# THE WATER TRANSPORT OF GRADIENT CURRENTS

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## Introduction.

In modern physical oceanography the observations of temperature and salinity are treated according to the method of BJERKNES [1910]. The pressure at the standard dynamic depths or, preferably, the dynamic depths of the standard isobaric surfaces are computed, and isobaric charts or dynamic topography charts are drawn. These charts show the stream-lines of the gradients currents, supposing that the friction can be disregarded.

The difference between the velocities of a current at different depths can be easily computed by means of the formula of HELLAND-HANSEN [1905]:

$$v - v_i = \frac{10}{\lambda L}(D'_A - D'_B) \qquad (1)$$

where $v$ is the average velocity component (in *metres* per second) at right angles to the vertical section between two stations, $A$ and $B$, at an upper isobaric surface, and $v_i$ the corresponding quantity at a lower isobaric surface; $D'_A$ and $D'_B$ are the thickness (in dynamic metres) of the water layer between the upper and lower isobaric surfaces in question at the two stations; $L$ is the distance (in *metres*) between the stations. $\lambda = 2\omega \sin\varphi$, where $\omega$ is the angular velocity of the earth's rotation and $\varphi$ the mean geographical latitude of the two stations.

Fig. 1 shows the meaning of $D'$, the thickness of the water layer between the upper and lower isobaric surfaces. When $D$ is the dynamic depth we have

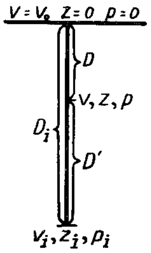

Fig. 1.

$$D = \int_0^p \alpha \, dp$$

where $p$ is the pressure at the dynamic depth $D$; $\alpha$ is the specific volume. With the designations of the figure we obtain:

$$D' = \int_p^{p_i} \alpha \, dp = \int_0^{p_i} \alpha \, dp - \int_0^{p} \alpha \, dp = D_i - D \qquad (2)$$

where $D$ is the dynamic *depth* of an isobaric surface under the sea-surface and $D'$ is the dynamic *height* of the same isobaric surface over the isobaric surface, $p = p_i$, where the velocity is $v_i$. $D_i$ is the dynamic depth of this isobaric surface, $p = p_i$.

The assumptions of HELLAND-HANSEN's formula are that stationary conditions exist and that the friction can be disregarded. It gives the differences between the velocities at the different levels, and if the velocity at one level is known, the velocities at the other levels can be found by means of the formula. The velocities thus found can be used for the construction of vertical sections of velocity. The areas between the different iso-lines of velocity are found *e. g.*, by means of a planimeter and the water transport through the section can thus be computed if the velocity, $v_i$, at the isobaric surface, $p = p_i$, to which the computations are extended, is zero. However, it is more convenient to compute the average velocity component through the section between the two stations and thus find the water transport directly. A further development of this leads to a method for the computation of the water transport of gradient currents which seems to have certain points of advantage over the method of EKMAN [1929] and also includes the method of WERENSKIOLD [1935]; by means of the latter, the water transport of coastal currents can be found on certain assumptions by means of a single station.

## The Volume Transport.

If $\bar{v}$ is the average velocity component between the sea-surface and the depth (in common metres) $z_i$ (where $v = v_i$) at a right angle to the vertical section between the two stations $A$ and $B$, we have:

$$\bar{v}\, z_i = \int_0^{z_i} v\, dz\,.$$

The volume of the water, which flows through the section in the unit of time, is:

$$V = L\, \bar{v}\, z_i = L \int_0^{z_i} v\, dz \quad \text{(cubic metres per second).}$$

By inserting (1) we obtain

$$V = L \int_0^{z_i} \left\{ v_i + \frac{10}{\lambda L} (D'_A - D'_B) \right\} dz$$

and further:

$$V = L\, v_i\, z_i + \frac{10}{\lambda} \left\{ \int_0^{z_i} D'_A\, dz - \int_0^{z_i} D'_B\, dz \right\} \tag{3 a}$$

$$= L\, v_i\, z_i + \frac{10}{\lambda} (Q_A - Q_B)\,,$$

where

$$Q = \int_0^{z_i} D'\, dz = \int_0^{z_i} \int_p^{p_i} \alpha\, dz\, dp = \int_{z_i}^{0} \int_{p_i}^{p} \alpha\, dz\, dp\,. \tag{4 a}$$

$Q_A$ and $Q_B$ are the values of the integral $Q$ at the two stations, $A$ and $B$. The dimensions of $Q$ are $L^3 T^{-2}$.

If $v_i = 0$, *i. e.*, if the water at the depth $z_i$ is motionless we obtain

$$V = \frac{10}{\lambda} (Q_A - Q_B) \ \text{m}^3/\text{sec.} \tag{3 b}$$

When transport computations are performed for a number of stations and the integration can be extended to the same "zero-level" ($z = z_i$, $p = p_i$) at all the stations, it is most convenient to use the last expression of $Q$ in (4 a), and perform the integrations from the "zero-level" upwards (by means of a calculating machine).

Equation (4 a) can also be written as follows:

$$Q = \int_0^{z_i} D'\, dz = \int_0^{z_i} (D_i - D)\, dz = D_i z_i - \int_0^{z_i} D\, dz \tag{4 b}$$

according to (2). If therefore the dynamic depth, $D$, of the standard isobaric surfaces is computed beforehand, as is usual in recent oceanographic papers, it is most convenient to use the formula (4 b) in connection with (3 b) for the transport calculations; the calculation of the various differences, $D_i - D$, giving the thickness of the water layers between the various standard isobaric surfaces and the isobaric surface $p = p_i$ is thus avoided.

In recent papers dealing with oceanographic observations, the specific volume, $\alpha$, and the dynamic depth, $D$, of the isobaric surfaces are most often given by their anomalies, $\triangle\alpha$ and $\triangle D$, *i. e.*, the differences between the actual values and the values which would have been found at the corresponding pressures if the temperature had been 0° C and the salinity 35 ‰ at all depths from the surface downwards.

$$\triangle\alpha = \alpha - \alpha_{35,\,0,\,p}$$

$$\triangle D = D - D_{35,\,0,\,p}$$

If $\triangle Q$ is the corresponding anomaly of the quantity $Q$, it is easily seen that we get the value of $\triangle Q$ if instead of $\alpha$ or $D$ we insert their anomalies, $\triangle\alpha$ or $\triangle D$, in the formulae of the quantity $Q$, (4 a) and (4 b).

As

$$\triangle Q_A - \triangle Q_B = Q_A - Q_B$$

the values of the anomaly, $\triangle Q$, can be inserted in the transport formulae, (3 a) and (3 b), instead of the values of the quantity $Q$ itself. This way of reasoning is also valid for the transport formulae mentioned below.

