ON EQUAL-AREA TRANSFORMATIONS OF THE INDICATOR DIAGRAM, AND A NEW AEROLOGICAL CHART

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1. For the exploitation of the observations made on aeroplane ascents or soundings, it has proved useful to plot the corresponding values of pressure and temperature on some kind of squared paper. For several purposes it is convenient to employ a simple pressure-volume diagram, like the indicator diagram used by engineers. On this diagram, the work done by the external forces in a cycle process can be found directly from the area of the figure traced on the paper. This property is useful in meteorological work, too; and by the help of the same chart, it is possible to evaluate the integral

$$g z = -\int_{p_1}^{p_2} v dp$$

giving the height z between two isobars p_1 and p_2. As usual g is the acceleration of gravity.

For aerological work, however, other diagrams may be more convenient, and quite a number of different papers have been constructed. We require that the paper shall retain the property of the p, v diagram: that the work done by the external forces in a cycle process is represented by the corresponding area on the plot. We are thus lead to consider the theory of equal area projection in general.

2. The condition for equal area transformation of the plane v, p upon the plane x, y is:

$$\frac{\partial x}{\partial p} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial p} = 1$$

Here x and y are functions of p and v. We will now make the supposition that y is known as a function of p, v, and find the value of x that satisfies this condition.

When y is a known function, the same is the case with the partial derivates:

$$M = \frac{\partial y}{\partial v} \quad \text{and} \quad N = \frac{\partial y}{\partial p}$$

The equation of condition becomes:

$$M \frac{\partial x}{\partial p} - N \frac{\partial x}{\partial v} = 1$$

The corresponding simultaneous system is:

$$\frac{dp}{M} = -\frac{dv}{N} = dx$$

From the two first terms we obtain:

$$N\, dp + M\, dv = 0$$

that is, integrated:

$$y = \text{const.} = y_0$$

From this last expression, v is obtained as a function of p and y_0, and this value is substituted in the expression M, which then assumes the form M_1.

Then

$$x = \int \frac{dp}{M_1} + F(y)$$

where F is an arbitrary function. In the integral, p is the only variable, y_0 is constant; but after the integration has been performed, we make the substitution:

$$y_0 = y\,(p, v)$$

and obtain the value of x, as a function of p and v.

In an analogous way, we can, from

$$y = y_0$$

find an expression for p, in terms of v and y_0, which is substituted in the function N

$$N = N\,(v, p) = N_1\,(v, y_0)$$

Then x is found from:

$$x = -\int \frac{dv}{N_1} + F(y)$$

Under the integration, y_0 is constant, afterwards we substitute:

$$y_0 = y(p, v)$$

whereupon x is obtained as a function of p, v.

By the help of these two formulae, we can determine the function x that satisfies the condition of equal-area transformation, when y is known as a function of p and v.

3. We will consider some examples. First, let

$$y = v^m p^n$$

Then:

$$M = m v^{m-1} p^n$$

Here v must be expressed by p and y. We have:

$$v = y^{1/m} p^{n/m}$$

and by substitution:

$$M_1 = m y^{(m-1)/m} p^{n/m}$$

and further:

$$x = \int \frac{dp}{M_1} = \frac{1}{m-n} y^{(1-m)/m} p^{(m-n)/m}$$

By substituting the value of y, we get:

$$x = \frac{1}{m-n} v^{1-m} p^{1-n}$$

From this follows:

$$xy = \frac{vp}{m-n} = \frac{RT}{m-n}$$

The argument breaks down for the case of $m = n$:

$$y = v^m p^m = R^m T^m$$

Then, by the same procedure, we obtain:

$$x = \frac{1}{m} v^{1-m} p^{1-m} \log p$$

$$= \frac{1}{m} R^{1-m} T^{1-m} \log p$$

also:

$$xy = \frac{1}{m} RT \log p$$

4. We will consider another case. Let

$$y = m \log v + n \log p$$

Then

$$M = m/v$$

but, as

$$v = e^{y/m} p^{-n/m}$$

we have:

