CONTRIBUTION TO THE THEORY OF FRONTOGENESIS

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CHAPTER I

GENERAL THEORY[1])

1. Definition and Criterion. The reason why the current theory of frontogenesis is incomplete is probably due to the fact that the term *frontogenesis* has not yet been properly defined. We shall, therefore, elaborate the definition and the criterion of frontogenesis before we proceed to the mathematical deductions.

Let us consider a scalar quantity α which has a continuous distribution in the horizontal plane (xy). α may then be represented by means of a map of its equiscalar curves. When time (t) varies, the α-curves will, in general, move relative to the co-ordinates of the chart. We then say *that we have frontogenesis if the α-curves move in such a way that they tend to produce a discontinuity along a line in the field.*

Frontolysis, being the negative of frontogenesis, needs no special definition.

The line along which frontogenesis takes place may be called the *line of frontogenesis.* This line may be a stationary or a moving curved or straight line.

Let $|\nabla \alpha|$ be the magnitude of the ascendant of α, and let $\frac{\delta}{\delta t}$ denote the time differentiation in a system of co-ordinates which is fixed to a particle of the moving line of frontogenesis. We then see that

$$(1) \qquad F = \frac{\delta |\nabla \alpha|}{\delta t}$$

expresses the variation in $|\nabla \alpha|$ per unit time in a system of co-ordinates which moves with the line of frontogenesis. The quantity F may then be taken as a measure of the frontogenetical effect, provided that frontogenesis takes place. (See below.)

If frontogenesis takes place along a line in the field, it follows that $|\nabla \alpha|$ must increase more rapidly on this line than elsewhere. This necessary criterion is expressed by:

$$(2) \qquad \frac{\partial F}{\partial N} = \frac{\partial}{\partial N}\left(\frac{\delta |\nabla \alpha|}{\delta t}\right) = 0,$$

where N measures length along an axis normal to the line of frontogenesis.

The quantity F has a maximum when $\frac{\partial F}{\partial N} = 0$ and $\frac{\partial^2 F}{\partial N^2} < 0$, and a minimum when $\frac{\partial F}{\partial N} = 0$ and $\frac{\partial^2 F}{\partial N^2} > 0$. The following cases may occur when $\frac{\partial F}{\partial N} = 0$:[2])

a) $F > 0$ and $\frac{\partial^2 F}{\partial N^2} < 0$. The ascendant of α increases more rapidly on the line considered $\left(\frac{\partial F}{\partial N} = 0\right)$ than elsewhere. In this case there is *frontogenesis.*

b) $F > 0$ and $\frac{\partial^2 F}{\partial N^2} > 0$. The ascendant of α increases less rapidly on the line considered than elsewhere. In this case there is neither frontogenesis nor frontolysis.

c) $F < 0$ and $\frac{\partial^2 F}{\partial N^2} < 0$. The ascendant of α decreases less rapidly on the line considered than elsewhere. In this case there is neither frontogenesis nor frontolysis.

d) $F < 0$ and $\frac{\partial^2 F}{\partial N^2} > 0$. The ascendant of α decreases more rapidly along the line considered than elsewhere. In this case there is *frontolysis.*

[1]) The first attempt to explain the formation of fronts was made by *Bergeron,* (Die dreidimensional verknüpfende Wetteranalyse. Geof. Publ. Vol. V, No. 6), who has made substantial contributions to our knowledge of fronts and air masses.

[2]) The case when $F = 0$ is trivial.

2. Conservative Field of Property. There can be little doubt that frontogenesis in the atmosphere is mainly a kinematical phenomenon: When air masses from different and distant source regions are brought into juxtaposition there will be formed a front in the conservative properties of the masses.

It is true that no property is strictly conservative. Physical and dynamical processes (radiation, conduction, mixing etc.) will in most cases *counteract* the kinematical frontogenesis. However, in order to study the nature of the kinematical (conservative) frontogenesis we shall in this paper neglect the non-conservative influences, and discuss frontogenesis on a strictly conservative basis.

Let α be a conservative property which is continuous in the horizontal plane. We may then write:

(1) $$\alpha = \alpha\,(x, y, t).$$

Since α is a conservative property we put:

(2) $$\frac{d\alpha}{dt} = 0,$$

and hence:

(3) $$\frac{\partial \alpha}{\partial t} = -\boldsymbol{v} \cdot \nabla \alpha,$$

where $\boldsymbol{v} = (u, v)$ is wind velocity.

We consider the air movement along the surface of the earth where the vertical component of

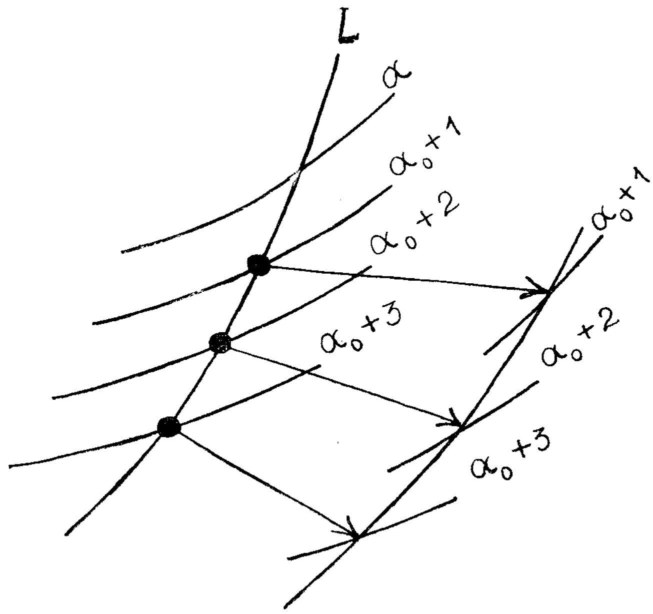


Fig. 1.
Showing the meaning of the velocity of the line of frontogenesis. This velocity is not the formal velocity of the line along its own normal.

$\boldsymbol{v}$ vanishes. It is then clear that all changes in α, either in a fixed system of co-ordinates or in a system of co-ordinates which is fixed to the moving frontogenetical line (if any) can only be caused by horizontal advection. From this and from the definition of frontogenesis it follows that the line of frontogenesis in a conservative field must be a substantial line consisting of the same individual particles, because otherwise the frontogenetical effect would steadily act on fresh air particles, and no frontogenesis would result. Furthermore, the equiscalar curves of α must also consist of the same particles because of the conservatism of the property. Since the process of frontogenesis primarily is a variation in the distribution of the ascendant of the property in question, we need only investigate how the ascendant of α varies on a moving air particle. The formulae thus deduced may then be applied to any individual substantial line such as lines of frontogenesis or equiscalar curves.

3. Deformation of the Field of Property. The variation in the magnitude of the ascendant of α is given by **1** (1), viz.,

(1) $$F = \frac{\delta \,|\nabla \alpha|}{\delta t}.$$

The time differentiation $\frac{\delta}{\delta t}$ in a moving system of co-ordinates is given by the symbolic equation:

(2) $$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla .$$

where $\boldsymbol{V}$ is the velocity of the system of co-ordinates. From (1) and (2) we obtain:

(3) $$F = \frac{\partial \,|\nabla \alpha|}{\partial t} + \boldsymbol{V} \cdot \nabla \,|\nabla \alpha|$$

It is convenient to eliminate $\frac{\partial}{\partial t}$ in order to obtain an expression which is independent of the time variation. In order to do this we may use the criterion of conservatism. Substituting for $\frac{\partial \,|\nabla \alpha|}{\partial t}$ by means of **2** (3), we obtain:

(4) $$F = \frac{-\nabla \alpha \cdot \nabla \boldsymbol{v} \cdot \nabla \alpha - \nabla \alpha \cdot \nabla \nabla \alpha \cdot (\boldsymbol{v} - \boldsymbol{V})}{|\nabla \alpha|},$$

which expresses the rate at which the magnitude of the ascendant of α increases per unit time in a moving system of co-ordinates when α is a conservative property.

Interpreting $\boldsymbol{V}$ as the velocity of a substansial line (e. g. line of frontogenesis or equiscalar curve), we may put:

$$\boldsymbol{v} \equiv \boldsymbol{V},$$

which substituted in (4) gives:

$$F = -\frac{\nabla a \cdot \nabla \boldsymbol{v} \cdot \nabla a}{|\nabla a|}. \tag{5}$$

The vector $\frac{\nabla a}{|\nabla a|}$ is the unit vector of ∇a. The vector $\frac{\nabla a}{|\nabla a|} \cdot \nabla v$ gives the variation in v along the vector lines of ∇a. F is then positive when this vector forms an angle with ∇a which is larger than $\frac{\pi}{2}$, and F is negative when the angle is less than $\frac{\pi}{2}$. In each particular case it is easy to map the distribution of F.

4. Criterion of Conservative Frontogenesis. According to the definition in § 1 there is frontogenesis when the equiscalar curves of a move in such a way that the magnitude of the ascendant of a increases more rapidly on a line in the field than outside this line. The necessary condition for frontogenesis is expressed by **1** (2), viz.,

$$\frac{\partial F}{\partial N} = 0,$$

where N measures length along a line normal to the line of frontogenesis. This equation expresses the necessary condition of partial maximum. The corresponding condition for total maximum is expressed by:

$$\nabla F = 0. \tag{1}$$

Substituting from **3** (5), we obtain:

$$-\nabla\left(\frac{\nabla a \cdot \nabla \boldsymbol{v} \cdot \nabla a}{|\nabla a|}\right) = 0. \tag{2}$$

This is a vector whose components are:

$$\begin{aligned} \frac{\partial F}{\partial x} &= -\frac{\partial}{\partial x}\left[\frac{\nabla a \cdot \nabla \boldsymbol{v} \cdot \nabla a}{|\nabla a|}\right] = 0, \\ \frac{\partial F}{\partial y} &= -\frac{\partial}{\partial y}\left[\frac{\nabla a \cdot \nabla \boldsymbol{v} \cdot \nabla a}{|\nabla a|}\right] = 0. \end{aligned} \tag{3}$$

Performing the partial differentiations of the above expressions with respect to x and y respectively, the following cases may occur:

a) $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ vanish identically, in which case F is constant throughout the field,

b) $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are constants or functions of t only, in which case F is a linear function of space co-ordinates.

c) $\frac{\partial F}{\partial x}$ and (or) $\frac{\partial F}{\partial y}$ contain space co-ordinates.

It is obvious that in the cases (a) and (b) there are no maxima or minima in F. There can, therefore, not be frontogenesis in these cases. In the case (c), however, there may be maxima or minima in the distribution of F. The necessary criterion for frontogenesis in a conservative field may then be expressed as follows: There is frontogenesis (positive or negative) when the equations (3) hold for finite real values of x and y, provided that the equations are not fulfilled throughout the field (case a).

5. Centre and Line of Frontogenesis. Solving **4** (3) with respect to x and y we obtain the coordinates of the point (x_0, y_0) where F has a maximum or minimum. This point may be called the *centre of frontogenesis.*

The line of frontogenesis, which is defined by **1** (2), runs through this point, and its position is most easily determined graphically. Developing **3** (5), we obtain:

$$F = -\left[\frac{\partial u}{\partial x} a_{10}^2 + a_{10} a_{01}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \right. \left. + \frac{\partial v}{\partial y} a^2_{01}\right](a^2_{10} + a^2_{01})^{-\frac{1}{2}}, \tag{1}$$

where u and v are the components of the wind vector, and

$$a_{pq} = \frac{\partial^{p+q} a}{\partial x^p \, \partial y^q}.$$

Giving F successive constant values we get the isolines of F. The line of frontogenesis runs through the centre and through the points where the equiscalar F-curves have maximum of curvature. (See fig. 2.) In a similar way the equation of the line of frontogenesis may be determined analytically.

When the two equations **4** (3) depend on one another in such a way that the coefficients are proportional, the two lines that determine the centre coincide, and the centre degenerates into a

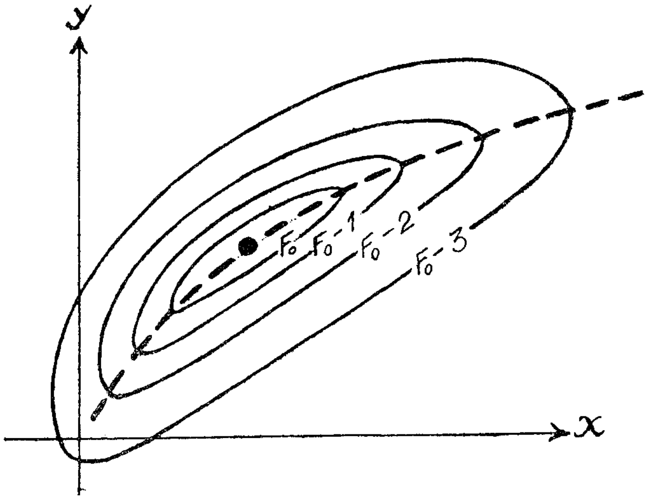


Fig. 2.