For the sake of brevity and simplicity in this account of the transport computation method, I have preferred not to introduce the anomalies in the formulae.

## The Mass Transport. The Method of Ekman.

By means of a slightly modified mode of procedure one can obtain an expression for the *mass* transport similar to the *volume* transport expression (3 b). The water mass which passes between the two stations $A$ and $B$ in one second is equal to the weight $U$ of a water column resting over the horizontal area $L\,\bar{v}$ at the isobaric surface $p = p_i$. We find:

$$U = \frac{10}{g} L\, \bar{v}\, p_i$$

and as $\bar{v}\, p_i = \int_0^{p_i} v\, dp$, we obtain by inserting (1):

$$U = \frac{100}{g\,\lambda} (Q'_A - Q'_B) \quad \text{(metric tons per sec.)} \tag{5}$$

where

$$Q' = \int_0^{p_i} D'\,dp = \int_0^{p_i} (D_i - D)\,dp \tag{6}$$

with the designations used above, when assuming $v_i = 0$. Here we have disregarded the possibility that layers with different density may have different velocity. The maximum error thus introduced has an order of magnitude of about 1 ‰. The dimensions of $Q'$ are $MLT^{-4}$.

This expression (6) is closely related to the stream function of EKMAN [1929], which, with the designations used above, will take the following form:

$$P = \int_0^{D_i} (-\varepsilon p_i + p)\,dD \tag{7}$$

where $\varepsilon$ is very nearly equal to $\frac{\varrho_i + \varrho_o}{2\varrho_i}$. $\varrho_o$ and $\varrho_i$ are the densities at the sea-surface and at the dynamic depth $D_i$ respectively.

EKMAN has given the following expressions for the components of the *amount of current* ("Strommenge") of the gradient current:

$$S'''_x = -\frac{100}{g\lambda}\frac{\partial P}{\partial y}; \quad S'''_y = \frac{100}{g\lambda}\frac{\partial P}{\partial x}.$$

These equations show that the vector $S'''$ is directed 90° *contra solem* to the ascendant of the potential $P$ and that the mass transport between the two stations $A$ and $B$ is:

$$U = \frac{100}{g\lambda}(P_A - P_B) \text{ (metric tons per second).} \tag{8}$$

A supposition of the formula is that the water at the dynamic depth $D_i$ is motionless ($v_i = 0$).

In developing his formula EKMAN has taken the friction into account while in HELLAND-HANSEN's simplified formula the friction is disregarded. To the same extent as EKMAN's formula, the formulae developed above take the friction into account as the frictional loss of velocity or of momentum of one water layer is gained by the adjacent layers, and thus will not influence the total transport. Of course, neither these formulae nor the formula (8) account for the frictional loss of energy converted to heat.

The quantity $P$ is negative, and consequently $P_A < P_B$ if $Q'_A > Q'_B$, as the quantity $Q'$ is positive.

EKMAN has applied his method for some stations from the "Armauer Hansen" 1913 [HELLAND-HANSEN and NANSEN 1926] and the "Michael Sars" 1910 [HELLAND-HANSEN 1930]. For the station "A. H." 4 a, he obtains $\triangle P = -398$ ($\triangle P = P_{35, 0, D} - P$) while the formula (6) gives $\triangle Q' = 374$. The difference between the numerical values is about 6 %. The mass transport between the two stations "A. H." 4 a and 10 computed by means of (8) is found to be 11.6 mill. tons per sec. and by means of (5) 10.9 mill. tons per sec. The difference is about 6 %. This difference occurs because $P$ (and $\triangle P$) is about 4 % too high and $Q'$ (and $\triangle Q'$) about 2 % too low, as will be shown later (Page 7).

When transport computations are performed according to the method of EKMAN, the pressures at the standard dynamic depths must be computed, whereas in the formulae derived above, the dynamic depths of the standard isobaric surfaces, which are also computed for other purposes and which are published in all recent papers in dynamic oceanography, are used. This is, of course, a great advantage.

## Calculation of the Water Transport of a Coastal Current by Means of the Observations at a Single Station.

The integral

$$D' = \int_{p_i}^{p} \alpha\,dp$$

(in Formula 4 a) can be represented formally by the shaded area in Fig. 2 and the double integral $Q$ by the volume in Fig. 3 as is easily seen. We see that

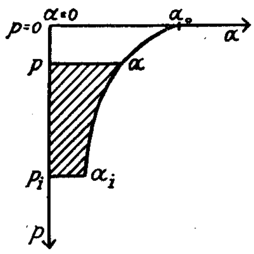


Fig. 2.

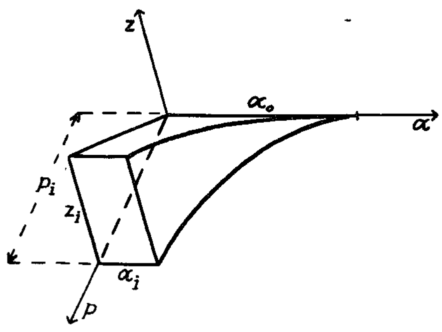


Fig. 3.

this volume can be expressed in two other ways when dividing the volume by planes perpendicular to the two other axes $\alpha$ and $p$. We obtain

$$Q=\int_{o}^{p_i} \alpha\, z\, dp=\int_{o}^{z_i} \alpha\, z\, dz=\tfrac{1}{2}\int_{o}^{z_i^2} \alpha\, d(z^2) \tag{9}$$

when the volume is divided by planes perpendicular to the $p$-axis and parallel to the rectangular side $(\alpha_i z_i)$ and:

$$Q=\tfrac{1}{2} p_i z_i \alpha_i+\tfrac{1}{2}\int_{\alpha_i}^{\alpha_o} p\, z\, d\alpha=\tfrac{1}{2} z_i^2 \alpha_i+\tfrac{1}{2}\int_{\alpha_i}^{\alpha_o} z^2\, d\alpha \tag{10}$$

when the planes are put perpendicular to the $\alpha$-axis and parallel to the triangular side $\left(\tfrac{1}{2} p_i z_i\right)$.

The replacement of the integration letter $p$ by $z$ in these and the following formulae only means that, when integrating, we use, instead of the actual pressure at the observation depth, the pressure (in decibars) which is expressed by the same figure as the observation depth (in metres), as usually done when performing the integration $\int \alpha\, dp$. The values of $Q$ and $V$ thus obtained will be only about 1 % too low down to a depth of 1000 metres. Formula 4 a will then take the following form

$$Q=\int_{z_i}^{o}\int_{z_i}^{z} \alpha\, dz\, dz\,. \tag{11}$$

The three expressions for $Q$: (9), (10) and (11) are identical.

If we introduce the expression (10) in the transport formula (3 b) we see that the first term on the right-hand side in (10), $\tfrac{1}{2} z_i^2 \alpha_i$ will disappear when the water at the depth $z_i$ and deeper, is homogeneous or horizontally layered (*barotropic*), as the term will then have the same value for both stations.