$$M_1 = m e^{-y/m} p^{n/m}$$

and

$$x = \frac{1}{m} e^{y/m} \int p^{-n/m} dp$$

$$= \frac{1}{m-n} e^{y/m} p^{(m-n)/m}$$

and by substituting the value of y:

$$x = \frac{1}{m-n} vp = \frac{RT}{m-n}$$

In the case of $m = n$ we obtain:

$$y = n \log (pv) = n \log (RT)$$

$$x = \frac{1}{n} vp \log p = \frac{RT}{n} \log p$$

It is believed that these cases comprise all practical diagram papers.

5. We will now consider some special cases. In (3), we put: $m = 0$ and $n = (\varkappa - 1)/\varkappa$, where $\varkappa$ denotes the quotient between the two heat capacities of air:

$$\varkappa = c_p/c_v$$

We then have:

$$y = p^{\frac{\varkappa-1}{\varkappa}}$$

$$x = \frac{\varkappa}{\varkappa - 1} v p^{\frac{1}{\varkappa}}$$

and

$$xy = \frac{\varkappa}{\varkappa - 1} RT = \frac{c_p T}{A}$$

A being the caloric equivalent.

This set of variables does not seem to have been applied in meteorological papers. We shall therefore return to this case later on.

Now, let $m = 1$ and $n = 1$, then,

$$y = vp = RT$$

$$x = \log p$$

(*Stüve* 1927, *Refsdal* 1930, «emagram».)

This is a well-known diagram; another form is:

$$y = RT$$

$$x = \log s \qquad s \text{ being the density.}$$

I case (4) we choose:

$$m = c_p \text{ and } n = c_v$$

then:

$$y = c_p \log v + c_v \log p = S = \text{entropy.}$$

$$x = \frac{RT}{c_p - c_v} = \frac{T}{A}$$

This temperature-entropy-diagram is well known both from physics and from meteorology (*Sir Napier Shaw*).

Putting $m = 1$ and $n = 1$ we have:

$$y = \log (R T)$$
$$x = R T \log p$$

which set of coordinates has been used by *Refsdal* an his «aerogram» paper (1937). On this paper, the evaluation of the integral

$$gz = -\int v\, dp$$

is very simple.

By adding a function of y, we can find still another form:

$$y = \log (R T)$$
$$x = R T \log p + F(y)$$

We now choose

$$F(y) = \frac{1}{\varkappa - 1} y e^y = \frac{1}{\varkappa - 1} R T \log (R T)$$

and obtain:

$$x = \frac{1}{\varkappa - 1} R T \log \left(v p^{\frac{1}{\varkappa}} \right) = \frac{T S}{A}$$
$$y = \log (R T)$$

Whether this diagram has ever been used, the writer does not know. It may perhaps have some application.

6. We now return to the set of coordinates:

$$y = p^{\frac{\varkappa - 1}{\varkappa}}$$
$$x = \frac{\varkappa}{\varkappa - 1} v p^{\frac{1}{\varkappa}}$$

with
$$xy = \frac{\varkappa}{\varkappa - 1} R T = \frac{c_p T}{A}$$

The computations and constructions are easy. (Fig. 1.) The values of the constants R and $\varkappa$, however, are found to differ somewhat in the various physical tables.