Showing the line of frontogenesis in relation to the equiscalar curves of F.

line along which F is constant. This line is then the line of frontogenesis. Such cases are frequent in nature, and we shall return to such cases in a later paragraph.

6. Degree of Frontogenetical Functions. We shall now prove the following theorems:

(a) *Frontogenesis is not possible when both the field of motion and the field of property are linear fields of x and y.*

(b) *Frontogenesis is only possible when the distribution of property and wind velocity along a profile (s) are such functions of s that the sum of their degrees is at least 4.*

The first theorem results directly from the equations **4** (2) or **4** (3). Inspecting these equations we see that $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ vanish for all values of x and y when v and α are linear functions of x and y, which proves the theorem (a).

Let us next consider the distribution of α and v along a profile s. Choosing the x-axis along the profile we may write:

$$\alpha = \alpha(x, t) \text{ and } u = u(x).$$

t may be eliminated as shown in § 2. The function F' corresponding to **3** (4) is then: (see **5** (1))

$$F' = -\frac{\partial u}{\partial x} \alpha_{10}.$$

The necessary condition for F having a maximum or minimum is:

$$\frac{\partial F'}{\partial x} = -\frac{\partial^2 u}{\partial x^2} \alpha_{10} - \frac{\partial u}{\partial x} \alpha_{20} = 0.$$

If a maximum or minimum shall occur it follows that equation (1) must contain space variables. Considering the number of differentiations we see that (1) can only contain space variables when neither α nor u are constants, and when the sum of the degrees of these functions is at least 4, which proves the second theorem.

The following table illustrates the content of the theorems:

Frontogenesis:

<table>
<tr><th>Field of v \ Field of α</th><th>Constant</th><th>Linear</th><th>2nd degree</th><th>3rd or higher degree</th></tr>
<tr><td>Constant</td><td colspan="4">Impossible</td></tr>
<tr><td>Linear</td><td colspan="3">Impossible</td><td>Possible</td></tr>
<tr><td>2nd degree</td><td colspan="2">Impossible</td><td colspan="2">Possible</td></tr>
<tr><td>3rd or higher degree</td><td>Impossible</td><td colspan="3">Possible</td></tr>
</table>

From this table we see that the linear field of motion can only produce frontogenesis when the field of property is of 3rd or higher degree. But even when the degree is sufficiently high, frontogenesis can only occur if the field of property obeys the more rigorous conditions enunciated in § 4. Moreover, since α cannot exceed a certain value when x increases, it follows that only periodic functions or functions of a special exponential character can be applied in order to study frontogenesis. We shall return to these questions in Chapter IV.

CHAPTER II.
FRONTOGENESIS IN LINEAR FIELDS OF MOTION

7. Introductory Remarks on the Linear Field of Motion. The idea of treating the field of motion as a linear field originates from the fact that the distribution of v may be represented by means of a Taylor series whose linear terms usually predominate within a certain area. However, with increasing distance from the point of development the non-linear terms become important because, otherwise, $\boldsymbol{v}$ would increase infinitely with increasing distance. This restricts the area within which it is permissible to neglect terms of higher order. It is, moreover, likely that the areas which contribute substance to the formation of fronts and the adjacent quasi-homogeneous air masses are much larger than the areas within which linear motion predominates.

On the other hand it is a fact that linear motion seems to predominate in the vicinity of the saddle points in the pressure distribution, and it is in such areas that fronts are most frequently observed. It, therefore, becomes important to examine whether such fronts really are formed in the linear field of motion, or if they are formed in the area of non-linear motion and afterwards brought into the linear area by the air currents.

We have seen in § 6 that a linear field of motion cannot produce frontogenesis except when the field of property is a function of high degree. We shall, therefore, examine the conditions under which frontogenesis takes place in linear fields of motion and non-linear fields of property.

A linear field of motion may in the most general case be represented by:

$$u = u_0 + u_1 x + u_2 y$$
$$v = v_0 + v_1 x + v_2 y$$

where u_0, v_0, u_1, v_1, u_2 and v_2 are constants. By appropriate choice of system of co-ordinates we may write:

$$u = u_0 + ax + bx - cy,$$
$$v = v_0 - ay + by + cx. \qquad (1)$$

Where (u_0, v_0) represents a translatory, $(ax, -ay)$ a deformative, (bx, by) a divergent, and $(-cy, cx)$ a rotational component.[1]) Writing the equations in this form we see clearly which rôle each partial field plays in the formation of frontogenesis.

It is well to note that the x-axis in (1) is chosen along the positive axis of deformation (the *axis of dilatation*). In the following paragraphs we shall always choose the x-axis in this way except when otherwise stated explicitly.

Usually u_0 and v_0 can be eliminated by parallel translation of the co-ordinate system. This is, however, only possible when the field of motion is such that u and v vanish in a point or along a line only. We shall, therefore, in the following general deductions retain the translatory components.

Owing to the choice of x-axis, a may be regarded as a positive quantity, whereas b and c may have either sign. $b > 0$ means divergence, and $b < 0$ convergence. $c > 0$ means positive rotation, i. e. rotation from the positive x-axis to the positive y-axis in a right hand system of co-ordinates. By appropriate choice of the constants a, b and c, the equations (1) will represent the linear terms in a Taylor series developed in the point where $u = u_0$ and $v = v_0$.

From (1) we obtain:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 2a$$
$$\text{div } \boldsymbol{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2b$$
$$\text{curl } \boldsymbol{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2c$$

which illustrates the nature of these constants.

The division of the linear field of motion in a deformative, a divergent and a rotational component is, naturally, a formal operation. Moreover it is important to note that the translation (u_0, v_0) depends entirely on the point which we choose as the origin. If the field is strictly linear the constants a, b and c are independent of the choice of origin, and the deformation, the divergence and the curl of $\boldsymbol{v}$ are constant throughout. Thus, wherever we choose the origin there will be an axis of dilatation and an axis of contraction running through the origin. If we choose another origin, the new axes of deformation will be parallel to the previous ones. Since a, by hypothesis, is constant throughout the field, the deformation is also constant. Furthermore, since the divergence and the rotation are symmetrical in all directions whereas the deformation is symmetrical only with respect to two axes, it follows that, for a given field, the *direction* of the axes of deformation is a reality, whereas the *position* of the axes depends on the point where we choose to place the origin of the co-ordinate system. This is most clearly seen when we consider a field of motion which consists of a deformation and a translation. It should also be mentioned that the axis of outflow in the *resultant* field of motion *need neither coincide with nor be parallel to the axis of dilatation.*

[1]) The method of determining the constants a, b and c is described in most text-books on vector analysis or field geometry.

Finally, it should be observed that the field of deformation, divergence and rotation are concentric fields, i. e. the velocities of each partial field vanish for the same values of x and y. If we superimpose linear partial eccentric fields on one another, we obtain:

$$\begin{aligned}&\text{Deformation: } u = a\,(x - x_1);\; v = -a\,(y - y_1),\\&\text{Divergence: } u = b\,(x - x_2);\; v = b\,(y - y_2),\\&\text{Rotation: } u = -c\,(y - y_3);\; v = c\,(x - x_3).\end{aligned}$$

The resultant of these eccentric partial fields is:

$$\begin{aligned}u &= \quad ax + by - cy - (ax_1 + bx_2 - cy_3).\\v &= -ay + by + cx - (-ay_1 + by_2 + cx_3).\end{aligned}$$

Putting:

$$\begin{aligned}-ax_1 - bx_2 + cy_3 &= u_0,\\ay_1 - by_2 - cx_3 &= v_0,\end{aligned}$$

we see that eccentric partial fields are equivalent to concentric fields plus a constant translatory movement. Thus, the separation of the resultant field in a constant translation and partial fields of deformation, divergence and rotation is a formal operation, and the centre of each partial field and the location of axes of deformation have no autonomy. What is a reality is the centre of the resultant field and the *direction* of the axes of deformation. This is in contrast to non-linear fields, where each centre and the location of the axes of deformation may be autonomous. (See Chapter IV).

8. Linear Deformation of the Field of Property. We shall now see how the magnitude of the ascendant of a varies in a linear field of motion. According to **3** (5) we have:

$$F = -\frac{\nabla a \cdot \nabla \boldsymbol{v} \cdot \nabla a}{|\nabla a|} \tag{1}$$

We put:

$$\nabla a = |\nabla a|\,(m\boldsymbol{i} + n\boldsymbol{j}), \tag{2}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors of the x- and y-axis respectively.

From **7** (1) we obtain:

$$\begin{aligned}\boldsymbol{v} = \boldsymbol{i}\,(u_0 + ax + bx - cy) &+\\+ \boldsymbol{j}\,(v_0 - ay + by + cx).&\end{aligned} \tag{3}$$

Substituting (2) and (3) in (1), we obtain:

$$F = -|\nabla a|\,[b + a\,(m^2 - n^2)]. \tag{4}$$

Let φ denote the angle between the x-axis (i. e. the axis of dilatation) and ∇a, and ψ the angle between the x-axis and the tangent to the isolines of a in the point which we are considering. We then have:

$$\begin{aligned}\cos\varphi &= -\sin\psi = m,\\\sin\varphi &= \quad\cos\psi = n,\end{aligned}$$

which substituted in (4) gives:

$$F = |\nabla a|\,(a\cos 2\psi - b). \tag{5}$$

From (4) and (5) we see that the translation (u_0, v_0) and the rotation $(-cy, cx)$ have no influence on the variation in $|\nabla a|$. Since F is a measure of the frontogenetical effect (see § 3), we see that *it is only the divergent and the deformative components of the field of motion which can produce frontogenesis, and these two components play equal parts in the production of fronts.*

From (4) and (5) we also see that F (ceteris paribus) has a maximum where $|\nabla a|$ has a maximum. The point where F has a maximum is the centre of frontogenesis, and the line along which F has a maximum with respect to the variation along the normal of the line is the line of frontogenesis. Thus, for a given field of motion (u, v), the centre of frontogenesis and the line of frontogenesis are determined by the magnitude of ∇a and the direction of the isolines of a relative to the axis of dilatation (i. e. the x-axis).

9. Frontogenetical and Frontolytical Sectors. Equation 8 (5) shows that the sign of F, for a given field of motion, depends entirely on the angle ψ between the x-axis and the tangent to the isolines. F vanishes along a line determined by:

$$\cos 2\psi' = \frac{b}{a} \tag{1}$$

If, at a given point, $\psi' > \psi > -\psi'$, then F is positive, and $|\nabla a|$ increases. On the other hand, when $\pi - \psi' > \psi > \psi'$, then F is negative, and $|\nabla a|$ decreases. Thus frontogenesis can only occur when the tangent to the isolines form an angle with the x-axis which in magnitude is less than $\pm\psi'$. Owing to the duplicity of equation (1) we see that there are, in general, two symmetrical sectors in which F is positive and two in which F is negative.

Fig. 3. illustrates the meaning of the term *sector of frontogenesis* and *sector of frontolysis*: Consider the tangent of the curve $a =$ constant at T_1. Transferring its direction to the origin, we see that it falls outside the sector indicated by $2\psi'$, which

is the sector of frontogenesis. F is then negative at the point T_1. At T_2, however, F is positive, because the tangent translated to the origin would fall inside the sector $2\psi'$. The line $F = 0$ divides

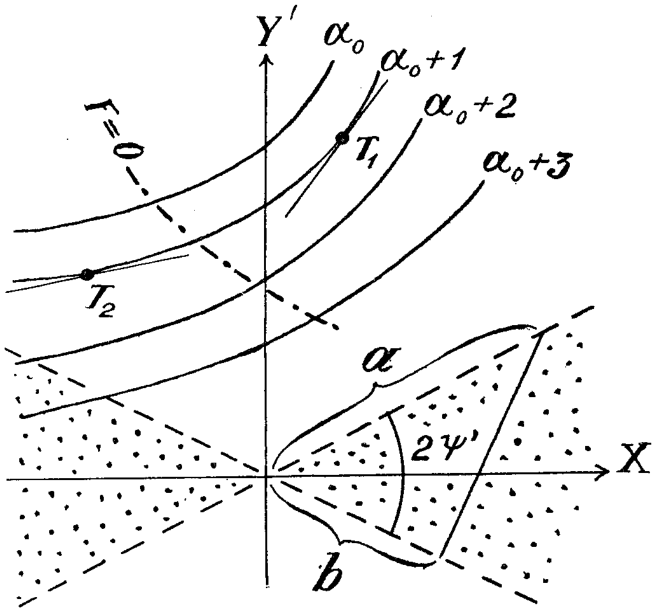


Fig. 3.

the field in areas of positive and negative F, or into *areas of frontogenesis* and *areas of frontolysis*. In the negative area the isolines of a diverge from the sector of frontogenesis and in the positive area the isolines converge towards the sector of frontogenesis.