It remains:

$$V=\frac{10}{\lambda}(q_A-q_B) \tag{12}$$

where

$$q=Q-\tfrac{1}{2} z_i^2 \alpha_i=\tfrac{1}{2}\int_{\alpha_i}^{\alpha_o} z^2\, d\alpha. \tag{13}$$

Assuming that the water at the station B has the specific volume $\alpha_i$ from the depth $z_i$ to the sea-surface, we will obtain $q_B=0$ and when introducing this in (12) we get

$$V=\frac{10}{\lambda} q_A=\frac{10}{\lambda}\left(Q_A-\tfrac{1}{2} z_i^2 \alpha_i\right) \tag{14}$$

or, by means of (9),

$$V=\frac{10}{2\lambda}\int_{o}^{z_i^2}(\alpha-\alpha_i)\, d(z^2)\,. \tag{15}$$

This formula, (15) differs very little from the formula:

$$V=\frac{g}{2\lambda\alpha_i}\int(\alpha-\alpha_i)\, d(z^2)$$

derived by Werenskiold (1935) by means of a quite different method of procedure. The constant factor of this formula will give about 1 % higher values than (15) and thus the formula gives nearly correct values because (15) gives about 1 % too low values as mentioned above. It will be shown in detail below.

By means of Werenskiold's formula or by means of (14) or (15) the water transport of a coastal current can be found by means of a single station, *i. e.* we find the water transport of the current outside of the station. When applying these formulae to the actual conditions in nature it is very important to be aware of the above mentioned assumptions of the formulae.

In coastal currents, lighter water is layered in heavier, more or less homogeneous water and the isostere $\alpha_i$, which at a station in the current is found in the approximately homogeneous water, is supposed to approach the sea-surface outside of the current. Actually, for the transport calculations, this assumption is nearly equivalent to the knowledge of the conditions at another station.

If (11) or (10) is inserted in (14) we obtain:

$$V=\frac{10}{\lambda}\int_{z_i}^{o}\int_{z_i}^{z}(\alpha-\alpha_i)\, dz\, dz \tag{16}$$

or

$$V=\frac{10}{2\lambda}\int_{\alpha_i}^{\alpha_o} z^2\, d\alpha\,. \tag{17}$$

These formulae are identical with (14) and (15) as to content, but different as to the method of computation. (15), (16) and (17) are of minor interest as (14) is superior regarding suitability for computations. In (14) $Q_A$ is computed for the station $A$ by means of (4 a) or (4 b) (the previous remarks about the choice between the two expressions of the quantity $Q$ being also valid here), and the calculation of the various differences $(\alpha - \alpha_i)$ is avoided. It is easily seen that this is the quickest method of performing the computations of the water transport of a coastal current by means of a single station.

A formula for the *mass* transport, corresponding to the *volume* transport formula (14), can be derived in a similar way on the basis of (5) and (6).

(As previously, we can use the anomalies of the specific volume and of the dynamic depth of the isobaric surfaces in these formulae, our supposition now being that the iso-line of anomaly, $\triangle\alpha_i$, reaches (or approaches) the sea-surface. The courses of the lines of constant $\triangle\alpha$ do not coincide with the courses of the isosteres, the difference between $\alpha$ and $\triangle\alpha$ being equal to $\alpha_{35, 0, p}$ and thus varying with the pressure, $p$, or, with the depth. Consequently, the courses of the iso-line of anomaly, $\triangle\alpha_i$, and of the isostere, $\alpha_i$, coincide at the depth $z_i$ only).

## The Water Transport over a Bank.

According to the method of NANSEN and HELLAND-HANSEN [HELLAND-HANSEN 1934] dynamic calculations can be performed down to the "zero-level" (or the isobaric surface $p = p_i$) even at a station taken on a bank where the depth is smaller than the depth of this "zero-level", supposing that (due to other observations) the distribution of the specific volume is sufficiently well known along the bottom down to the "zero-level". The integration $D' = \int_p^{p_i} \alpha \, dp$ is performed along the vertical of the bank station to the bottom and along the bottom to the depth $z_i$ where $p = p_i$. Thus the dynamic heights of the standard isobaric surfaces over the isobaric surface $p = p_i$ are found and these values can be used in the transport formulae. Obviously the values of the specific volume, $\alpha$, along the bottom, can also be used directly in the formulae. HELLAND-HANSEN (*loc. cit.*) points out that "the computation will give a correct result if the velocity of the current at the bottom is nil, otherwise some uncertainty arises". Of course, this statement is also valid for the application of the method to the transport calculations.

## Errors in the Calculation of the Water Transport.

*Methodical Errors of Computation.* In practice, when the computation of the quantities $P$, $Q$ and $Q'$, in the formulae above, are performed by means of numerical integration, the values of density and specific volume recorded for $a$ metres are used without reduction as valid also for $a$ dynamic metres and for a depth where the pressure is $a$ decibars. By this means an error is made. If observations have been taken at depths of $a$[1] metres or $D = \frac{g}{10}\, a$ dynamic metres and we perform the integration $p = \int \varrho \, dD$ as if the observations had been taken at the dynamic depths $a$ we will get a value of $p$ about 2 % too high, and we make another error of about 2 % in the same direction when the integration in formula (7) is performed, so that the numerical values of $P$ will be about 4 % too high, totally.

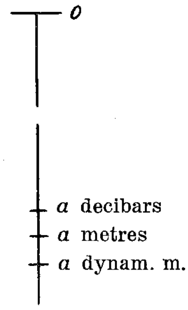

Fig. 4.

In the same way we make an error of about 1 % (down to 1000 metres) when we use observations from the depths $a$ metres as valid for the isobaric surfaces $a$ decibars, when the integration $D = \int \alpha \, dp$ is performed. $D$ will be about 1 % too low, and $Q$ also about 1 % too low as $z$ (in metres) is used as integration letter when the second integration is performed. $Q'$ will be 2 % too low as $p$ is the integration letter twice. (The expressions $Q$ and $Q'$ give the same numerical values). The total difference in the values for the transport obtained by means of $P$ and $Q'$ will be about 6 %.

*Errors Caused by Inaccuracy of Observations.* If an error $\varepsilon_n$ occurs in $\alpha$ at a depth $z_n$ it will produce errors $\varepsilon'_n$ and $\varepsilon''_n$ in $D'$ and $Q$ respectively. When $z_{n+1}$ and $z_{n-1}$ are the nearest observation depths below and above $z_n$ we find:

$$\varepsilon'_n = \frac{\varepsilon_n}{2}(z_{n+1} - z_{n-1}) \tag{18}$$

and

$$\varepsilon''_n = \frac{\varepsilon_n}{4}(z_{n+1} - z_{n-1})(z_{n+1} + z_{n-1}) = \frac{\varepsilon_n}{4}\left(z^2_{n+1} - z^2_{n-1}\right) \tag{19}$$

[1] The letter $a$ represents the figures giving the various standard depths of observation (in metres), standard dynamic depths (in dynamic metres) and standard isobaric surfaces (in decibars).