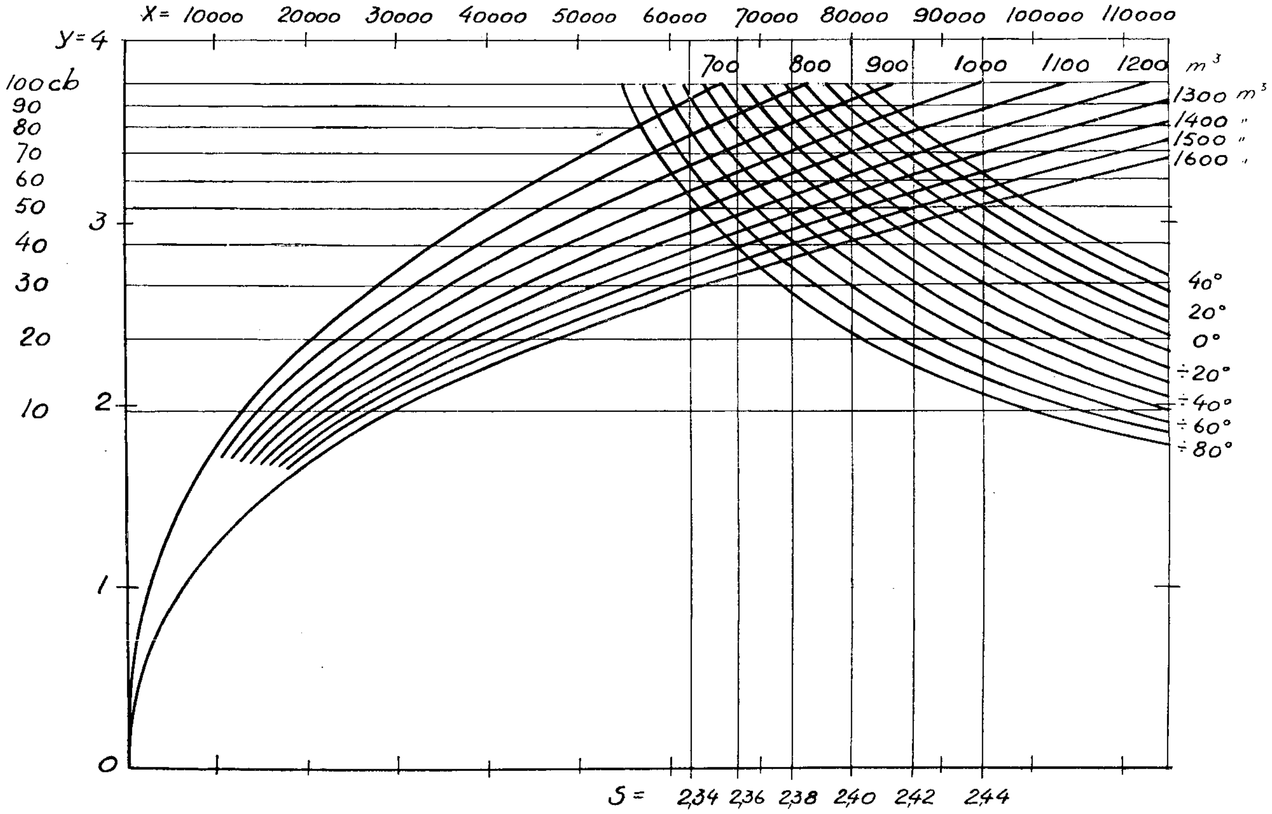

Fig. 1. The adiabate-pressure chart.

Scales: extreme left, for pressure p, in centibars;
along left margin, for coordinate y;
on top, for coordinate x;
below upper margin, and continued down to the right, specific volume v, in cubic metres;
to the right, temperature in centigrades;
down below, entropy S.

Curves are drawn for pressure p, entropy S, temperature T, and specific volume v.

ordinate x_m for this part of the observation curve; the ordinate will intersect the scale at a point, and the reading will give the height required, in dynamical metres. (Fig. 3.)

As a matter of fact, the numbers of the scale give the area of the strip between the neighbouring principal isobars, from the ordinate $x = 0$ to the ordinate $x = x_m$, in the proper units.

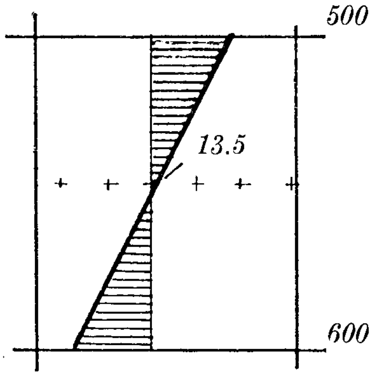

Fig. 3. The mean ordinate intersects the scale of small crosses at the point 13.6, giving the thickness of the layer between isobars 600 and 500° = 1360 dyn. metres.