The terms frontogenetical and frontolytical sectors are probably not very logical. The terms frontogenetical and frontolytical angles might be more adequate names, but the term sector is preferred in order to facilitate the discussion in § 13.

Returning again to (1) we see that when $b = 0$ (i. e. no divergence) ψ' is $\pm \frac{\pi}{4}$. Since (1) is independent of a contingent existence of translatory and rotational components of the field of motion, we see that, when there is no divergence, the sector of frontogenesis is $\frac{\pi}{2}$ and symmetrical with respect to the axis of dilatation. Likewise, the sector of frontolysis is $\frac{\pi}{2}$ and symmetrical with respect to the axis of contraction. A contingent curl, therefore, does not influence the position and the magnitude of the sectors. The axis of dilatation bisects the frontogenetical sector; and the axis of contraction bisects the frontolytical sector.

The curl, however, acts in such a way that the *axis of outflow* of the *resultant* field forms an angle with the axis of dilatation. It is important to note this for the discussion in § 13.

When b increases from zero to a (a being always positive), the sector of frontogenesis decreases from $\frac{\pi}{2}$ to 0. Thus, when $b = a$, F is positive nowhere in the field, and there cannot be frontogenesis anywhere. This means that the isolines move in such a way that $|\nabla a|$ decreases, except when the isolines of a are parallel to the x-axis, in which case $F = 0$, and $|\nabla a| =$ constant.

When $b > a$, (1) cannot hold, and F does not vanish anywhere, it being negative throughout the field.

When b decreases from zero to $-a$ the sector of frontogenesis widens and becomes equal to π. and the two symmetrical sectors cover the whole area. In this case the isolines move in such a way that $|\nabla a|$ increases everywhere, except where the isolines of a are perpendicular to the x-axis, in which case $F = 0$, and $|\nabla a|$ constant.

When $-b > a$, (1) becomes meaningless, which means that F does not vanish anywhere.

From the above discussion we learn that the x-axis (axis of dilatation) decides the orientation of the frontogenetical and the frontolytical sectors. The width of the sectors relative to $\frac{\pi}{2}$ is determined by the coefficient of divergence. *Translation and curl influence neither the orientation nor the width of the sectors.*

We shall return to this discussion in § 13 in connection with various types of linear fields of motion, and we shall then see that the curl component plays an important part in deciding the movement of the line of frontogenesis.

10. Determination of the Centre and Line of Frontogenesis. We shall first suppose that the field of motion and the field of property are given in the shape of mathematical equations. It is then most convenient to use the equation 8 (1) for determining the centre of frontogenesis. Substituting 8 (3) in 8 (1), we get the following equations for determining the centre of frontogenesis: (see § 5).

(1)
$$\frac{\partial F}{\partial x} = |\nabla \alpha|^{-3}[(a+b)(-\alpha^3_{10}\alpha_{20} - 2\alpha_{10}\alpha^2_{01}\alpha_{20} + \alpha^2_{10}\alpha_{01}\alpha_{11}) + (b-a)(-\alpha^3_{01}\alpha_{11} - 2\alpha^2_{10}\alpha_{01}\alpha_{11} + \alpha_{10}\alpha^2_{01}\alpha_{20})] = 0,$$

(1)
$$\frac{\partial F}{\partial y} = |\nabla \alpha|^{-3}[(a+b)(-\alpha^3_{10}\alpha_{11} - 2\alpha_{10}\alpha^2_{01}\alpha_{11} + \alpha^2_{10}\alpha_{01}\alpha_{02}) + (b-a)(-\alpha^3_{01}\alpha_{02} - 2\alpha^2_{10}\alpha_{01}\alpha_{02} + \alpha_{10}\alpha^2_{01}\alpha_{11})] = 0.$$

These equations can only exist simultaneously when the determinant

$$|\nabla \alpha|^{-6}\begin{vmatrix} -\alpha^3_{10}\alpha_{20} - 2\alpha_{10}\alpha^2_{01}\alpha_{20} + \alpha^2_{10}\alpha_{01}\alpha_{11}, & \alpha^3_{01}\alpha_{11} - 2\alpha^2_{10}\alpha_{01}\alpha_{11} + \alpha_{10}\alpha^2_{01}\alpha_{20} \\ -\alpha^3_{10}\alpha_{11} - 2\alpha_{10}\alpha^2_{01}\alpha_{11} + \alpha^2_{10}\alpha_{01}\alpha_{02}, & \alpha^3_{01}\alpha_{02} - 2\alpha^2_{10}\alpha_{01}\alpha_{02} + \alpha_{10}\alpha^2_{01}\alpha_{11} \end{vmatrix} = 0,$$

which developed gives:

(2)
$$\frac{\alpha_{10}\alpha_{01}(\alpha_{20}\alpha_{02} - \alpha^2_{11})}{\alpha^2_{10} + \alpha^2_{01}} = 0.$$

This equation contains all possible solutions of (1). In the general case (1) determines a centre which in special cases may degenerate into a line (i. e. the line of frontogenesis). This happens when

$$\alpha_{20}\alpha_{02} - \alpha^2_{11} \equiv 0,$$

or when the isolines of α are parallel and straight lines. Imaginary centres and singularities may also occur.

It is, however, of but little use to discuss these equations without any specifications as to the nature of the function α, because only very special types of functions will represent the field of property as it occurs in nature. We shall, therefore, turn to the case when the distribution of α is given as a map of its equi-scalar curves. It is then most convenient to base the discussion on the equation 8 (5).

As will be shown in § 13 the direction of the axis of dilatation and the ratio $\frac{b}{a}$ may be determined from the analysis of the wind chart. Substituting **9** (1) in 8 (5) we obtain:

(3)
$$F = a\,|\nabla \alpha|\,(\cos 2\psi - \cos 2\psi').$$

The variables $|\nabla \alpha|$ and ψ may be evaluated from the α-chart, and the distribution of F is easily found.

A few general results may be deduced from (3). Differentiating partially with respect to x and y and equating to zero, we obtain the following equations for determining the centre of frontogenesis:

(4)
$$\frac{\partial F}{\partial x} = a\frac{\partial|\nabla \alpha|}{\partial x}(\cos 2\psi - \cos 2\psi') - 2a|\nabla \alpha|\sin 2\psi\frac{\partial \psi}{\partial x} = 0,$$
$$\frac{\partial F}{\partial y} = a\frac{\partial|\nabla \alpha|}{\partial y}(\cos 2\psi - \cos 2\psi') - 2a|\nabla \alpha|\sin 2\psi\frac{\partial \psi}{\partial y} = 0.$$

When $\frac{\partial \psi}{\partial x} \equiv \frac{\partial \psi}{\partial y} \equiv 0$, the isolines of α are parallel and straight lines. In this case the centre of frontogenesis is determined by:

$$\frac{\partial|\nabla \alpha|}{\partial x} = \frac{\partial|\nabla \alpha|}{\partial y} = 0.$$

But since the isolines are parallel and straight lines it follows that $|\nabla \alpha|$ is constant along each isoline. The centre of frontogenesis, therefore, degenerates into a line which is the line of frontogenesis. *The line of frontogenesis is then situated along the line where the magnitude of the ascendant of α has a maximum.*

The case of quasi-parallel isolines occurs very frequently in nature when fronts develop between two different source regions of property. (See Chapter IV).

11. Illustrative Examples. At this point it might be useful to consider a couple of examples in order to demonstrate, from a practical point of view, the way to proceed in order to detect frontogenesis. Fig. 4a shows a map of a hypothetical distribution of property of a type which often occurs in nature, the distribution being characterized by a region of quasi-homogeneous low value of property (say temperature) in the north and north west and a similar region with high value in the south. We suppose that the direction of the axis of dilatation and the angle ψ' has been evaluated from the wind chart as described in § 13. The direction of the axis of dilatation and the angle ψ' are indicated in the upper left corner of the figs. 4c and 4d. In the present case ψ' is 52°, or $\frac{b}{a} = -\frac{1}{4}$, which corresponds to a slight convergence superimposed on a deformation. The problem is to find the distribution of F. (Equation **10** (3).)

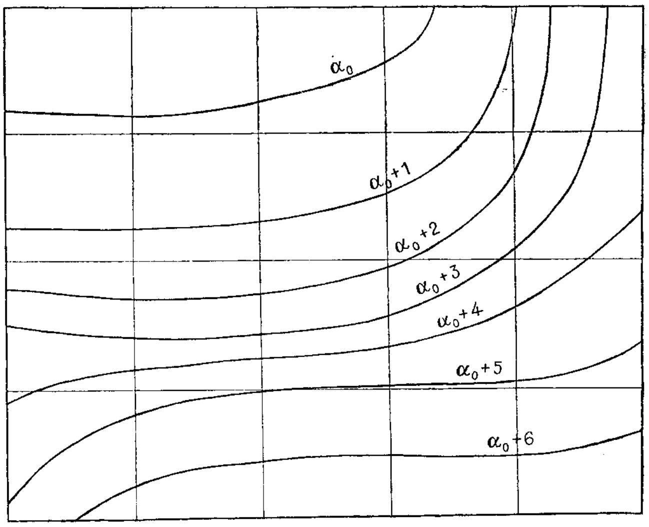


Fig. 4 a.

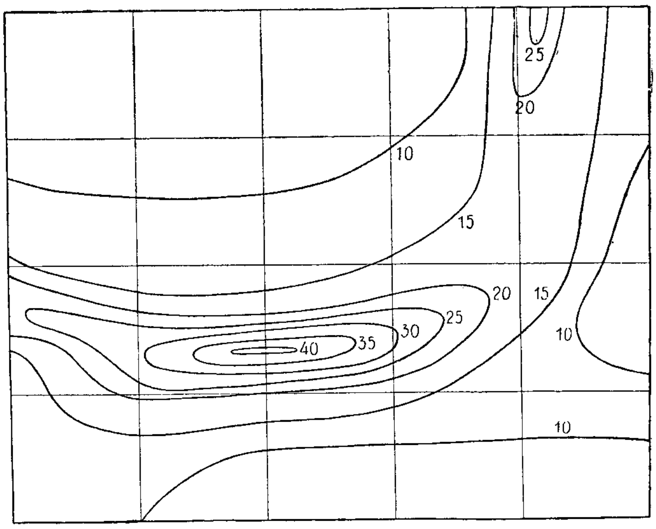


Fig. 4 b.

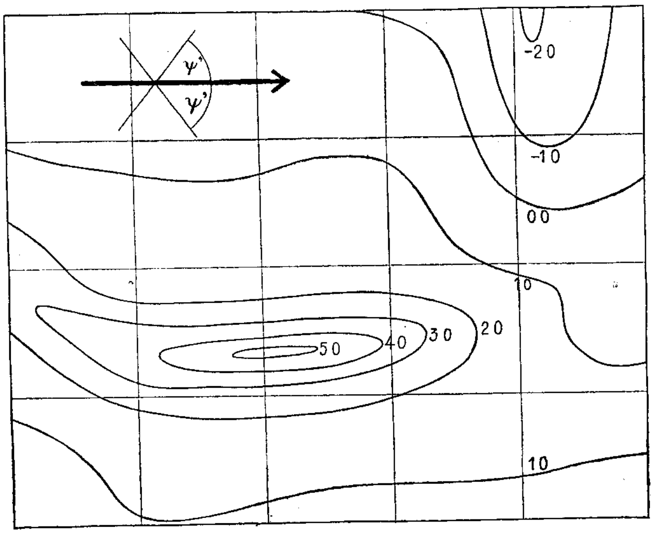


Fig. 4 c.

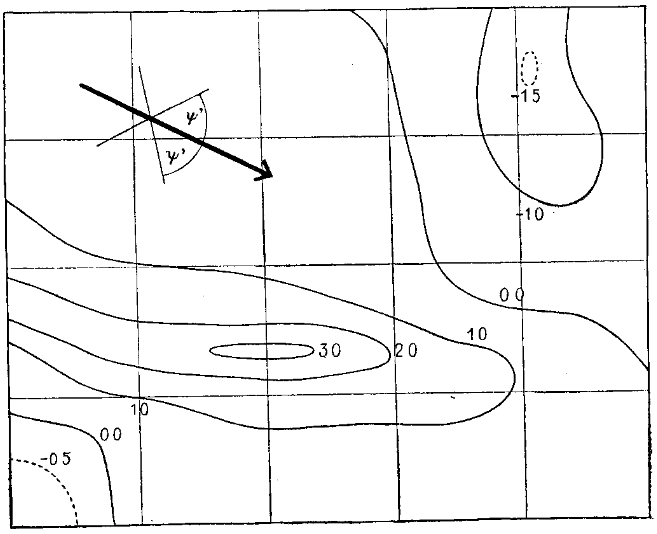


Fig. 4 d.