If an error in $\alpha$, $\varepsilon_i$, occurs at the deepest observation depth $z_i$ the errors in $D'$ and $Q$ are:

$$\varepsilon'_i = \frac{\varepsilon_i}{2}(z_i - z_{i-1})$$

$$\varepsilon''_i = \frac{\varepsilon_i}{4}(z_i - z_{i-1})(z_i + z_{i-1}) = \frac{\varepsilon_i}{4}\left(z_i^2 - z_{i-1}^2\right).$$

These formulae are found by considering the errors which an erroneous observation produce in the individual steps of the numerical integrations, when assuming the variations in $\alpha$ and $D'$ to be linear between the depths of observation.

If an error in $\alpha$, $\varepsilon$, occurs at all depths of observation, the errors in $D'$ and $Q$ are:

$$\varepsilon' = \varepsilon z_i$$

$$\varepsilon'' = \frac{\varepsilon}{2} z_i^2$$

as is easily found from (9).

(As before, $z_i$ is the depth to which the integrations are performed).

Formulae 18 and 19 show that the influence exerted on $D'$ (or on the computed velocities) by an error in $\alpha$, is proportional to the interval of depth, $\frac{z_{n+1} - z_{n-1}}{2}$, for which the observation is valid and that the influence on $Q$ is proportional to this interval multiplied by $\frac{z_{n+1} + z_{n-1}}{2}$ which is equal to the depth of observation, $z_n$, or slightly different from it. In other words: the accuracy of the observations from the greater depths is of great importance, as the error produced in $Q$ by an erroneous observation is not only proportional to the interval for which the observation is valid but also proportional to the depth of observation. In practice, the intervals between the depths of observation are much greater in the deeper than in the upper water strata, and thus the influence of an erroneous deep-water observation on the transport calculations is considerably increased.

If *e. g.*, observations have been made at the depths 75, 100, 150, 800, 1000, 1200, 4500, 5000 and 5500 metres, we can compute the errors in the water transport calculations produced by the same error in $\alpha$ at the depths 100, 1000 and 5000 metres at one of the two stations $A$ and $B$, and we find 4219, 200 000 and 2 500 000 $m^3$/sec. respectively, if a station at a latitude of about 43° is considered $\left(\frac{10}{\lambda} = 100\,000\right)$ and the error in $\alpha$ is $10^{-5}$.

Obviously, the accuracy reached in the salinity observations (0.01 ‰ corresponding to nearly $10^{-5}$ in the specific volume) is not sufficient to permit exact transport calculations if the zero-level (or the motionless water) is situated very deep. However, as the horizontal variation of the salinity in the deeper layers is very small, the observed values of salinity (or the computed values of specific volume) can be smoothed if a sufficient number of stations have been taken within a limited area. By this means the relative accuracy of the specific volume in the deep-water at the various stations can be increased, as these calculations are only influenced by the differences between the values of specific volume belonging to the same depth at the different stations. This could perhaps be expressed more clearly in the following way: The "horizontal accuracy" in $\alpha$ is increased by means of the smoothing, and it is this "horizontal accuracy" which is of importance for the transport calculations (and also, to a smaller extent, for the velocity calculations). The formulae for transport calculations are of course more applicable for a sea bordered by high ridges forming a basin which is filled with homogeneous and motionless water. In such a sea the motionless water is found at relatively small depths and the inaccuracy of the observations will only influence the accuracy of the transport calculations to a small extent.

The Norwegian Sea might be supposed to be a sea of this type with a deep basin isolated by the Scotland—Faeroe—Iceland—Greenland Ridges (saddle depth 576 metres) and the Nansen Ridge between Spitsbergen and North-East Greenland (saddle depth about 1100 metres). The deep-water of this sea is, however, formed in the sea itself during the winter [NANSEN 1906] and thus the deeper water layers are probably not sufficiently motionless, at least in winter.

Another source of error is the term $L v_i z_i$ in (3 a). If the water is not motionless at the "zero-level" a considerable error may be introduced. This error is proportional to the velocity, $v_i$, at the depth $z_i$ (the "zero-level") and to this depth itself. (The transport below the depth $z_i$ is disregarded). Although this error is proportional to the depth of the "zero-level", it must be remembered that the velocity usually decreases with increasing depth and thus the error $L v_i z_i$ may nevertheless decrease with increasing depth.

## Charts of Amount of Current.

Obviously the functions $Q$ (and $Q'$) can be used to draw charts of amount of current.

Considering a sufficiently small area, the factor $\frac{10}{\lambda}$ is approximately constant and the function $V$ in (3 b) can be thus divided:

$$V = \frac{10}{\lambda} Q_A - \frac{10}{\lambda} Q_B .$$

For each station the value of $\frac{10}{\lambda} Q$ can then be computed and entered on the chart and the iso-lines of amount of current drawn.

If we consider a large area with great differences in the latitudes of the stations, it might be tempting to compute the value of $\frac{10}{\lambda} Q$ for each single station and use the values of $\frac{10}{\lambda}$ corresponding to the latitudes of the various stations. However, by this means an error is introduced which may be very great. We obtain:

$$\begin{aligned} V &= \frac{10}{\lambda_A} Q_A - \frac{10}{\lambda_B} Q_B \\ &= \left(\frac{10}{\lambda_A} - \frac{10}{\lambda_B}\right) Q_A + \frac{10}{\lambda_B}(Q_A - Q_B) \end{aligned} \tag{20}$$

where the last term to the right is unessentially different from the correct:

$$V = \frac{10}{\lambda_{AB}}(Q_A - Q_B)$$

when the difference of latitude between the stations, $A$ and $B$, is not very great.

$$\lambda_A = 2\,\omega \sin \varphi_A$$

$$\lambda_B = 2\,\omega \sin \varphi_B$$

$$\lambda_{AB} = 2\,\omega \sin \frac{\varphi_A - \varphi_B}{2}$$

$Q_A$ is supposed to be greater than $Q_B$.

The first term to the right in (20), $\left(\frac{10}{\lambda_A} - \frac{10}{\lambda_B}\right) Q_A$, is, however, a most important source of error, when the difference of latitude between the stations, $A$ and $B$, is not quite inconsiderable. The error may be very great when the "zero-level" of dynamic calculations is situated very deep and $Q_A$ consequently will be great.