By a slight modification of the same procedure it is possible to determine the height of any point on a curve plot.

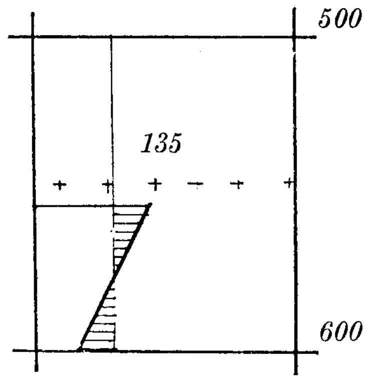

Fig. 4. The mean ordinate for the part of the curve shown on the plot, gives the reading 13.41. By measurement, the breadth of the whole strip is found to be 4.1 cm. and the distance from the end point of the curve, to the isobar 600, is 1.9. Then the height of the end point, above the said isobar 600, is $1341 \cdot (19/41) = 625$ dyn. metres. The other method, using the temperature scale, gives $100 \cdot (16{,}4 - 10.1) = 630$ dyn. metres.

It is required to find the height of the end point of a plot, above the principal isobar immediately next below.

The mean ordinate is drawn for the part of the curve in question, and the number n is read off the cross-scale. By means of a centimeter scale, the breadth of the strip between the two neighbouring principal isobars is found to be q, and the distance from the end point of the curve, to the principal isobar immediately next bolow, is p. Then obviously the dynamical height Φ of the end point, above the said isobar, is: $\Phi = n \cdot (p/q)$.*)

10. If the observations made during an ascent are plotted on the diagram paper, we can determine the stability of the air from the slope of the curve.

The stability, as defined by *Hesselberg*, is:

$$E = \frac{g}{T}(m_0 - m)$$

where m is the vertical lapse rate

$$m = -\frac{\partial T}{\partial z}$$

But on our paper we have:

$$-\frac{m}{T} = \frac{\partial T}{T \partial z} = \frac{1}{c_p}\frac{\partial S}{\partial z} - \frac{1-\varkappa}{\varkappa}\frac{\partial p}{p \partial z}$$

$$= \frac{1}{c_p}\frac{\partial S}{\partial z} - \frac{\varkappa - 1}{\varkappa}\frac{v \partial p}{v p \partial z} = \frac{1}{c_p}\frac{\partial S}{\partial z} - \frac{\varkappa - 1}{\varkappa}\frac{g}{RT}$$

For an adiabatic lapse rate $dS = 0$, and

$$m_0 = \frac{\varkappa - 1}{\varkappa}\frac{g}{R}$$

Then we obtain:

$$E = \frac{g}{T}(m_0 - m) = \frac{m_0}{A}\frac{\partial S}{\partial z}$$

Now, along an adiabate $S = \text{const.}$ or $x = \text{const.}$ we have:

$$dT = -m_0 \, dz$$

and accordingly:

$$E = -\frac{{m_0}^2}{A}\frac{dS}{dT}$$

*) *Table* of inverse values of distances between principal isobars on chart:

mb.: 1000 — 900 — 800 — 700 — 600 — 500 — 400 — 300 — 200 — 100

$1/q$:	0.715	0.660	0.604	0.545	0.483	0.419	0.349	0.274	0.187
q:	1.399	1.515	1.657	1.835	2.068	2.388	2.862	3.652	5.337

According to *Landolt-Börnstein*
$R = 286.8$ and $\varkappa = 1.402$, then

$$(\varkappa - 1)/\varkappa = 0.28673, \quad \frac{\varkappa}{\varkappa - 1} R = 1000.2$$

