-Hansen & Nansen for the upper 500 metres; in order to get comparable values. I have cut the diagrams at 250 000. This is not quite exact, but the result seems to justify the method.

It is also possible to calculate the momenta numerically; the result will be almost practically the same. The columns of the squares of the depth are summed up, but of the numbers at top and bottom only one half is taken. We get the values placed immediately above the «Area» in the table. This method is not directly applicable to the Lofoten section, which comprises only the upper 500 metres in Helland-Hansen and Nansen's computation.

If the distribution of density in an oceanographical section is given by lines of equal specific volume, the calculation of the water transport can be carried out in precisely the same way. In our rather rough calculation, no regard has been taken to the change in density due to the different compressibility of the water, but this is of no great importance, as compared with so many other sources of error.

The Stream Line Function. The total transport of the current can thus be found from observations at one station only. But, in the same way, we can determine the transport through the part of a section outside any vertical line from observations at one station only; and further, we can find the transport between any two vertical lines.

If the section cuts the current obliquely, we still get the true transport by the same procedure.

If the function U has been computed for two stations, the difference between the two values will represent the total transport of the current per second through the section between these stations. If two neighbouring stations have the same value of U, then obviously no transport takes place across the connecting section. It follows, that if a line can be drawn on the chart for stations with equal values of U, then the transport of the current will follow the direction of this line.

If the function U has been determined for a great many stations scattered over a part of the sea, it will be possible to draw such contour lines for successive values of U. As the transport through a section between two verticals is equal to the difference in the values of U at the two stations, the transport along a stripe between any two fixed contour lines is constant. Moreover, the momentum $u = \varrho v$

is inversely proportional to the distance between the contour lines. Here is meant the total momentum from the surface to the bottom water. The contour lines for equal values of U will then obviously be stream lines for the transport, integrated from surface to the bottom water.

In a paper from 1929, *V. W. Ekman* came to similar results. (Über die Strommenge der Konvektionsströme im Meere. — Acta Universitatis Lundensis, II, 25.) His line of reasoning is in short as follows. The transport S through a vertical line has the components:

$$S_x = \int_0^H \varrho\, v_x dz$$

$$S_y = \int_0^H \varrho\, v_y dz$$

The Z-axis is pointing downwards, and H is a depth well into the bottom water, which is at rest. If acceleration and friction are disregarded, the equations for the motion are:

$$\frac{\partial p}{\partial x} + \varrho \lambda v_y = 0$$

$$\frac{\partial p}{\partial y} - \varrho \lambda v_x = 0$$

From these relations is obtained:

$$\lambda S_x = \int_0^H \frac{\partial p}{\partial y} dz$$

$$\lambda S_y = -\int_0^H \frac{\partial p}{\partial x} dz$$

If now a function P is introduced, defined by:

$$P = \int_0^H p dz$$

it follows that:

$$\lambda S_x = \frac{\partial P}{\partial y}$$

$$\lambda S_y = -\frac{\partial P}{\delta x}$$

Accordingly, P is the stream line function, well known from theoretical hydrodynamics. If contours are drawn for constant values of P, the transport along a stripe between any two of these lines is constant. Dr. Ekman has also drawn a chart, showing the course of the Gulf Stream along the edge of the Newfoundland Bank.

To these theoretical considerations, the following may be added. If the homogeneous bottom water, with density ϱ_0, is supposed to reach the surface somewhere in the open sea, the value of the function P at this place will be:

$$P_0 = \tfrac{1}{2} g \varrho_0 H^2$$

At a station near the coast, a value P_1 has been computed. Then the total transport of the current between the two stations is

$$U = \frac{1}{\lambda}(P_1 - P_0)$$

There is thus a very close connection between the functions P and U:

$$P = \int_0^H p dz = \int_0^H \left(\int_0^z g\varrho dz \right) dz$$

and $$U = \frac{g}{2\lambda} \int_0^{H^2} (\varrho_0 - \varrho)\, dz^2$$

The expression U seems to be the most convenient one for numerical computation

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Printed March 9, 1935.