In his paper, EKMAN [1929] has computed the values of the quantity $P$ at the above-mentioned stations of the "Michael Sars" 1910 and the "Armauer Hansen" 1913, and has on a chart drawn iso-lines of the quantity $P$. THORADE [1933] has given an account of the method of EKMAN and has in a chart given the iso-lines of the amount of current itself, corresponding to the mentioned $P$-line chart of EKMAN. It seems as if, when preparing his chart, he has computed the value of the quantity $\frac{100}{g\lambda} P$ for each station using the value of $\frac{100}{g\lambda}$ corresponding to the latitude of the station, whereby an error is introduced. For the transport between the stations "M. S." 88 a and "A. H." 10 he has got a value which seems to be about 25 % too high. The above mentioned methodical error of the $P$ computations, 4 %, is included, (see later).

A correct method of procedure for the construction of charts of amount of current should be the following: All the stations are numbered from 1 to $n$ in a convenient sequence and $\frac{10}{\lambda_{AB}}$ and $Q_A - Q_B$ is computed between all consecutive stations and also between the $n$'th and the 1st station. Then successive summation of the values $\frac{10}{\lambda_{AB}}(Q_A - Q_B)$ is performed.

For an arbitrary station, $k$, we obtain

$$\Sigma_k = \sum_{\substack{A=2\\B=1}}^{\substack{A=k\\B=k-1}} \frac{10}{\lambda_{AB}}(Q_A - Q_B) . \tag{21}$$

When the summation is carried through to the 1st station we obtain:

$$\Sigma_1 = \sum_{\substack{A=2\\B=1}}^{\substack{A=1\\B=n}} \frac{10}{\lambda_{AB}}(Q_A - Q_B) = 0 .$$

This equation will be fulfilled (at least approximately) if the summation has been correctly performed.

The summation is very easily performed by means of a calculating machine, and when the values of $\Sigma$ have been entered on a chart the iso-lines of amount of current are drawn. The difference between the values of $\Sigma$ for two stations gives directly the water transport between the stations.

(The summation by means of a calculating machine is thus performed: $Q_1$ is inserted in the quoting dial (multiplier) of the machine and $\frac{10}{\lambda_{21}}$ in the check dial (multiplicand) while the register dial (product) shows nought. The machine is now worked so that the quoting dial shows $Q_2$ and the register dial will then give $\Sigma_2$. Then $\frac{10}{\lambda_{32}}$ is inserted in the check dial and the machine worked so that $Q_2$ is replaced by $Q_3$ in the quoting dial and now the register dial will give $\Sigma_3$, and so on.)

To avoid negative values of $\Sigma$ one may also choose the initial value in the sequence, $\Sigma_1$, sufficiently high to make all the values positive, or one may simply choose that station as the initial one which has the lowest value of the quantity $Q$ (as in Instances 1 and 2 below).

For a coastal current area it will be natural to choose $\Sigma = 0$ for an initial station outside the current (as in Instance 2 below) or near the coast so that the value of $\Sigma$ will give the water transport of the current outside or inside of the station.

Of course the quantity $\Sigma$ must be constant along the coastline as there can be no transport across

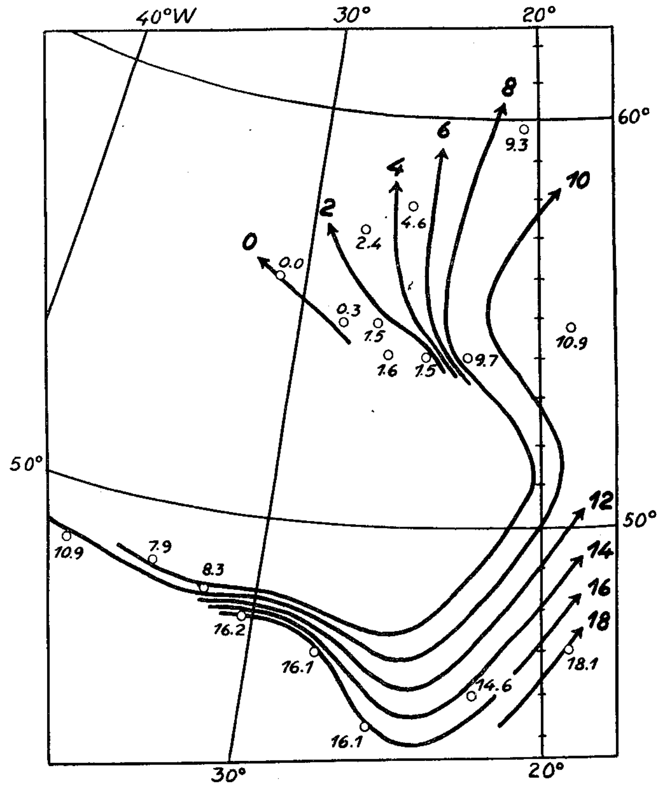


Fig. 5. Iso-lines of amount of current, "Michael Sars" 1910 and "Armauer Hansen" 1913. Between two neighbouring lines are transported 2 million tons per second.

Table 1.

*Transport calculations, "Michael Sars" 1910 and "Armauer Hansen" 1913.*

| Stat. | Lat. N | Long. W | $\triangle Q'$ | $\frac{1}{(g\lambda)_{AB}}$ | $\Sigma$ |
|---|---|---|---|---|---|
| MS 82 | 48° 26′ | 37° 00′ | 378 | | 10.9 |
| | | | | 934 | |
| » 83 | 48° 30′ | 33° 55′ | 346 | | 7.9 |
| | | | | 937 | |
| » 85 | 47° 58′ | 31° 41′ | 350 | | 8.3 |
| | | | | 945 | |
| » 86 | 47° 29′ | 30° 20′ | 434 | | 16.2 |
| | | | | 954 | |
| » 87 | 46° 48′ | 27° 44′ | 433 | | 16.1 |
| | | | | 972 | |
| » 88 a | 45° 12′ | 25° 46′ | 432 | | 16.1 |
| | | | | 980 | |
| » 89 | 45° 55′ | 22° 24′ | 417 | | 14.6 |
| | | | | 965 | |
| » 90 | 46° 58′ | 19° 06′ | 454 | | 18.1 |
| | | | | 901 | |
| AH 4 a | 54° 42′ | 18° 44′ | 374 | | 10.9 |
| | | | | 858 | |
| » 5 | 54° 06′ | 23° 00′ | 359 | | 9.7 |
| | | | | 862 | |
| » 6 | 54° 02′ | 24° 34′ | 265 | | 1.5 |
| | | | | 862 | |
| » 7 a | 54° 05′ | 26° 08′ | 266 | | 1.6 |
| | | | | 858 | |
| » 8 | 54° 49′ | 26° 57′ | 265 | | 1.5 |
| | | | | 855 | |
| » 9 | 54° 51′ | 28° 15′ | 251 | | 0.3 |
| | | | | 848 | |
| » 10 | 55° 55′ | 31° 07′ | 247 | | 0.0 |
| | | | | 837 | |
| » 12 | 57° 08′ | 27° 46′ | 275 | | 2.4 |
| | | | | 827 | |
| » 13 | 58° 00′ | 25° 32′ | 303 | | 4.6 |
| | | | | 816 | |
| » 14 a | 59° 42′ | 20° 39′ | 361 | | 9.3 |

the shore (the unessential supply from land disregarded). When the quantity $\Sigma$ has been computed for a number of stations near a coast we expect to find nearly the same value of $\Sigma$ at all the stations. If one (or more) of the $\Sigma$ values do not agree well with the others, the reason for this must be that the water is not motionless at the depth $z_i$ at the station, and then the velocity at this depth may be found.