Bjerknes uses the values:
$R = 287.05$, and $\varkappa = 1.403$, then

$$(\varkappa - 1)/\varkappa = 0.28724, \quad \frac{\varkappa}{\varkappa - 1} R = 999.3$$

It seems, that with the same certainty, we may use the values:

$$R = 287, \ (\varkappa - 1)/\varkappa = 0.287, \quad \frac{\varkappa}{\varkappa - 1} R = 1000$$

We then have the formulas for the numerical computation:

$$\log y = 0.287 \log p$$

$$x = \frac{1000\, T}{y}$$

7. Instead of the coordinate x we can introduce the entropy S. Remembering that:

$$S = c_p \log v + c_v \log p$$

or

$$S = c_p \log \left(v p^{\frac{1}{\varkappa}} \right)$$

we obtain:

$$x = \frac{\varkappa}{\varkappa - 1} e^{S/c_p}$$

Thus, our paper can also be used as an entropy-temperature diagram.

8. The work done by the pressure under a circle process is:

$$F = \int v\, dp = \int y\, dx$$

the integrals taken within the same boundaries. We will compute the value of this integral for an area, bounded by two isotherms T_1 and T_2, and two ordinates x_1 and x_2. The computation is easy, and leads to:

$$F = \frac{c_p}{A} (T_2 - T_1) (\log x_2 - \log x_1)$$

$$= \frac{1}{A} (T_2 - T_1) (S_2 - S_1)$$

For unit differences $T_2 - T_1 = 1$, $S_2 - S_1 = 1$, the value of the integral is

$$f = 1/A$$

The work done by the exterior forces during a circle process is equal to the number of unit meshes in the S, T pattern, divided by A; the numerical value is

$$\frac{1}{A} = 4184$$

using the meter-ton-sec.-system.

9. We will compute the dynamical height or the gravity potential Φ from the observations of temperature T and pressure p:

$$\Phi_2 - \Phi_1 = -\int v\, dp = -\int x\, dy$$

$$= (y_2 - y_1)\, x_m = y_2 x_m - y_1 x_m$$

Here x_m is a mean value, which can be determined approximately on the paper. But now:

$$xy = \frac{\varkappa}{\varkappa - 1} RT = 1000\, T$$

and accordingly

$$\Phi_2 - \Phi_1 = 1000\, (T_1 - T_2) \quad \text{(dyn. dm)}$$

Here T_1 and T_2 are the temperatures corresponding to the points x_m, y_1 and x_m, y_2 respectively. (Fig. 2.) If we want the height in meters we have

$$H_2 - H_1 = \frac{1}{m_0} (T_1 - T_2) \ \text{(m.)} = 102\, (T_1 - T_2) \ \text{(m.)}$$

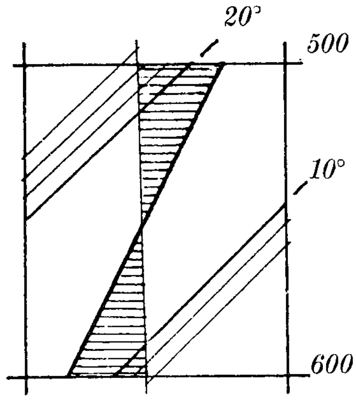

Fig. 2. Part of curve plot between isobars 600 and 500 mb. Against upper and lower end points of the mean ordinate are read off the temperatures —22.1 and —8.5 respectively. The difference is 13.6, giving a thickness of the layer of 1360 dyn. metres.

where m_0 ist the adiabatic vertical lapse rate of temperature:

$$m_0 = \frac{\varkappa}{\varkappa - 1} \frac{g}{R} = \frac{1}{102}$$

For the computation of results of aerological observations, the heights of the principal isobars: $p = 1000$, 900, 800 .... mb have generally to be determined. To faciliate this operation, a scale is inserted in each interval, approximately in the middle between two and two consecutive principal isobars, in accordance with *Väisälä's* method.