Fig. 4 b shows the distribution of $|\nabla\alpha|$ in relative numbers depending on the choice of units. (Arbitrary orthogonal co-ordinate lines are drawn in order to facilitate the comparison between the illustrations.) Mapping the quantity $(\cos 2\psi - \cos 2\psi')$ and multiplying graphically by fig. 4 b we obtain fig. 4 c, which shows the distribution of F when the direction of the axis of dilatation is as indicated in the upper left corner. The centre of frontogenesis (proper) is situated within the 50-isoline of F, and the centre of frontolysis is situated within the — 20-isoline. The line of frontogenesis (or frontolysis) runs through the centres and through the points where the the F-lines have maximum of curvature, but the position of the line is indeterminate in the vicinity of the curve $F = 0$.

Comparing fig. 4 b and 4 c, we see that the centres are situated where $\nabla\alpha$ has maxima. In the regions where the α-lines are quasi-parallel the line of frontogenesis coincides with the line along which $|\nabla\alpha|$ has a maximum. Even outside this region the line of frontogenesis is very close to the maximum line in $|\nabla\alpha|$.

Fig. 4 d shows the distribution of F when the α-field is as shown in fig. 4 a, and the direction of the axis of dilatation as indicated by the arrow. Comparing the figs. 4 c and 4 d we see that, owing to the unfavourable direction of the axis, frontogenesis is much less active than in the previous case (fig. 4 c). However, the line of frontogenesis has mainly the same position as in fig. 4 c, and in the area of quasi-parallel α-curves the line of frontogenesis is uninfluenced by the change in the direction of the axis. In all practical cases it suffices to place the line of frontogenesis by means of the simple rule that *it is quasi-coincident with the line along which* $|\nabla \alpha|$ *is maximum.* This rule holds with sufficient accuracy except in the vicinity of the line along which $F = 0$, but in this area frontogenesis is negligible.

For practical purposes it is highly important to note the angle between the α-curves and the axis of dilatation, Thus, along the east coast of Greenland there is usually a zone of maximum temperature gradient, and, therefore, there is frequently favourable conditions for frontogenesis. But, if the axis of dilatation points towards SE there will usually be frontolysis. If the same axis points towards NE there will usually be frontogenesis. When the angle between the axis and the isotherms is near the critical value ϑ' there is neither frontogenesis nor frontolysis.

However, a line of frontolysis will in general move and it will usually pass the critical direction and then change into a line of frontogenesis. It is, therefore, necessary not only to make out the instantaneous distribution of F, but also to find out the movement of the line of frontgenesis and its chances of development. In order to do this we must discuss the various types of stream lines.

12. Classification of Stream Line Patterns. Since our aim is to investigate the possibilities of frontogenesis in the various types of linear fields, it is important to distinguish the various types of stream line patterns. It is then natural to classify the patterns according to the number and nature of straight stream lines.

The field of motion is given by 7 (1) viz.

$$(1)\qquad \begin{aligned} u &= u_0 + (b+a)\,x - cy \\ v &= v_0 + \quad cx \qquad + (b-a)\,y. \end{aligned}$$

The field has a centre when u and v vanish simultaneously in one point (x_0, y_0) only. Putting $u = v = 0$ and solving (1) with respect to x and y, we obtain:

$$(2)\qquad x_0 = \frac{\begin{vmatrix} -u_0, & -c \\ -v_0, & b-a \end{vmatrix}}{\begin{vmatrix} b+a, & -c \\ c, & b-a \end{vmatrix}} \qquad y_0 = \frac{\begin{vmatrix} a+b, & -u_0 \\ c, & -v_0 \end{vmatrix}}{\begin{vmatrix} a+b, & -c \\ c, & b-a \end{vmatrix}}$$

The following three cases may occur:
(A) x_0 and y_0 are finite. The field of motion has a centre. (B) x_0 and (or) y_0, are indeterminate. The field has no centre, but there is a line along which there is no flow. (C) x_0 and y_0 are infinite, in which case there is a line along which the flow is constant.

A. Central Patterns. The condition for having a centre is expressed by

$$\begin{vmatrix} b+a, & -c \\ c, & b-a \end{vmatrix} \gtrless 0,$$

or:

$$(3)\qquad b^2 - a^2 + c^2 \gtrless 0.$$

By parallel translation of the system of coordinates to the centre we obtain (indices dropped):

$$(4)\qquad \begin{aligned} u &= (b+a)\,x - cy \\ v &= cx + (b-a)\,y. \end{aligned}$$

The condition for having straight stream lines through the centre is:

$$\frac{dy}{dx} = \frac{y}{x}.$$

Substituting in (4), we obtain

$$(5)\qquad \frac{y}{x} = \frac{a \pm \sqrt{a^2 - c^2}}{c}$$

independent of b.

Four classes of central stream line patterns may be distinguished:

1°: $a = c = 0$. According to (3) we have $b \gtrless 0$. The field is one of pure divergence, and there is an infinite number of straight stream lines through the centre. (Figs. 5 and 6).

2°: $a^2 > c^2$. This in connection with (3) gives:

$$(6)\qquad b^2 \lesseqgtr a^2 - c^2 > 0.$$

There are two straight stream lines through the centre.

The inequality (6) may be written:

$$b^2 < a^2 - c^2 > 0 \quad \text{or} \quad b^2 > a^2 - c^2 > 0.$$

The first type gives stream lines of hyperbolic character with two straight stream lines as asymptotes. The second gives parabolic stream lines.

Let us first consider the hyperbolic type of stream lines. Since $0 \overline{\gtrless} b^2 < a^2 - c^2$, we see that a must be larger than b and c. From (5) we see that the slope of the straight stream lines is independent of b. From 8 (5) we see that the sector of frontogenesis is independent of c. Therefore, the relative magnitude of b and c is of no consequence for the type of stream lines. It, therefore, suffices to examine 9 types of hyperbolic stream lines. These are illustrated in § 13. (Figs. 7—15).

It is a characteristic feature of all hyperbolic patterns that the sector of frontogenesis always is less than π, and that the line of outflow is situated in the sector of frontogenesis, and the line of inflow in the sector of frontolysis. Put $a = k_1 c = k_2 b$. Since a is larger than the magnitudes of c and b, we see that k_1 and k_2 are larger than 1. Let φ be the angle between the x-axis and the nearest straight stream line. We then have:

$$\tan \varphi = k_1 \pm \sqrt{k_1^2 - 1}$$

$$\tan \psi' = \sqrt{\frac{k_2 - 1}{k_2 + 1}}$$

Putting $\psi' = \varphi$, we obtain:

$$k_1 = \frac{k_2}{\sqrt{k_2^2 - 1}},$$

which does not agree with the condition $b^2 < a^2 - c^2 > 0$, which involves:

$$k_1 < \frac{k_2}{\sqrt{k_2^2 - 1}},$$

whence one sees that $\varphi < \psi'$. Therefore, in the hyperbolic types of stream line patterns the asymptote which is the line of outflow is always situated in the sector of frontogenesis, and the other asymptote, which is the axis of inflow, is always situated in the sector of frontolysis. This feature is of crucial significance for the estimation of the possibilities of frontogenesis, as will be demonstrated in § 13.

Turning now to the parabolic type of stream lines which is characterized by:

$$b^2 > a^2 - c^2 > 0,$$

we see that there are 22 possible combinations of a, b and c which satisfy the criterion. (See Table I.)

Table I.

No.	$b^2 > a^2 - c^2 > 0$
1— 2	$\pm b > a > c = 0$
3— 6	$\pm b > a > \pm c > 0$
7—10	$\pm b = a > \pm c > 0$
11—14	$a > \pm b > \pm c > 0$
15—18	$a > \pm c = \pm b > 0$
19—22	$a > \pm c > \pm b > 0$

However, since the relative magnitude of b and c is of no consequence for the type of stream lines, and since $\pm b > a$ and $\pm b = a$ give either frontolysis ($b > 0$) or frontogenesis ($b < 0$) all over the field, we see that only a few cases need be examined. The chances for frontogenesis in these patterns are discussed in § 13. (Figs. 16—19.)

It is well to note that parabolic stream lines are rarely observed in nature, and are improbable as dynamically stable systems. Inspecting Table I, we see that the cases 1—10 are characterized by excessive divergence. In these cases a field of curl would develop rapidly and the stream lines would rapidly change into a new type. (See § 14).

The cases 11—22 in Table I also are rare and improbable because they can only exist within narrow intervals of b, owing to the condition heading the table.

3°. $a = \pm c > 0$. In this case $b \gtrless 0$. There is only one straight stream line through the centre. The angle between the x-axis and the straight stream line is $\frac{\pi}{4}$ or $-\frac{\pi}{4}$ according to whether c is positive or negative. Four typical cases are discussed in § 13. (Figs. 20—23.)

Fields of this type are rare because they can only exist when $a = \pm c$.

4°. $a^2 < c^2$. This in connection with (6) gives $a^2 - c^2 < 0 \gtreqless b^2$. There are no straight stream lines through the centre (see (5)).

This class of stream lines is highly important because it represents all sorts of motion around cyclonic and anticyclonic centres.

Table II shows the various types which may occur:

Table II.

$a^2 - c^2 < 0 \leqq b^2$			
1— 2	$\pm c > a = b = 0$	17—20	$\pm c > \pm b > a = 0$
3— 4	$\pm c > a > b = 0$	21—24	$\pm c = \pm b > a = 0$
5— 8	$\pm c > a > \pm b > 0$	25—28	$\pm c = \pm b > a > 0$
9—12	$\pm c > a = \pm b > 0$	29—32	$\pm b > \pm c > a > 0$
13—16	$\pm c > \pm b > a > 0$	33—36	$\pm b > \pm c > a = 0$

Since the relative magnitude of b and c is of no consequence for the type of stream lines, and since $a = \pm b$ is not qualitatively different from $a > \pm b$, it suffices to examine only a small number of cases. These are discussed in detail in § 13. (Figs. 24—35.)

B. Straight Stream Lines. Let us next consider the case when the equations (2) are indeterminate, or when:

$$b^2 - a^2 + c^2 = 0 \tag{7}$$

$$\frac{v_0}{u_0} = \frac{c}{b+a} = -\frac{b-a}{c}$$

This substituted in (1) shows that u and v vanish everywhere on a straight line whose slope is $\frac{b+a}{c} = -\frac{c}{b-a}$. By parallel translation of the system of co-ordinates so that the origin falls on this line we obtain: (indices dropped)

$$u = (b+a)\,x - cy, \tag{8}$$

$$v = cx + (b-a)\,y.$$

The slope of the stream lines is $\frac{dy}{dx} = \frac{v}{u}$. Substituting from (7) and (8) we obtain:

$$\frac{dy}{dx} = \frac{c}{b+a} = -\frac{b-a}{c} = \frac{v_0}{u_0},$$

independent of x and y. All stream lines are straight lines throughout the field. Eight qualitatively different cases occur. These are discussed in § 13. (Figs. 36—43.)

C. Curved Stream Lines without Centre. Curved stream lines without centre occur when the numerators in (2) are different from zero, and the denominators equal to zero. We then have:

$$\frac{v_0}{u_0} \gtrless -\frac{b-a}{c} = \frac{c}{b+a} \tag{9}$$

Comparing this with (7) it is easily seen that the field of motion in this case may be divided into two parts: one which is congruent with the cases discussed in B (straight stream lines) and another which is a constant translation superimposed on and forming an angle with the straight stream lines.

In this case the field of motion results from Figs. 36—43 by adding a constant translation which has a component normal to the straight stream lines. These cases are discussed in § 13. (Figs. 44—51.)

The aim of the somewhat lengthy discussions in this paragraph has been to show how the various types of stream lines depend on the combination of translation, deformation, divergence and curl. In § 9 we have seen that frontogenesis depends on the angle between the axis of dilatation and the equiscalar α-curves. But the axis of dilatation is usually neither coincident with nor parallel to the axes in the resultant field of motion. It is, therefore, of crucial significance to develop methods for locating the direction of the axes of deformation when the field of motion is obtained as a result of the analysis of weather charts and not given in the shape of mathematical equations. The deductions in this paragraph furnish us with the necessary means of locating the direction of the axes of dilatation and contraction, but it is convenient to postpone the discussion to § 13.