*Instance 1.* Corresponding to the above-mentioned $P$-line chart of EKMAN and the chart of amount of current of THORADE, I have drawn a chart of amount of current, based on the same stations.

The computations are performed according to Formulae 6 and 21 giving the mass transport when the factor $\frac{10}{\lambda_{AB}}$ in the last formula is replaced by $\frac{100}{(g\lambda)_{AB}}$ and $Q$ by $\triangle Q'$.

Table 2. *Transport calculations, "Belgica" 1905.*

| Stat. | Lat. N | Long. | $\triangle Q$ | $\frac{1}{10\,\lambda_{AB}}$ | $\Sigma$ | $\frac{1}{10\,\lambda}$ | $V$ |
|---|---|---|---|---|---|---|---|
| 3 | 79° 40′ | 6° 29′ E | 8.8 | | 0.32 | 696 | 0.33 |
| | | | | 696 | | | |
| 11A | 52′ | 10° 42′ | 10.1 | | 0.41 | 696 | 0.42 |
| | | | | 696 | | | |
| 12 | 80° 05′.5 | 9° 40′ | 6.6 | | 0.17 | 696 | 0.17 |
| | | | | 696 | | | |
| 13 | 13′.5 | 7° 42′ | 8.0 | | 0.27 | 696 | 0.27 |
| | | | | 696 | | | |
| 14 | 17′.5 | 5° 40′ | 7.5 | | 0.23 | 696 | 0.24 |
| | | | | 696 | | | |
| 15 | 03′ | 2° 47′ | 9.1 | | 0.34 | 696 | 0.35 |
| | | | | 696 | | | |
| 16 | 79° 56′ | 1° 29′ | 9.2 | | 0.35 | 696 | 0.35 |
| | | | | 696 | | | |
| 17 | 34′ | 2° 40′ | 8.5 | | 0.30 | 697 | 0.31 |
| | | | | 697 | | | |
| 18 | 12′ | 1° 52′ | 9.2 | | 0.35 | 698 | 0.36 |
| | | | | 698 | | | |
| 19 | 78° 43′ | 0° 00′ | 7.6 | | 0.24 | 699 | 0.24 |
| | | | | 699 | | | |
| 21A | 20′ | 4° 27′ W | 7.5 | | 0.23 | 700 | 0.24 |
| | | | | 700 | | | |
| 21B | 14′ | 30′ | 7.2 | | 0.21 | 700 | 0.22 |
| | | | | 701 | | | |
| 22 | 05′ | 5° 21′ | 11.4 | | 0.51 | 701 | 0.51 |
| | | | | 702 | | | |
| 23 | 77° 25′ | 4° 03′ | 4.2 | | 0.00 | 703 | 0.01 |
| | | | | 704 | | | |
| 24A | 76′ 55′ | 3° 30′ | 4.7 | | 0.04 | 704 | 0.04 |
| | | | | 705 | | | |
| 25 | 44′ | 55′ | 7.3 | | 0.22 | 705 | 0.23 |
| | | | | 705 | | | |
| 26 | 28′.5 | 4° 54′ | 4.9 | | 0.05 | 706 | 0.06 |
| | | | | 706 | | | |
| 27B | 75° 56′ | 8° 35′ | 13.5 | | 0.66 | 707 | 0.66 |
| | | | | 707 | | | |

| Stat. | Lat. N | Long. | $\triangle Q$ | $\frac{1}{10\,\lambda_{AB}}$ | $\Sigma$ | $\frac{1}{10\,\lambda}$ | $V$ |
|---|---|---|---|---|---|---|---|
| 28 | 75° 55′ | 9° 00′ W | 13.1 | | 0.63 | 707 | 0.64 |
| | | | | 708 | | | |
| 29A | 35′ | 10° 23′ | 11.6 | | 0.52 | 708 | 0.53 |
| | | | | 708 | | | |
| 30 | 39′ | 12° 00′ | 22.7 | | 1.31 | 708 | 1.32 |
| | | | | 708 | | | |
| 31A | 47′.5 | 59′ | 20.9 | | 1.18 | 708 | 1.19 |
| | | | | 707 | | | |
| 32 | 58′.5 | 14° 08′ | 24.3 | | 1.42 | 707 | 1.43 |
| | | | | 706 | | | |
| 33 | 76° 30′ | 47′ | 26.8 | | 1.60 | 706 | 1.60 |
| | | | | 705 | | | |
| 34 | 46′ | 33′ | 30.0 | | 1.82 | 705 | 1.83 |
| | | | | 705 | | | |
| 35 | 33′.5 | 58′ | 31.3 | | 1.91 | 705 | 1.92 |
| | | | | 705 | | | |
| 36A | 37′ | 18° 22′ | 26.8 | | 1.60 | 705 | 1.60 |
| | | | | 704 | | | |
| 37 | 77° 30′ | 34′ | 24.4 | | 1.43 | 703 | 1.43 |
| | | | | 702 | | | |
| 38 | 35′.5 | 12′ | 24.3 | | 1.42 | 702 | 1.42 |
| | | | | 702 | | | |
| 39A | 47′.5 | 17° 11′ | 27.0 | | 1.61 | 702 | 1.61 |
| | | | | 701 | | | |
| 40 | 78° 13′.5 | 14° 18′ | 26.9 | | 1.60 | 700 | 1.60 |
| | | | | 700 | | | |
| 41 | 09′ | 01′ | 27.1 | | 1.62 | 701 | 1.61 |
| | | | | 701 | | | |
| 42 | 06′.5 | 15° 06′ | 27.4 | | 1.64 | 701 | 1.63 |
| | | | | 701 | | | |
| 43 | 13′ | 16° 31′ | 26.3 | | 1.56 | 700 | 1.55 |
| | | | | 701 | | | |
| 44 | 77° 57′ | 17° 00′ | 27.6 | | 1.65 | 701 | 1.65 |
| | | | | 703 | | | |
| 47 | 76° 47′ | 15° 21′ | 24.8 | | 1.46 | 705 | 1.46 |

The same "zero-level" as used by EKMAN (and by THORADE), 1000 decibars (corresponding to 1000 dynamic metres or to the observations from 1000 metres), is chosen. (It is not possible to say to what extent the water at this depth is really motionless, probably the velocity is very small, but even a small velocity at this depth may cause a considerable error. However, the intention is only to compare the methods, and the absolute values obtained are of slight interest as the computations of EKMAN (and of THORADE) are also based on the supposition of motionless water at this depth).