To find the thickness of a layer between two consecutive principal isobars, draw the middle

This expression can be determined with the help of our chart. Through the point in question, draw an adiabate and an isotherm, and the tangent to

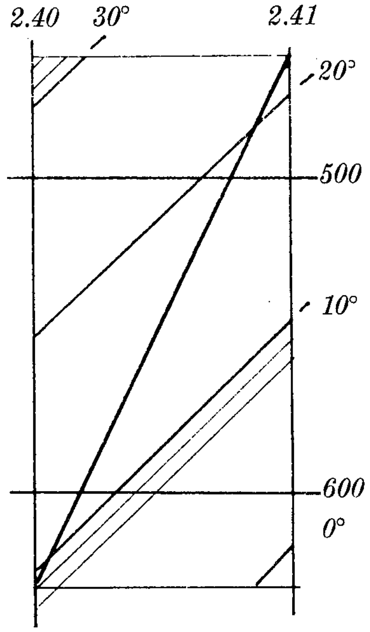

Fig. 5. The stability determined from the formula $E = \frac{\triangle S}{\triangle T} \cdot 0.4$. On the figure, we have: $\triangle S = 0.01$, and $\triangle T = 32.5 - 9.5 = 23$. Then $E = 0.4 \cdot (0.01/23) = 1.75 \cdot 10^{-4}$.

the curve plot. If we now proceed upwards along the adiabate, a distance corresponding to the unit interval of temperature, and then horizontally to the tangent, we can at once read off the value of dS/dT. The numerical value of the factor m_0^2/A is 0.4.

11. It is possible to express the stability by the help of the variables x and y.

In the formula:

$$E = \frac{g}{T}(m_0 - m)\,, \quad m = -\frac{dT}{dz}$$

we introduce:

$$dT = \frac{\varkappa - 1}{\varkappa R}(x\,dy + y\,dx)$$

$$dz = -\frac{1}{g}\,x\,dy$$

Then:

$$m = -\frac{dT}{dz} = \frac{\varkappa - 1}{\varkappa}\,\frac{g}{R}\left(1 + \frac{y\,dx}{x\,dy}\right)$$

and, accordingly:

$$E = \frac{m_0 g}{T}\,\frac{y\,dx}{x\,dy}$$

If now u is the angle between the tangent to the curve plot, and the vertical (or adiabate):

$$\text{tg}\,u = \frac{dx}{dy}$$

and w the angle between the tangent to the isotherm through the same point, and the vertical:

$$\text{tg}\,w = \frac{x}{y}$$

we obtain:

$$E = \frac{m_0 g}{T}\,\frac{\text{tg}\,u}{\text{tg}\,w} = K\frac{\text{tg}\,u}{\text{tg}\,w}$$

This expression is independent of the scale of the paper. The numerical value of the coefficient K, which represents the stability for an isothermal atmosphere, is:

−70	−60	−50	−40	−30	−20	−10	0	10	20°	C.
4.7	4.5	4.3	4.1	3.95	3.8	3.65	3.5	3.4	3.3 · 10^{-4}	

12. By the help of the same diagram, we may determine the vertical lapse rate of temperature:

$$m = -\frac{dT}{dz} = m_0\,\frac{\text{tg}\,w - \text{tg}\,u}{\text{tg}\,w}$$

If the lapse rate is referred to the geopotential we have:

