ON THE THEORY OF TIDAL OSCILLATIONS IN OCEANS WITH SOLID BOUNDARIES

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In a previous paper[1]) I have with the aid of variational methods given a new treatment to the problem of tidal oscillations. In particular variational methods are well known from their successful applications to atomic problems. Therefore, considering the similarity of many wave mechanical and hydrodynamical problems, variational methods suggest themselves to a more extensive use in hydrodynamics than has so far been the case. The application of variational methods in the theory of tidal oscillations was, however, above all introduced in order to overcome serious difficulties of satisfying given boundary conditions.

Continuing the work the advantage of using variational methods in tidal theory has been even more stressed. As will be shown it is possible to formulate the problem of tidal motion in an ocean with given arbitrary boundaries in such a way that boundary conditions, either of cinematical or of dynamical type, are all comprised in a single variational principle. The explanation of this remarkable fact is that the variational integral formulating the problem may be put in a form in which the so-called *natural boundary conditions* of the problem are identical with boundary conditions derived from dynamical and cinematical considerations.

The main aim of the foregoing paper was to make clear the remarkable effect of the natural stable stratification of sea water. As has been shown by *Solberg*[1]) the familiar two-dimensional theory of tidal motion given by *Laplace* cannot be deduced from the general three-dimensional hydrodynamic equations for *homogeeous water*, due to the fact that in the equations of motion it is not legitimate to neglect the vertical component of Coriolis' acceleration. On the contrary, due to vertical Coriolis' forces, certain types of motion must occur which are quite unforeseen by the Laplacian tidal theory.

However, if we start from the hydrodynamical equations for *stable water* the situation becomes quite different. The water particles are driven back to their original equilibrium positions, not only by the elevation or lowering of the above-lying point of the sea surface, but also by their own vertical displacements relative to the surroundings. By normal mean stability of sea water these vertical quasi-elastic forces are much stronger than the vertical components of Coriolis' forces.

The mean downward density gradient of the oceans is rather small, its order of magnitude being 0,001—0,002 per kilometer. It is, however, strong enough to create quite a new situation. This is easily seen from the modified equations of motion. Due to the natural stability of sea water the vertical Coriolis' force component, or rather the component due to the horizontal component of the earth's rotational vector, looses its importance in the theory of tidal motions in large oceans. Apart from small

[1]) *E. A. Hylleraas:* Über die Schwingungen eines stabil geschichteten durch Meridiane begrenzten Meeres. *Astrophysica Norvegica* Vol. III, No. 6, 1939.

[1]) *H. Solberg:* Über die freien Schwingungen einer homogenen Flüssigkeitsschicht auf der rotierenden Erde *Astrophysica Norvegica.* Vol. I, No. 7, 1936.

modifications of the waves of an order of magnitude which surely cannot be detected by tidal observations we are justified in neglecting this component of Coriolis' force. It is the aim of this paper to make that side of the problem clear.

Before passing over to the two-dimensional Laplacian theory, there remains another interesting point to discuss. The quasielastic forces of stable water, which were able to suppress vertical Coriolis' forces, have also the power of maintaining waves of a special type which are well known from theoretical and practical investigations by *Fjeldstad*[1]) as *internal waves*.

In a deep and large ocean with flat bottom or slow variations of the depth free internal waves may theoretically exist. However, due to the form of the waves (with a multitude of zero lines in horizontal direction) they must remain almost unaffected by tidal forces. Furthermore, even if they were strongly affected by ordinary tidal forces, they could not be generated with any perceptible amplitude except in the case of practically exact resonance of their eigen-frequencies with frequencies of tidal forces. We may therefore consider internal waves, though theoretically not excluded, as having a vanishing small probability of being generated by tidal forces (cf. equ. (90)).

This is in agreement with the view of *Fjeldstad* who considers the generation of such waves as mainly due to irregularities of the sea bottom or of the sea boundaries as for instance sharp edges and narrow fjords, the generating forces being in this particular case the periodical tidal motion of adjacent larger sea basins. Also from purely theoretical considerations it is clear that, in order to adjust the wave motion to cinematical boundary conditions by rapid variation of the depth or of side boundaries of an ocean, it would be necessary to take into account elements of motion corresponding to complicated internal waves. As to the motion of the sea surface, however, most disturbances originating from such irregularities of the sea basin can surely be accounted for by the Leplacian tidal theory.

Corresponding to the numbers of zeros in vertical direction we may divide internal waves into "zeroth order" and higher order waves, the latter being characterized as proper internal waves. In the Laplacian theory all higher order waves are neglected.

We shall start with the equations of motion for a stable liquid on a rotating earth of spherical shape

$$(1) \qquad \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + 2\boldsymbol{\Omega} \times \frac{\partial \boldsymbol{u}}{\partial t} + \sigma g\, u_r \frac{\boldsymbol{r}}{r} = -\nabla (p - g\Phi),$$

$\boldsymbol{u}$ being the displacement of a water particle from its mean undisturbed position, u_r its radial component, and $\boldsymbol{\Omega}$ the vectorial angular velocity of the earth. σ is the mean relative downward increase of water density per unit length corresponding to a density distribution $\varrho = \varrho_0 e^{-\sigma(r-a)}$. For the sake of simplicity we have taken the density ϱ_0 at the earth's surface $r = a$ to be equal to unity. Further $-g\Phi$ is the potential of tidal forces from the moon or the sun, g being the acceleration of gravity. By this definition Φ means the elevation of the sea surface in the static Newtonian tidal wave. Finally p is the variation of pressure or pressure perturbation.