13. Frontogenesis in Various Types of Linear Fields. The stream line equations discussed in § 12 have been integrated for all qualitatively different types of linear stream line patterns. The integration is performed for selected values of the constants involved. The results of the integrations are contained in the figures 5—51, the frontogenetical sectors being indicated by dotted areas.

We now propose to discuss the chances for frontogenesis in the various types of stream line patterns. It is then well to remember that the line of frontogenesis is independent of the linear field of motion, its position being determined by the distribution of the field of property, as shown in § 10 and 11. When the line of frontogenesis has been found, two problems arise: a) where will the line of frontogenesis move, and b) what is the distribution of the frontogenetical effect along the line?

The line of frontogenesis, being a substantial line, will move with the air current. If the field of motion contains straight stream lines, the line of frontogenesis will move in such a way that it approaches and becomes parallel to one of the straight stream lines.

The frontogenetical effect on the line depends on the angle between the axis of dilatation and the tangent to the equiscalar curves. This axis is only rarely visible as a straight stream line. In all the following figures the axis of dilatation is chosen as the x-axis.

When the angle between the x-axis and the isolines is larger than ψ' there is frontolysis on the line, and when the angle is smaller than ψ' there is frontogenesis. The intensity of frontogenesis is largest when the line of frontogenesis is parallel to the axis of dilatation. If a straight stream line is situated between the axis of dilatation and the line of frontogenesis, then the line of frontogenesis will become stationary along the straight stream line and never become parallel to the axis of dilatation.

We now turn to the various types of stream lines and we discuss them in the same order as in § 12.

1°. Divergence. Fig. 5: The line of frontogenesis is exposed to frontolysis independent of the direction of the line. The line of frontogenesis will move out of the field and dissolve, except when it happens to run through the centre, in which case it remains stationary while it dissolves.

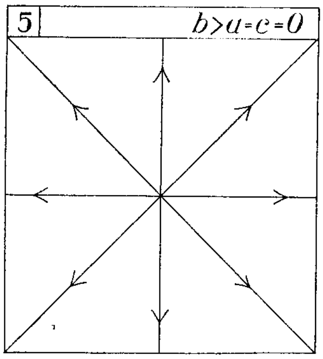


Fig. 5. Divergence.

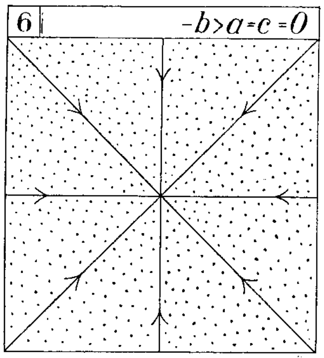


Fig. 6. Convergence.

Fig. 6: Any line of frontogenesis will increase in intensity independent of its direction. The line will move towards the centre of motion and become stationary. If the line of frontogenesis is a straight line, it will remain straight. If it is a curved line, its curvature will increase and the line will finally become stationary along two straight stream lines, and eventually become a front between two sectorially distributed air masses.

Pure divergence or convergence cannot exist by itself as a lasting phenomenon because there is no pressure field which corresponds to the motion. In nature convergence and divergence can only exist as lasting phenomena when superimposed on a field of motion which corresponds to a possible field of pressure. But it is well to note the fact that convergence helps to increase the frontogenetical process, and to suck the line of frontogenesis into the centre of motion.

2°. Hyperbolic Stream Lines. (Figs. 7—15.) Fig. 7 shows the case of pure deformation. If the line of frontogenesis happens to be parallel to the

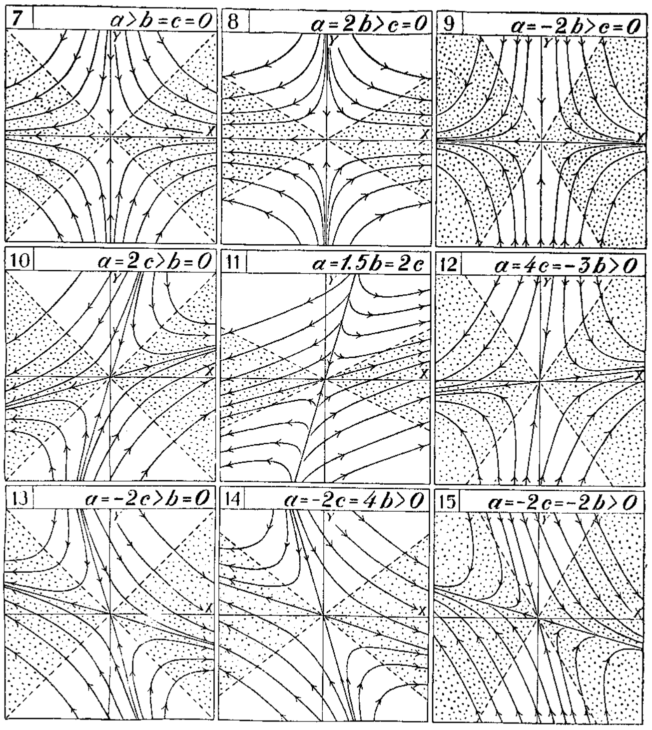


Fig. 7—15, showing the types of stream lines when $b^2 < a^2 - c^2 > 0$.

axis of contraction it will always remain so. The line will then move out of the field and dissolve, except when it happens to run through the centre, in which case it remains stationary and dissolves.

If the line forms an angle with the axis of contraction it will move and ultimately become parallel to the axis of dilatation, and it will approach this axis asymptotically. Since $b = 0$, the sector of frontogenesis is $\frac{\pi}{2}$ with the axis of dilatation as bisector. The line will, therefore, be exposed to frontolysis until the angle ψ is $\pm \frac{\pi}{4}$. When this critical angle is passed, the line will be exposed to frontogenesis which increases in intensity until the line becomes parallel to the axis of dilatation. Theoretically, an infinite interval of time is needed to turn the line parallel to the axis of dilatation. Therefore, a line of frontogenesis which does not coincide with the axis of dilatation will never coincide with it, but it will steadily approach it.

Since the field of motion does not last eternally, we see that a line of frontogenesis which forms a small angle with the axis of contraction will have poor chances of developing a front, because it will first have to move for a long interval of time in the frontolytical sector. Therefore, the more parallel it is to the axis of dilatation the greater is the chance of developing a front. Moreover, the chance for developing a front is larger when a is large, because the line will then move more quickly and sooner come into the frontogenetical sector, and more quickly approach the axis.

Fig. 8 shows a field of divergence superimposed on the deformative field and forming a hyperbolic pattern. We see that the sector of frontogenesis is small. The chances for frontogenesis are, therefore, poor because of the width of the frontolytical sector. On the other hand a line of frontogenesis which is exposed to frontolysis will relatively rapidly be turned so that it becomes exposed to frontogenesis because there is a stronger current parallel to the x-axis than parallel to the y-axis. In other respects this field is similar to fig. 7.

Fig. 9 shows a field of convergence superimposed on a field of deformation. The sector of frontogenesis is large and the chances for frontogenesis are large. On the other hand, when the line of frontogenesis forms an angle with the x-axis which is larger than $\pm\psi'$, it will turn slowly towards this axis, because the flow along the x-axis is smaller than normal to it.

Fig. 10 shows a field of curl superimposed on the field of deformation and forming a hyperbolic pattern. We see that the line of outflow is deviated from the axis of dilatation, and the line of inflow from the axis of contraction. Since $b = 0$, the frontogenetical sector is $\frac{\pi}{2}$ bisected by the axis of dilatation. Let us imagine that the y-axis points northwards. We may then say that a line of frontogenesis that comes from the north will stop at the line of outflow. The effective sector of frontogenesis is, therefore, small in the NE-quadrant but large in the NW-quadrant. Conversely, a line of frontogenesis that comes from the south will cross the eastern half of the axis of dilatation, but not the western half. The effective sector of frontogenesis is therefore large in the SE-quadrant but small in the SW-one.

Fig. 11 shows a field of divergence superimposed on fig. 10. The frontolytical sectors are large, and the effective sectors of frontogenesis are excessively small in the NE and SW-quadrants, but moderately large in the other quadrants.

Fig. 12 shows a field of convergence superimposed on deformation and curl.

Fig. 13—15 shows stream line patterns corresponding to figs. 10—12 but with negative curl.

In each particular case the line of frontogenesis is determined by the field of α, and must be found by studying the equiscalar α-curves. When the line is found as shown in § 10 and 11, it is easy to estimate the possibilities of effective frontogenesis. In doing so the following facts, which are in common for all hyperbolic patterns, should be noted:

(1) *The field of motion need not be stationary in order to produce frontogenesis.* Since the line of frontogenesis is independent of the distance from the axes of deformation and only dependent on the angle ψ, and since the deformation and divergence are constant, it follows from 8 (5) that the frontogenetical effect will proceed in the same way whether the axes of deformation are stationary or moving parallel to themselves.

(2) If the axis of outflow rotates in such a way that the angle between the axis and the line of frontogenesis decreases, then the frontogenetical effect is increased, because the line of frontogenesis will then sooner become parallel to the axis.

(3) If the axis of outflow rotates the other way, the frontogenetical process is retarded.

(4) The hyperbolic stream line patterns correspond to similar patterns of isobars. The changes in the field of pressure may be estimated or computed according to the methods developed in previous papers.[1]) In this way the changes in the field of motion may be anticipated.

(5) Fields with divergence develop negative curl, and fields with convergence develop positive curl. (See § 14.)

(6) When the stream line pattern is given as a result of the analysis of weather charts the direction of the axes of deformation may be determined in this way: Halve the angles between the straight stream lines, and draw the lines which form the angles $\pm\frac{\pi}{4}$ with the halving lines. These lines are axes of deformation. The axis of dilatation

[1]) Sverre Petterssen: Kinematical and Dynamical Properties of the Field of Pressure etc. Geof Publ. Vol. X, No. 2.
Sverre Petterssen: Practical Rules for Prognosticating the Movement and Development of Pressure Centres. Procès-Verbaux des séances de l'Association de Météorologie. U. G. et G. Int. Paris 1935.

is the one that is nearest to the line of outflow. The axis of contraction is the one that is nearest to the line of inflow. (This follows from the deductions in § 12.)

(7) The width of the sector of frontogenesis is determined in this way: The slope of the straight stream lines relative to the axis of dilatation is given by **12** (5) viz.,

$$\frac{y}{x} = \frac{a \pm \sqrt{a^2 - c^2}}{c} \tag{1}$$

The sector of frontogenesis is given by: **9** (1) viz.,

$$\cos 2\psi' = \frac{b}{a} \tag{2}$$

The slope of the stream lines is:

$$\frac{dy}{dx} = \frac{v}{u} = \frac{(b-a)\,y + cx}{(b+a)\,x - cy} = \frac{(b-a)\frac{y}{x} + c}{b + a - c\frac{y}{x}} \tag{3}$$

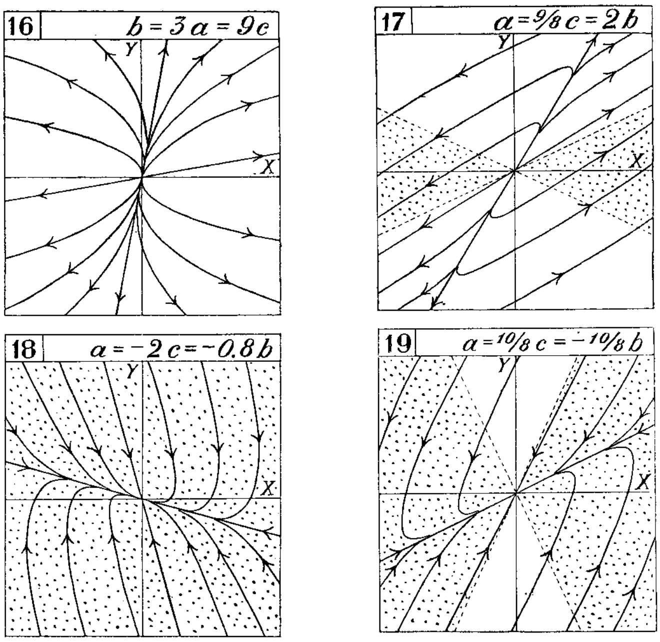
and constant along any straight line through the centre. Evaluate the ratio $\frac{a}{c}$ from (1) as explained above. Evaluate the constant $\frac{v}{u}$ along a straight line $\left(\frac{y}{x} = \text{const.}\right)$ through the centre (not coinciding with the straight stream lines). Take the means. Substitute in (3) and obtain $\frac{b}{a}$, and find ψ' from (2).

Thus for all hyperbolic stream line patterns the direction of the axes of deformation and the width of the sector of frontogenesis may be obtained from the analysis of the wind charts.