$$\mu = \frac{1}{1000}\,\frac{\text{tg}\,w - \text{tg}\,u}{\text{tg}\,w}$$

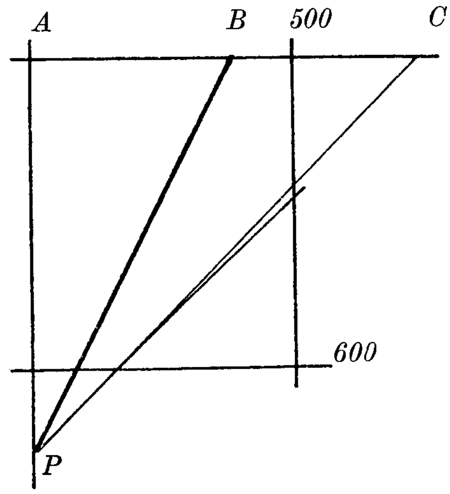

Fig. 6. Determination of stability and vertical lapse rate of temperature. PB is the tangent to the curve plot, and PC is the tangent to the isotherm. The distance AB = 2.6 cm, and AC = 5.0 cm. Then the stability $E = 3.65 \cdot 10^{-4} \cdot (2.6/5.0) = 1.9 \cdot 10^{-4}$. The vertical lapse rate is found in a similar way; as BC = 2.4 we have: $m = (2.4/5.0) \cdot 1/102 = 4.7°$ pr. 1000 m, or $\mu = 4.8°$ pr. 1000 dyn metres.

The graphical construction of this expression is easy. Through the point P in question, draw the isentrope, and the tangents to the curve plot, and to the isotherm. These three straight lines cut an isobar in three points, A, B and C respectively. Putting $AC = a$, and $BC = b$, we have:

$$m = \frac{1}{102}\frac{b}{a}$$

$$\mu = \frac{1}{1000}\frac{b}{a}$$

The same figure (Fig. 6) renders a geometrical representation of the stability, too. We have:

$$\operatorname{tg} u = \frac{AB}{AP} \text{ and } \operatorname{tg} w = \frac{AC}{AP}$$

Putting $AB = c$ we obtain:

$$E = \frac{m_0 g}{T} \cdot \frac{c}{a} = K \cdot \frac{c}{a}$$

13. These simple constructions are valid for all transformations of p-v-diagram. But on papers with curved lines, the tangents in the point P must be used for the determination of the various points of intersection, A, B, C. In this case the procedure is:

Through the point P on the observation curve, draw the tangents to the (1), isobar, (2) the isentrope, (3) the observation curve, (4) the isotherm. At a convenient distance, draw a straight line parallel to the tangent to the isobar. This line intersects the three other tangents in points A, B, C. Putting, as before: $AC = a$, and $BC = b$, the lapse rate of temperature:

$$m = \frac{1}{102}\frac{b}{a} \text{ (pr. m) or } \mu = \frac{1}{1000}\frac{b}{a} \text{ pr. dyn. metres.}$$

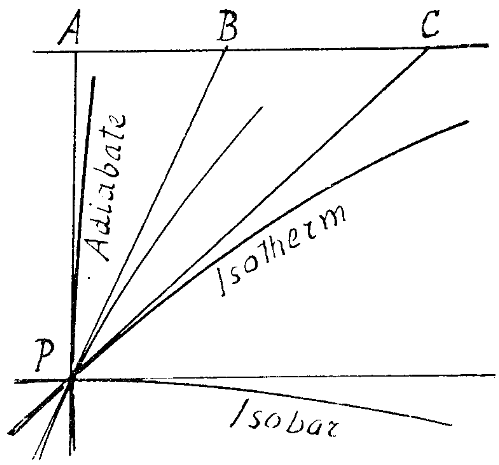

Fig. 7.

This construction is then valid for the simple p-v-diagram, for *Sir Napier Shaw*'s «tephigram», for *Stüve*'s «emagram», for *Refsdal*'s «aerogram», and for all other diagrams belonging to the same class of transformations. (Werenskiold 1937.)

14. So far, we have only considered the dry-air conditions for the construction of adiabates, stability and energy relations. It is of course possible to draw a set of wet-air adiabates too, on the new paper. This work must however be postponed.