Station AH 10 is chosen as initial station and is given the value *zero*. From this station the summation is performed towards Stat. AH 4 a and further to MS 82, as well as towards AH 14 a in the sequence of Table 1.

Fig. 5 shows the chart where the values of $\Sigma$ have been entered and where the iso-lines of amount of current are drawn. The water transport of the current is 2 million metric tons per sec. between two neighbouring lines.

It is seen that the transport between Stats. AH 10 and MS 90 is 18.1 million tons per sec., but as the quantity $Q'$ (Formula 6) gives about 2 % too low values, the correct value will be 18.6 mill. tons per sec. As the quantity $P$ (Formula 7) gives about 4 % too high values, the chart of THORADE should give 19.3 mill. tons per sec. between these stations while it gives 23.3 or about 21 % more. This seems to be due to THORADE's method of computing the values of the chart.

*Instance 2.* For the stations of the "Belgica" Expedition in 1905 between Spitsbergen and North-East Greenland [HELLAND-HANSEN and KOEFOED 1909] I have used Formulae 4 a and 21 and performed the summation from Stat. 23 (initial value *zero*) towards Stat. 3 and towards Stat. 47 in the sequence of Table 2. The integration in Formula 4 a is performed to the "zero-level" $z_l = 300$ metres (or decibars). For the

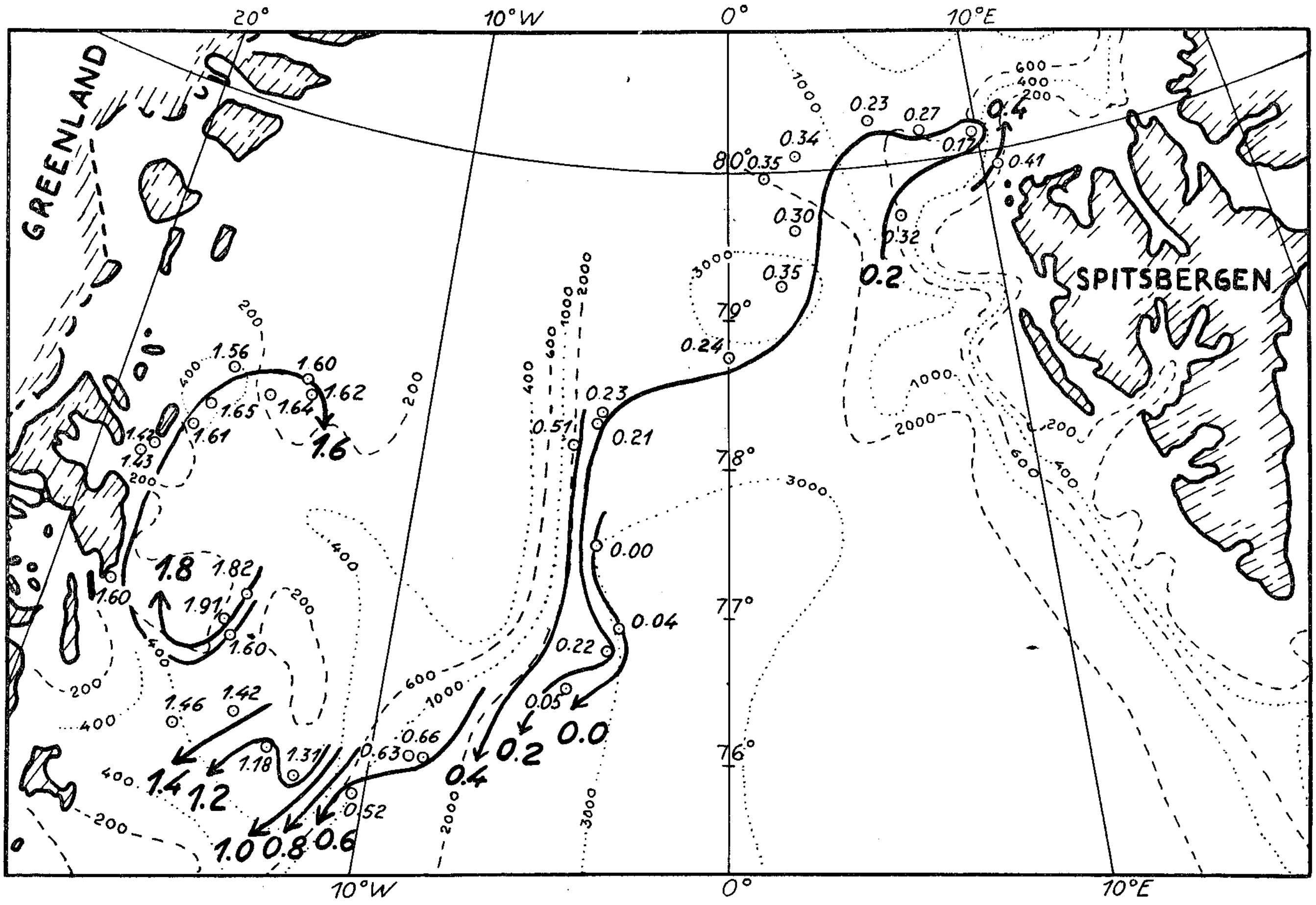


Fig. 6. Iso-lines of amount of current, "Belgica" 1905. Between two neighbouring lines are transported 0.2 million cubic metres per second.

stations on the banks with depths less than 300 metres the integrations have been performed along the bottom to another station with depth greater than 300 metres according to the mentioned method of NANSEN and HELLAND-HANSEN. At this depth the water is motionless only to a certain degree, but it has not been possible to extend the computations to a greater depth as the distribution of the anomaly of specific volume is not known sufficiently well along the deeper slope of the banks, because stations are lacking there, and thus the said method could not have been used for inclusion of the bank stations in the computations and on the chart of amount of current. It may be assumed, however, that Table 2 gives nearly correct values as the depth $z_i$ is small and the velocity at this depth must also be small (the error which is introduced if the velocity at this depth is $v_i$ instead of *zero* is $L\, v_i\, z_i$). The values of $\Sigma$ are entered on the chart in Fig. 6 and the iso-lines of amount of current[1] drawn, showing the water transport of the East Greenland Polar Current in its northernmost area[2]. As the "zero-station", Stat. 23, is apparently situated outside the current (no trace of Polar water is found at the station) the $\Sigma$ values of the other stations will give the water transport of the current outside of the station. As the Polar Current is a "coastal current" I have also computed this quantity by means of Formula 14 where $Q_A$ is the same $\triangle Q$ as computed before for the various stations. If in this formula $\triangle \alpha_i$ is chosen equal to $9 \cdot 10^{-5}$ ($z_i$ is 300 metres as before) we obtain $\frac{1}{2}\, z_i^2\, \triangle \alpha_i = 4.1$ and the

[1] Strictly, the term "amount of current" corresponding to EKMAN's "Strommenge" should be used regarding the mass transport, but no confusion can arise if it is also used for the volume transport.