3°. Parabolic Stream Lines. Figs. 16—19. As mentioned in § 12, these stream line patterns occur rarely in nature. As in the hyperbolic case there are two straight stream lines through the centre. The sector of frontogenesis is independent of the position of the straight stream lines.

Fig. 16 shows a case of excessive divergence with small positive curl. (For negative curl the straight stream lines would be situated in the 2nd and 4th quadrant). There is no sector of frontogenesis.

Fig. 17 shows a case with moderate divergence. A line of frontogenesis will move and become parallel to the straight stream line which is nearest to the axis of dilatation. The line would pass through the sector of frontogenesis and become stationary along the straight stream line which, in the present case, is situated in the sector of frontolysis. Frontogenesis would, therefore, be only a temporary phenomenon, and the line would ultimately dissolve.



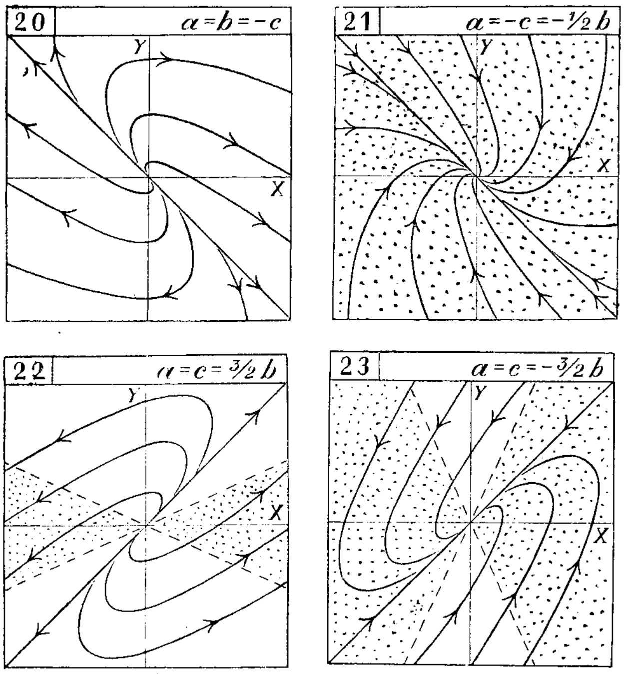
Figs. 16—19,
showing the types of stream lines when $b^2 > a^2 - c^2 > 0$.

Fig. 18 shows a case with large convergence. The sector of frontogenesis covers the whole field. Any line of frontogenesis will drift towards the line of converging stream lines. The frontogenetical effect would be most intense when the line is parallel to the x-axis (i. e. axis of dilatation). Fields of this type would soon develop a positive curl and the character of the field would change rapidly (see § 14).

Fig. 19 shows a case with moderate convergence. Excepting the narrow sector of frontolysis, the case is similar to fig. 18.

The axis of dilatation and the sector of frontogenesis, may be evaluated from the weather charts exactly as described under 2°.

4°. S-shaped Stream Lines. Figs. 20—33. The conditions with respect to frontogenesis are similar to the parabolic cases discussed above. Since there is only one straight stream line, the line of fronto-



Figs. 20—23,
showing the types of stream lines when $a = \pm c$, and $b \gtrless 0$.

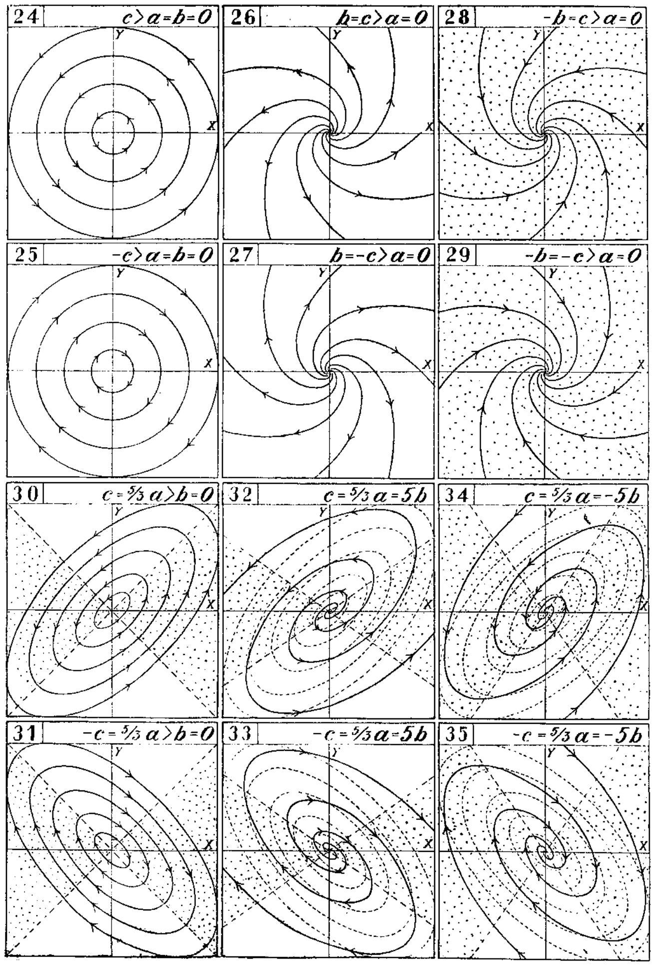
genesis will gradually become parallel to this line. The straight stream line forms plus or minus $\frac{\pi}{4}$ with the positive axis of deformation according to the sign of curl $\boldsymbol{v}$. Frontogenesis will only be a temporary phenomenon when $b > 0$, and a lasting phenomenon when $b < 0$. This is easily seen from the stream lines.

Patterns of this type occur very rarely in nature, and when they occur they do not last because they can only exist when $a = \pm c$ and $b \gtrless 0$.

The axis of dilatation is $\frac{\pi}{4}$ to the right of the straight stream line when $c > 0$, and $\frac{\pi}{4}$ to the left when $c < 0$. The sector of frontogenesis may be evaluated as in the previous cases.

5°. *Central Patterns without Straight Stream Lines.* (Figs. 24—35.) Figs. 24 and 25 show fields of positive and negative curl respectively. Since $a = b = 0$, these fields are indifferent relative to frontogenesis.

Figs. 26—29 show combinations of positive or negative curl and divergence or convergence. Since $a = 0$ the whole field is either frontolytical ($b > 0$) or frontogenetical ($b < 0$).



Figs. 24—35,
showing the types of stream lines when $a^2 - c^2 < 0 \gtreqless b^2$.

Fig. 30 shows a combination of deformation and positive curl. The stream lines are elliptical with the longest axes $\frac{\pi}{4}$ from the axis of dilatation. The situation corresponds to the geostrophic wind round an elliptical pressure centre. It is instructive to imagine lines of frontogenesis in various places relative to the centre and to discuss the chances for frontogenesis in the various cases. We shall here discuss only the case when the line of frontogenesis runs through a cyclonic centre on the northern hemisphere, and, to simplify the discussion, we suppose that the y-axis points northwards and towards the colder air masses. We then see that a line of frontogenesis that moves towards increasing curvature of the stream lines will be exposed to frontogenesis, and a line that moves towards decreasing curvature is exposed to frontolysis. If the centre is stationary the line of frontogenesis will be rolled up round the centre and periodically be exposed to frontogenesis and frontolysis. If the centre moves along its longest axis (as it usually does[1])) the line of frontogenesis may be exposed to frontogenesis or frontolysis for long intervals of time.

The following rules are useful:

(a) Fronts that move towards a trough increase in intensity.

(b) Fronts that leave a trough dissolve.

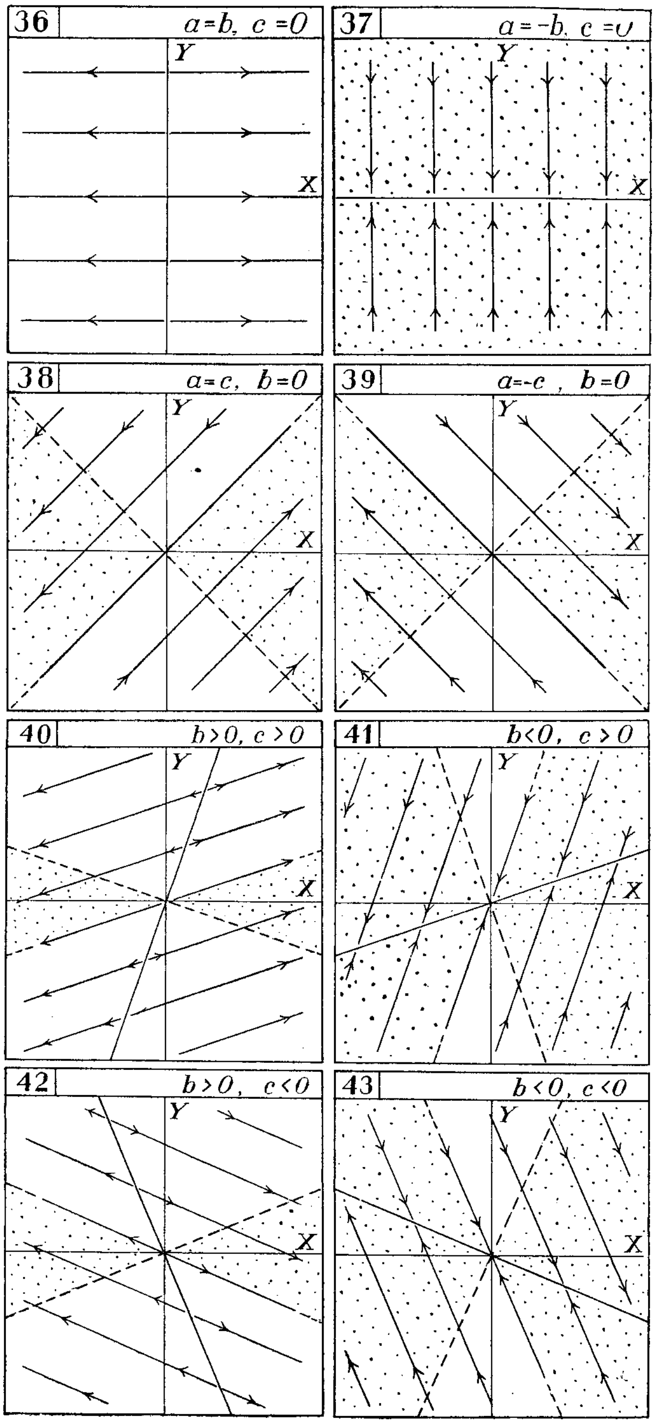
The last rule covers the case when a cold front moves with maximum of cyclonic curvature in its rear. The cold front will then dissolve while the bent back occlusion which approaches the trough will increase in intensity.[2]) A cold front that moves through the southern frontolytical sector will again increase in intensity when it comes into the eastern sector of frontogenesis, and the resulting occlusion may be a front of considerable intensity.[2])

Fig. 31 shows combination of deformation and negative curl. The situation corresponds to cyclonic movement on the southern hemisphere or anticyclonic movement on the northern hemisphere. The most interesting case of anticyclonic movement is when a line of frontogenesis comes from the north. It will then become stationary parallel to the longest axis of the anticyclone and develop a «Schleifzone» parallel to the isobars.

Figs. 32—35 show divergence or convergence superimposed on figs. 30 and 31. These patterns correspond to the general air movement round pressure centres in the northern or southern hemisphere. We see that divergence narrows the sectors of frontogenesis, and convergence widens these sectors. Otherwise the conditions are similar to the previous cases (figs. 30—31). There is, however, the principal difference that a line of frontogenesis which does not run through the centre will be sucked into the centre when there is convergence, and it will be expelled from the centre when there is divergence.

When the stream line patterns (figs. 32—35) are obtained from the analysis of weather charts the direction of the axes of deformation and the width of the frontogenetical sectors can only be evaluated by measuring the slope of the stream lines along two or more straight lines through the centre and subsequent computation of the ratios $\frac{a}{c}$ and $\frac{b}{a}$. (See page 19).

6°. *Straight Stream Lines.* Figs. 36—43 show the various cases of straight stream lines, and the



Figs. 36—43,
showing the types of stream lines when $\frac{v_0}{u_0} = \frac{c}{b+a} = -\frac{b-a}{c}$.

[1]) Sverre Petterssen: Practical Rules etc. loc. cit.

[2]) Sverre Petterssen: Kinematical and Dynamical Properties loc. cit. §§ 25, 26.

sectors of frontogenesis. The figs. 36, 37 and 40—43, which are improbable as lasting wind systems, are of no particular meteorological interest. Fig. 38, however, is interesting. It occurs in nature when the coefficient of deformation is approximately equal to the coefficient of curl. The two currents, which are parallel and of opposite direction, correspond to a cold and warm current in juxtaposition. The flow is zero on the straight line through the origin which bounds the sector of frontogenesis. The sector is $\frac{\pi}{2}$ symmetrical with respect to the axis of dilatation.