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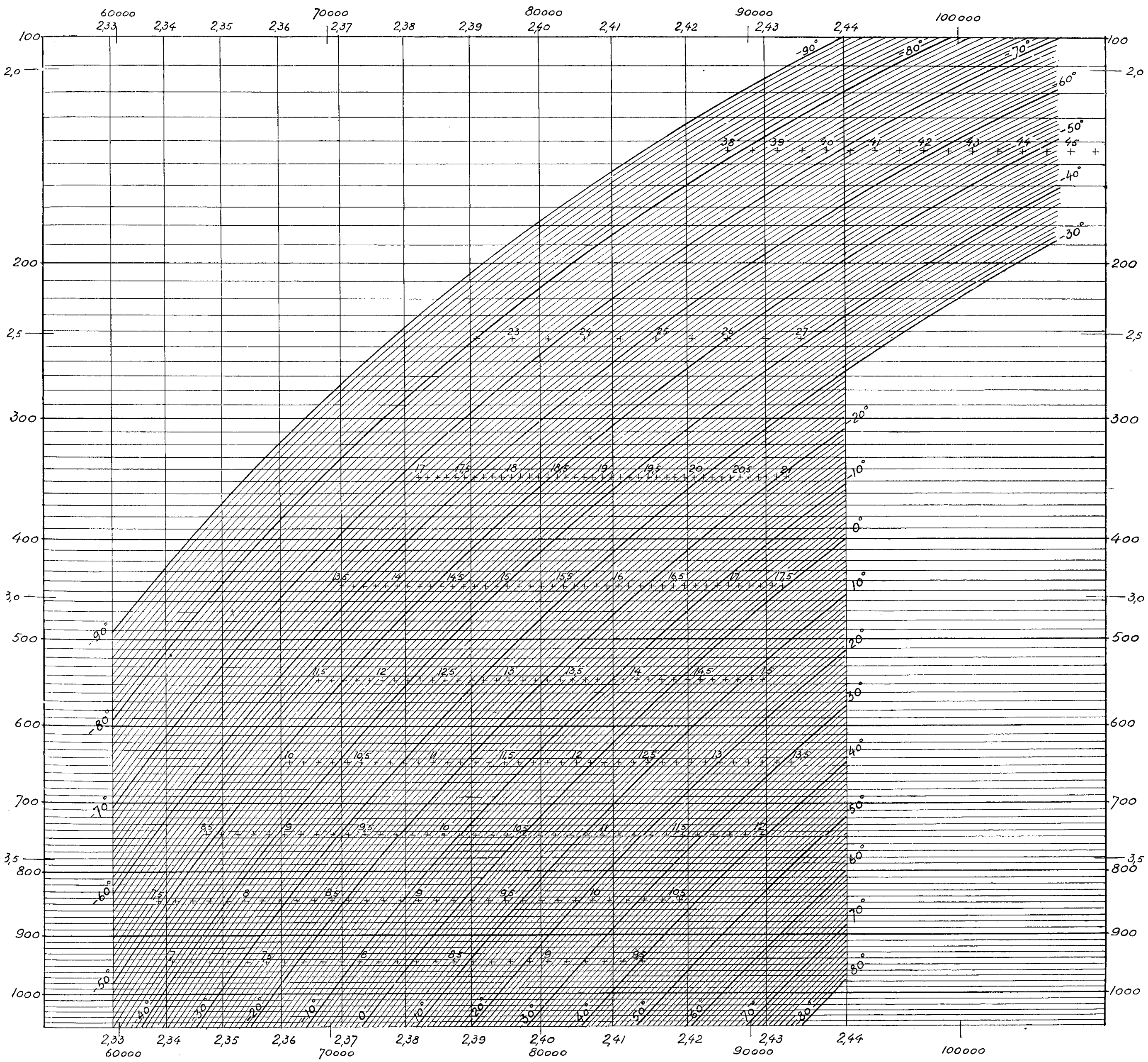

Description of chart.

The horizontal straight lines represent the pressure, with intervals of ten millibars, and the principal isobars marked by heavier lines; scales along right and left margin. Here also the coordinate y is indicated. — The vertical straight lines give the entropy, with scales along upper and lower margin; here also the coordinate x is indicated. The system of slanting lines represent the temperature in centigrades, with scales along upper, right, and lower margins. — The scales of small crosses indicate the area