[2] The values of $\Sigma$ at the stations near Spitsbergen are probably not correct as the velocity at a depth of 300 metres is probably not sufficiently small in that area.

Table 3.

| $\varphi^{\circ}$ | $\frac{1}{10 \cdot \lambda}$ | $\frac{1}{g \cdot \lambda}$ | $\varphi^{\circ}$ | $\frac{1}{10 \cdot \lambda}$ | $\frac{1}{g \cdot \lambda}$ | $\varphi^{\circ}$ | $\frac{1}{10 \cdot \lambda}$ | $\frac{1}{g \cdot \lambda}$ | $\varphi^{\circ}$ | $\frac{1}{10 \cdot \lambda}$ | $\frac{1}{g \cdot \lambda}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | ∞ | ∞ | 25 | 1 622 | 1 657 | 50 | 895 | 912 | 75 | 710 | 722 |
| 1 | 39 287 | 40 165 | 26 | 1 564 | 1 597 | 51 | 882 | 899 | 76 | 707 | 719 |
| 2 | 19 647 | 20 086 | 27 | 1 510 | 1 542 | 52 | 870 | 887 | 77 | 704 | 716 |
| 3 | 13 101 | 13 394 | 28 | 1 461 | 1 491 | 53 | 859 | 875 | 78 | 701 | 713 |
| 4 | 9 830 | 10 049 | 29 | 1 414 | 1 444 | 54 | 848 | 863 | 79 | 698 | 710 |
| 5 | 7 867 | 8 043 | 30 | 1 371 | 1 400 | 55 | 837 | 853 | 80 | 696 | 708 |
| 6 | 6 560 | 6 706 | 31 | 1 331 | 1 359 | 56 | 827 | 842 | 81 | 694 | 706 |
| 7 | 5 626 | 5 752 | 32 | 1 294 | 1 321 | 57 | 818 | 833 | 82 | 692 | 704 |
| 8 | 4 927 | 5 036 | 33 | 1 259 | 1 285 | 58 | 809 | 823 | 83 | 691 | 703 |
| 9 | 4 383 | 4 481 | 34 | 1 226 | 1 252 | 59 | 800 | 815 | 84 | 689 | 701 |
| 10 | 3 949 | 4 036 | 35 | 1 195 | 1 220 | 60 | 792 | 806 | 85 | 688 | 700 |
| 11 | 3 593 | 3 673 | 36 | 1 167 | 1 190 | 61 | 784 | 798 | 86 | 687 | 699 |
| 12 | 3 298 | 3 371 | 37 | 1 139 | 1 162 | 62 | 777 | 791 | 87 | 687 | 698 |
| 13 | 3 048 | 3 115 | 38 | 1 114 | 1 136 | 63 | 770 | 783 | 88 | 686 | 698 |
| 14 | 2 834 | 2 897 | 39 | 1 090 | 1 112 | 64 | 763 | 777 | 89 | 686 | 697 |
| 15 | 2 649 | 2 707 | 40 | 1 067 | 1 088 | 65 | 757 | 770 | 90 | 686 | 697 |
| 16 | 2 488 | 2 542 | 41 | 1 045 | 1 066 | 66 | 751 | 764 | | | |
| 17 | 2 345 | 2 397 | 42 | 1 025 | 1 045 | 67 | 745 | 758 | | | |
| 18 | 2 219 | 2 267 | 43 | 1 005 | 1 025 | 68 | 740 | 753 | | | |
| 19 | 2 106 | 2 152 | 44 | 987 | 1 007 | 69 | 734 | 747 | | | |
| 20 | 2 005 | 2 048 | 45 | 970 | 989 | 70 | 730 | 743 | | | |
| 21 | 1 913 | 1 955 | 46 | 953 | 972 | 71 | 725 | 738 | | | |
| 22 | 1 830 | 1 870 | 47 | 938 | 956 | 72 | 721 | 734 | | | |
| 23 | 1 755 | 1 793 | 48 | 923 | 941 | 73 | 717 | 729 | | | |
| 24 | 1 686 | 1 722 | 49 | 909 | 926 | 74 | 713 | 726 | | | |

values of $V$ will be nearly the same as the $\Sigma$ values. This is due to the fact that the variations of $\lambda = 2\omega \sin\varphi$ are small at these high latitudes. (When $\frac{10}{\lambda}$ in Formula 14 is chosen according to the latitude of the station $A$, it is also supposed that the station $B$, at which $q_B = 0$, is situated at the same latitude).

By this method, we save the work of finding the mean latitudes between the different stations in the sequence of Formula 21 but we must form the differences $\triangle Q_A - \frac{1}{2} z_i^2 \triangle a_i$ for each station. One of the methods may be as convenient as the other but the $\Sigma$ values are more correct than the $V$ values of Formula 14, especially at lower latitudes, if the supposition, that $q_B = 0$ at the same latitude as Stat. $A$, is not correct.

Fig. 6 shows how the water transport is most intense along the slope of the continental shelf and how an anticyclonic eddy is formed over the banks at about 77° to 78° N off the coast of East Greenland. The total water transport of the current in a southerly direction is probably about 1.60 million m$^3$ per sec. (Stat. 36 A). Stats. 37 and 38 near the coast give smaller values but the deeper observations at Stat. 37 were obviously erroneous, and as Stat. 38 is only 50 metres deep, extrapolation and the method of integration along the bottom have been applied to both these stations. The values of $\Sigma$ at the two stations may therefore be considered as doubtful.

*Table 3.* Ekman [*loc. cit.*] has computed a table giving the value of $\frac{1}{g\lambda}$ for every 5th degree of latitude. In the *Oceanographical Tables* of Subow [1931] are given the values of $\frac{1}{\sin\varphi}$ and $2\omega\sin\varphi$ for every single degree, but, unfortunately, no table for $\frac{1}{2\omega\sin\varphi}$. I have therefore computed the values of $\frac{10}{\lambda} = \frac{10}{2\omega\sin\varphi}$ and $\frac{100}{g\lambda} = \frac{100}{g \cdot 2\omega\sin\varphi}$ for each degree. For brevity, I have omitted the two last figures and have thus tabulated $\frac{1}{10\lambda}$ and $\frac{1}{g\lambda}$ in Table 3.

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