Any line of frontogenesis which is parallel to the stream lines will remain stationary and be exposed to neither frontogenesis nor frontolysis. If the line of frontogenesis has any other direction it will spin round and obtain a maximum of frontogenetical effect when it is parallel to the x-axis, and then continue to spin round till it coincides with the line of zero flow, where there is neither frontogenesis nor frontolysis.

It is of interest to note the position of the frontogenetical sectors relative to the direction of the flow.

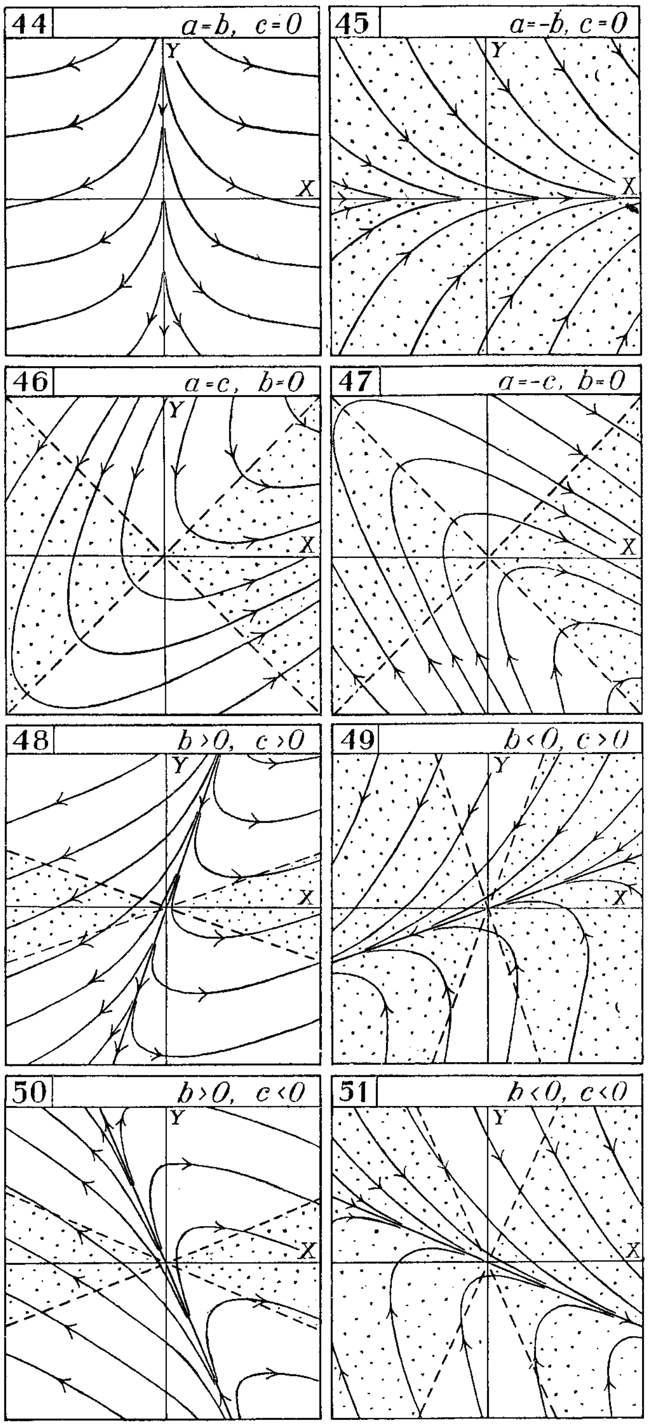
Fig. 39 shows the corresponding case when the curl is negative.

7°. Curved Stream Lines without Centre. (Figs. 44—51.) These patterns result from the figures 36—43 when a translation is added, the direction of which does not coincide with the straight stream lines. The sectors of frontogenesis are congruent with the corresponding sectors discussed under 6°.

When these stream line patterns are given as results of the analysis of weather charts the direction of the axis of dilatation may be obtained as follows:

When the stream lines are symmetrical with respect to a straight stream line the axis of dilatation is perpendicular to the straight stream line when the stream lines diverge (fig. 44), but coincident with the straight stream line when the stream lines converge (fig. 45). In the first case the sector of frontolysis covers the whole field, and in the second case the sector of frontogenesis covers the whole field.

When the stream lines are parabolic (figs. 46, 47), the axis of dilatation is deviated $\pm \frac{\pi}{4}$ from the line of symmetry according to whether the curl is positive or negative.



Figs. 44—51,
showing the types of stream lines when $\frac{v_0}{u_0} \gtrless -\frac{b-a}{c} = \frac{c}{b+a}$.

When the stream lines are asymmetrical (figs. 48—51) there is one straight stream line and an asymptotical direction to the curved stream lines.

Halve the angle between the straight stream line and the asymptote, and draw the lines $\pm \frac{\pi}{4}$ with the halving line. These lines are axes of deformation.

The width of the frontogenetical sector is determined precisely as described under 2°.

14. The Sequence of Stream Line Patterns. In the previous paragraphs we have seen how the distribution of frontogenesis may be evaluated, and in § 13 we have seen how the instantaneous movement of the line of frontogenesis may be found. The future state of frontogenesis will then depend on the changes in the field of motion.

Usually the field of motion will not remain stationary. In fact, when the divergence is different from zero, the field is bound to change its structure. The question then arises: How will the stream lines change with time? The complete discussion of this problem will be given in another paper dealing with the relation between the field of pressure and the field of motion when the field moves and changes its structure. A few comments on the question may be of interest here.

The equations of motion in the horizontal plane may be written:

$$\dot{\boldsymbol{v}} = -s\,\nabla p - \boldsymbol{\lambda} \times \boldsymbol{v}$$

where s is specific volume, p atmospheric pressure, $\boldsymbol{\lambda}$ a vertical vector whose magnitude is $2\omega \sin\varphi$, ω being the angular velocity of the earths rotation, and φ latitude. The effect of friction is neglected. Multiplying vectorially by ∇, and arranging the terms, we obtain:[1])

$$(1) \quad \frac{d}{dt}(\mathrm{curl}\,\boldsymbol{v}) = -\mathrm{div}\,\boldsymbol{v}\,(\boldsymbol{\lambda} + \mathrm{curl}\,\boldsymbol{v}) + \nabla s \times (-\nabla p).$$

The last term is the number of solenoids in the horizontal plane. This term is usually very small and may be neglected, except in the frontal zone, where it may be important when the divergence is very small. We shall here neglect the solenoidal term and see how the stream line patterns will change in accordance with the above relation between curl and divergence.

[1]) The orders of magnitude of the various terms in this equation have been discussed by Hesselberg and Friedmann, Veröff. des Geoph. Inst. d. Univ. Leipzig, zweite Serie, Heft 6, 1914.

Having neglected the solenoidal term we see from (1) that a special case occurs when

$$\mathrm{curl}\,\boldsymbol{v} = -\boldsymbol{\lambda}.$$

Curl $\boldsymbol{v}$ is then constant (i. e. zero in absolute movement) independent of div $\boldsymbol{v}$.

Returning to fig. 5, we see that a field of pure divergence must develop a negative curl, and the straight stream lines will be changed into spirals: fig. 27 on the northern hemisphere, and fig. 26 on the southern hemisphere. A field of convergence (fig. 6) will in the same way develop a positive curl and change into the spiral patterns of figs. 28 and 29. These are the most simple cases of anticyclogenesis and cyclogenesis through divergence and convergence respectively.

There is however a significant difference between the anticyclogenesis and the cyclogenesis: When negative curl develops, the vector curl $\boldsymbol{v}$ will subtract from the vector $\boldsymbol{\lambda}$, and there will be a limit, (curl $\boldsymbol{v} = -\boldsymbol{\lambda}$), which the anticyclonic curl cannot surpass. On the other hand, in the case of cyclogenesis, the positive curl adds to $\boldsymbol{\lambda}$, and the cyclonic curl may grow to any value.

In the same way all the previous stream line patterns may be discussed. We shall here only discuss the hyperbolic class, since this is the most important one with regard to frontogenesis. Figs. 7, 10 and 13 will have a tendency to be maintained because $b = 0$, and, therefore, $\frac{d}{dt}$ (curl $\boldsymbol{v}$) $= 0$, (except for the influence of the solenoidal term, which is here neglected). Since these patterns have a tendency to preserve their structures, the frontogenesis may last for a long interval of time.

Let us next consider fig. 8. Since $b > 0$, a negative curl will develop, and the stream lines will change into a pattern similar to fig. 14, and with increasing negative curl the stream lines will change through a series of patterns which are illustrated qualitatively in fig. 52. We thus see that the stream line patterns of figs. 8 and 11 will have a tendency to develop into states which not only disagree with cyclonic development, but even

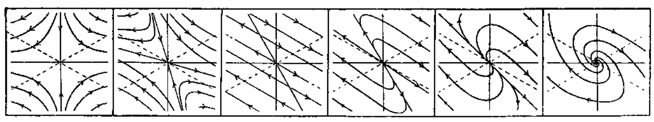


Fig. 52.
Anticyclogenesis through divergence.

may result in complete anticyclogenesis. In these cases, also, there is a limit (curl $\boldsymbol{v} = -\lambda$) which the negative curl cannot exceed. It should be noted that the 3rd, 4th and 5th pattern in fig. 52 can only exist for short intervals of time.

The hyperbolic type of stream line patterns with tendency to cyclonic development is illustrated by figs. 9, 12 and 15. Since $b < 0$, a positive curl will develop: figs. 9 and 12 will change into fig. 15. By increasing positive curl these patterns will run through a number of typical stages which are illustrated qualitatively in fig. 53. We see that hyper-

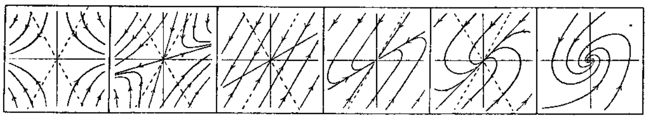


Fig. 53.
Cyclogenesis through convergence.

bolic patterns with convergence have a tendency to develop into cyclonic patterns. In these cases there is no limiting value for the amount of curl that can develop. The 3rd, 4th and 5th pattern in fig. 53, which can only exist for very short intervals of time, occur when the pressure field changes from a hyperbolic to a cyclonic pattern.

The above results, *that there is an upper limit which the anticyclonic curl cannot exceed, but no such limit for the cyclonic curl,* agree with what is observed in the weather charts.

Naturally, this tendency towards development either into cyclonic or anticyclonic systems may not last long enough to insure the ultimate result, but the instantaneous tendency is always clear from the kinematical analysis, and the above rules are very useful both for the prognostication of cyclogenesis and anticyclogenesis and for the estimation of the duration of the frontogenesis because they show what kind of motion is likely to develop. The problem then is to make out the distribution of convergence and divergence, and to discover the first signs of beginning convergence and divergence. In practical cases this is most conveniently done by studying the distribution of the isallobaric gradients, as has been shown in previous papers.[1])

The above rules hold when there is no solenoidal term to modify the process. Since the solenoidal term is negligible when there is no frontal zone, we see that cyclogenesis and anticyclogenesis may take place as a result of convergence and divergence. *Cyclogenesis need, therefore, not be connected with fronts, and fronts may form in frontless cyclones,* as has been shown in § 13.

[1]) Sverre Petterssen: Practical Rules etc. loc. cit.

CHAPTER III.

FRONTOGENESIS IN LINEAR FIELDS OF PROPERTY

15. General Remarks. In Chapter II we studied the conditions under which frontogenesis takes place in a linear field of motion, and we found that a linear field of motion could only produce frontogenesis when the field of property was non-linear. It might then be expected that frontogenesis also could occur in a linear field of property when, simultaneously, the field of motion is non-linear. It appears, however, that this case is only of secondary importance, and we shall, therefore, limit the discussion to some general remarks.

We suppose that the field of property is a linear function of x and y. Choosing the x-axis along the isolines of α, and the positive y-axis towards increasing α, we may write:

(1) $$\nabla\alpha = |\nabla\alpha|\,\boldsymbol{j}$$

where $|\nabla\alpha| > 0$, and $\boldsymbol{j}$ is the unit vector of the y-axis. Substituting this in **3** (5), we obtain:

(2) $$F = -|\nabla\alpha|\frac{\partial v}{\partial y}$$

where v is the y-component of the velocity. We then see that F is positive when $\frac{\partial v}{\partial y} < 0$ and F is negative when $\frac{\partial v}{\partial y} > 0$. The line $\frac{\partial v}{\partial y} = 0$, or the line along which v is maximum or minimum, divides the field in frontogenetical and frontolytical areas.

Substituting (1) in **4** (3), we obtain:

(3) $$\frac{\partial F}{\partial x} = -|\nabla\alpha|\frac{\partial^2 v}{\partial x \partial y} = 0$$
$$\frac{\partial F}{\partial y} = -|\nabla\alpha|\frac{\partial^2 v}{\partial y^2} = 0,$$

which equations determine the centre of frontogenesis, provided that the conditions enunciated in § 1 hold. We see that the centre of frontogenesis is the place where $\frac{\partial v}{\partial y}$ has a maximum. The line of frontogenesis is to be sought in the frontogenetical area where the variation in velocity normal to the isolines is a maximum.

We note that it is necessary that the equations (3) contain space variables, or, in other words, that v must be a function of 3rd or higher degree, which agrees with theorem (b) in § 6.

Frontogenesis in linear fields of property is a very simple phenomenon, but, nevertheless, it is only of secondary interest, because the field of property cannot *remain* linear when v is non-linear. If α is linear at a certain instant it will immediately develop into a non-linear field. The discussion of such a field should then be based on the equations **3** (5) and **4** (3) in the general case, or on 8 (5) when it is permissible to neglect the non-linear terms of the motion.

It is, however, important to note that a non-linear field of motion may start the non-linearity of the field of property which is essential in order to make frontogenesis active when the non linear area comes under the regime of linear or quasi-linear motion.

Other ways of producing non-linear variations in the distribution of property may be sought in the radiation and cooling effects which results from the distribution of land and sea, and similar radiation discontinuities.[1] No doubt these radiation discontinuities are responsible for the «embryos» of many fronts; but even on a uniform earth there would be frontogenesis because the equator-pole variation in temperature on a uniform globe *must* be non-linear, and non-linear distribution of property is the only essential condition for frontogenesis in linear fields of motion, and in non-linear fields of motion non-linear fields of property will in general intensity frontogenesis.

CHAPTER IV.
FRONTOGENESIS BETWEEN SOURCE REGIONS

16. Source Regions. The phrase source region[2]) is generally used to denote an extensive portion of the earth's surface in which the air masses approximate horizontal homogeneity. The formation of source air masses takes place on the surface of the earth whenever the air remains at rest or moves for a considerable interval of time over areas of quasi-homogeneous source properties. Examples of source regions are the subtropical anticyclones and the arctic (antarctic) or continental anticyclones over snowcovered areas.

The transition from one such source region to another is often continuous, but sometimes the air masses from different source regions are brought into juxtaposition, and, owing to the quasi-conservativism of the air mass properties, fronts are formed by advective processes.

Usually, the linear field of motion has but slight bearing on the actual problem because no linear field of velocity will cover both the source regions and the zone of transition.

In order to study the mechanism of frontogenesis in such cases we shall endeavour to introduce some functions for the field of property and the field of motion which may represent actual conditions much more acurately than the linear fields.

In order to represent the field of property we introduce a function $\alpha(x, y)$ where α approaches the source region values asymptotically. Usually the gradient of α is directed from the one source region to the other, the isolines of α being almost parallel straight lines. In such cases the field of α may be represented by means of a function of a simple exponential type. Tentatively we examine the function[1])

$$(1) \qquad \alpha = \frac{\alpha_1 + \alpha_2 e^{rs}}{1 + e^{rs}},$$

where α_1, α_2 and r are constants with respect to space variation, and s measures length along a straight line. We then see that α is continuous for finite values of r. α approaches α_2 and α_1 asymptotically when s increases or decreases infinitely. When (rs) is small, α is a quasilinear function of s. When r increases (e. g. with time) the asymptotical approach to α_1 and α_2 is more pronounced. When r increases infinitely, α becomes discontinuous for $s = 0$, its value then springing from α_1 to α_2.

Thus the formula (1) will represent an approximation to the conditions which are characteristic

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[1]) E. Gold: Fronts and Occlusions. Q. J. R. Meteor. Soc. **61** p. 156. 1935.

[2]) Tor Bergeron: Die dreidimentional verknüpfende Wetteranalyse. Teil I. Geof. Publ. Vol. V., No. 6, and H. C. Willet: American Air Mass Properties. Papers in Physical Oceanography and Meteorology. Cambridge, Mass. 1933

[1]) Sverre Petterssen: Kinematical and Dynamical Properties etc. loc. cit. § 20.

for source regions and the transitional zone during the entire process of frontogenesis. a_2 and a_1 may be called the source region values of property, and $r = r(t)$ will make the a-field vary in the same way as during a conservative frontogenesis.

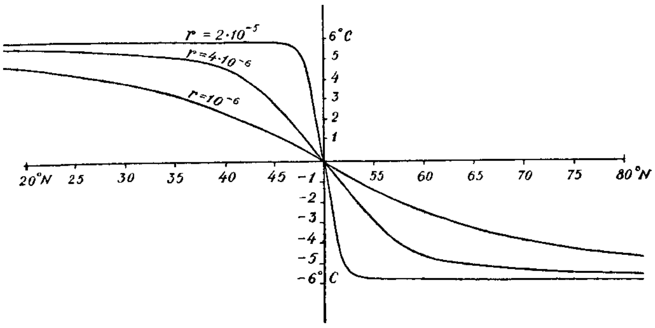

Fig. 54,
showing a section through two adjacent source regions for three different values of r. The field of property is temperature. A is 12° C. r is given in m^{-1}.

Fig. 54 shows the distribution of a along the axis s for appropriate values of r.

In order to combine the above function with the field of motion we introduce space variables (x, y) and put $a_2 - a_1 = 2A$, and $a_2 + a_1 = 2a_0$. We may then write:

$$a = a_0 + A \frac{Ne^{px+qy} - 1}{1 + Ne^{px+qy}}, \tag{2}$$

where N is a constant depending on the choice of system of co-ordinates. $N = 1$ when the origin is chosen on the line where $a = a_0$.

The next problem is to find functions which approximate a continuous distribution of velocity between two source regions. Since the linear field gives velocities which increase infinitely with increasing distance, we shall try to introduce a field of motion which is linear for small values of x and y and approaches a constant value with increasing distance. Such a field of velocity will in most cases be applicable to a larger area of the charts than is the linear field.

In choosing such a field we use the quality of the above function and the analogy with the linear field of motion. The following table shows a set of such functions which may be used to approximate the actual conditions.

From this table we see that the velocity components in the last column are identical to the linear components when x and y are small.[1]) When x and y increase, the velocity components will increase less rapidly than the linear components and approach certain maximum values with increasing distance. Thus, by appropriate choice of the constants involved, these formulae may be used for representing the field of motion within the transitional zone between the source regions.

[1]) Develop in a Taylor series.

Velocity components.

Linear Field	Approximation to actual conditions
Translation u_0, v_0	u_0, v_0
Deformation ax, $-ay$	$a_1 \frac{e^{nx} - 1}{1 + e^{nx}}$, $-a_1 \frac{e^{ny} - 1}{1 + e^{ny}}$
Divergence bx, by	$b_1 \frac{e^{nx} - 1}{1 + e^{nx}}$, $b_1 \frac{e^{ny} - 1}{1 + e^{ny}}$
Rotation $-cy$, cx	$-c_1 \frac{e^{ny} - 1}{1 + e^{ny}}$, $c_1 \frac{e^{nx} - 1}{1 + e^{nx}}$

The resultant field of motion is then expressed by:

$$(3) \quad \begin{aligned} u &= u_0 + (b_1 + a_1) \frac{e^{nx} - 1}{1 + e^{nx}} - c_1 \frac{e^{ny} - 1}{1 + e^{ny}} \\ v &= v_0 + (b_1 - a_1) \frac{e^{ny} - 1}{1 + e^{ny}} + c_1 \frac{e^{nx} - 1}{1 + e^{nx}} \end{aligned}$$

We shall presently use these formulae in order to discuss how the deviation from linear conditions acts with respect to frontogenesis. Obviously, only a few general conclusions can be obtained on the basis of these formulae.

17. Frontogenesis between Source Regions. Substituting **16** (2) and **16** (3) in **3** (5) and **4** (3) we get some complicated expressions for F, $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$. We shall not write down the deductions here, but it is well to note that F, $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ depend as much on the field of motion as on the field of property. Usually F has a maximum where the product of the property gradient and the wind gradient has a maximum. But it may also happen that each of the fields creates its own maximum line in the distribution of F, in which case *two* frontal zones, or a *double* front develops.

The most effective frontogenesis is obtained when the zone of transition of the a-field runs through the intersection point between the axes of deformation (the centre of deformation). In this case we may put $N = 1$, and **16** (2) gives:

(1) $$\alpha = \alpha_0 + A \frac{e^{px+qy} - 1}{1 + e^{px+qy}}.$$

Substituting (1) and **16** (3) in **3** (5), we obtain:

(2) $$F = \frac{4nA}{\sqrt{p^2+q^2}} Z[p^2(a_1+b_1)X + pqc_1(X-Y) - q^2(a_1-b_1)Y]$$

where

$$X = \frac{e^{nx}}{(1+e^{nx})^2},\quad Y = \frac{e^{ny}}{(1+e^{ny})^2} \text{ and } Z = \frac{e^{px+qy}}{(1+e^{px+qy})^2}$$

We see that $F = 0$ when the angle between the x-axis and the isolines of α is ψ', as given by

$$\tan\psi' = -\frac{p}{q} = \frac{c_1(X-Y) \pm \sqrt{c_1^2(X-Y)^2 + 4(a_1^2-b_1^2)XY}}{2(a_1+b_1)X}$$

The angle ψ' defines the sector of frontogenesis as explained in § 9. In the general case the angle ψ' is a function of x and y. However, in the quasi-linear area of the field of motion, we may put: $X = Y = \frac{1}{4}$, and hence:

$$\cos 2\psi' = \frac{b_1}{a_1},$$

which agrees with **9** (1). Thus, it is only in the case of linear movement that there is a sector of frontogenesis which is characteristic of the whole field.

Differentiating (2) with respect to x and y we see that

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \text{ only when } x = y = 0.$$

The centre of frontogenesis thus coincides with the centre of deformation. This is in contrast to the results obtained for linear fields of motion where the centre of frontogenesis may have any position relative to the axes of deformation. There is another difference too: The isolines of the above α-function are parallel and straight lines. In the linear field of motion the centre of frontogenesis degenerates into a line (the line of frontogenesis); but in the non-linear field of motion which we are here considering, the frontogenetical effect will in general dwindle with increasing distance along the line of frontogenesis (because $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ can only vanish when both x and y are zero).

Only in a special case will the centre of frontogenesis degenerate into a line of frontogenesis. This happens when the isolines of α are parallel to one of the axes. We consider the case when $p = 0$ (α-lines parallel to the x-axis). (2) then gives:

(3) $$F = -4nqA(a_1-b_1)\frac{e^{qy}}{(1+e^{qy})^2}\frac{e^{ny}}{(1+e^{ny})^2}.$$

We see that F is constant along any straight line parallel to the x-axis, and that F has a maximum along the x-axis.

We choose the positive y-axis towards decreasing α. We then see that $-qA > 0$ when $p = 0$. (3) then shows that the x-axis is the line of frontogenesis when $a_1 > b_1$ (deformation larger than divergence); $F = 0$ everywhere when $a_1 = b_1$; and, finally, the x-axis is the line of frontolysis when $b_1 > a_1$. In these cases there is no difference between the linear field of motion and the non-linear field which we are considering.

From the above discussion we learn that frontogenesis is a much more complicated phenomenon when the field of motion is non-linear than when it is linear. In fact, it is only when the zone of transition in the α-field is parallel to the axis of deformation in the non-linear field that there is complete analogy with the linear field.

The chief difference between linear and non-linear motion is that frontogenesis in the linear field is invariant relative to parallel translation of the axes of deformation, whereas, in the non-linear field of the type described, frontogenesis is only effective in the vicinity of the stress axis. Viewed from a practical standpoint, this is fortunate because, in the complicated cases when the field of motion is non-linear, one need in most cases only look for fronts in the vicinity of the axis of dilatation. And, since non-linear fields of motion which are characteristic of the transition from one source region to another are quasi-linear in the vicinity of the axis of dilatation, it follows that the results obtained in Chapter II usually have a much wider range of applicability than one might expect. However, no theory of kinematical frontogenesis is complete unless the considerations are based on the existence of source regions. Usually, the non-linear field of motion acts in such a way that it tends to produce homogeneity within the air masses near the source regions and to bring out the contrasts in the transitional zone.

The above discussion of non-linear fields should only be regarded as a first attempt to discuss frontogenesis from a wider point of view. Further practical and theoretical researches will be needed in order to divulge the intricacies of frontogenesis in non-linear fields of motion.

Vervarslingi på Vestlandet. Bergen. April 1935.