Of importance to the treatment of our problem is the magnitude of the quantity σg as compared to the square of the double angular velocity of the earth $(2\Omega)^2 \sim 2 \cdot 10^{-8}$. Taking $\sigma \sim 2 \cdot 10^{-8}$ and $g \sim 1000$ we have

$$(2) \qquad \sigma g \sim 1000\,(2\Omega)^2.$$

To the equation (1) we have to add the equation of continuity for an incompressible fluid

$$(3) \qquad \operatorname{div} \boldsymbol{u} = 0,$$

further cinematical conditions at solid boundaries (bottom and coasts)

$$(4) \qquad u_n = 0,$$

u_n denoting the normal displacement component, and finally the dynamical boundary conditions

$$(5) \qquad p = g u_r$$

at the surface of the sea $r = a$.

We now proceed to formulate the problem by the aid of a variational principle (writing $\boldsymbol{uv}$ etc. for scalar products of two vectors)

$$(6) \qquad I = \int_{t_0}^{t_1} \left\{ \left[\frac{1}{2} \left(\frac{\partial \boldsymbol{u}}{\partial t} \right)^2 - \boldsymbol{u} \left(\boldsymbol{\Omega} \times \frac{\partial \boldsymbol{u}}{\partial t} \right) - \frac{1}{2} \sigma g u_r^2 - \right.\right.$$
$$\left.\left. - \boldsymbol{u} \nabla (p - g\Phi) \right] dV + \frac{1}{2g} \int_{r=a} p^2 df \right\} dt = \text{Extr.},$$

in which $r = a$ denotes that the surface integral is restricted to the surface of the sea. By an extremum

[1]) *J. E. Fjeldstad:* Interne Wellen. Geofysiske Publikasjoner, Vol. X, No. 6, 1933.

value we shall here only mean a stationary value of the integral, condition of which the is $\delta I = 0$ for arbitrary infinitesimal variations $\delta \boldsymbol{u}$ and δp of the dependent variables $\boldsymbol{u}$ and p. It has to be understood that these variables also have to satisfy some boundary conditions at the limits of the time variable. Now the familiar condition of given values at the limits t_0 and t_1 may be replaced by the condition of periodicity. For periodic solutions we may assume $t_1 - t_0$ to contain either a whole number of periods or, as will also be sufficient, a very large number of periods. Owing to the linearity of equation (1) we may treat separately tidal forces of different frequencies, taking $e^{\pm i\omega t}$ as time factors.

We shall now demonstrate that a stationary value of the integral (6) is possible only if the equations of motion (1) and the additional cinematic and dynamic equations (3, 4, 5) are simultaneously satisfied. First giving arbitrary variations to the displacement vector $\boldsymbol{u}$ and integrating $\delta \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial}{\partial t} \delta \boldsymbol{u}$ by parts we get

$$(7) \quad \delta I = \int_{t_0}^{t_1} \left\{ \int \left[-\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - 2\left(\Omega \times \frac{\partial \boldsymbol{u}}{\partial t}\right) - \sigma g u_r \frac{\boldsymbol{r}}{r} - \right.\right.$$
$$\left.\left. - \boldsymbol{u} \nabla (p - g\Phi) \right] \delta \boldsymbol{u}\, dV \right\} dt = 0.$$

Owing to the periodicity the limits t_0 and t_1 contribute nothing to the integral. This equation is equivalent to (1) since $\delta \boldsymbol{u}$ is an arbitrary vectorial quantity.

Next consider the variation of pressure δp. The corresponding variation of the integral (6) becomes

$$(8) \quad \delta I = \int_{t_0}^{t_1} \left\{ -\int \boldsymbol{u} (\nabla \delta p)\, dV + \frac{1}{g} \int_{r=a} p \delta p df \right\} dt =$$
$$= \int_{t_0}^{t_1} \left\{ \int \operatorname{div} \boldsymbol{u}\, \delta p\, dV - \int u_n \delta p df + \frac{1}{g} \int_{r=a} p\, \delta p\, df \right\} dt = 0,$$

using Gauss' theorem $\int \operatorname{div}(\boldsymbol{u} \delta p)\, dV = \int u_n \delta p\, df$.

Here we find at once equs. (3) and (4) as necessary conditions for $\delta I = 0$ if δp is an arbitrary quantity. For the sea surface we write $u_n = u_r$ thus getting the dynamical boundary condition (5).

Corresponding to an elimination of the time factors $e^{\pm i\omega t}$ in the differential equation (1) we may dispense with time integration in our variational integral. The real quantities of the foregoing equations can always be split up into two parts, a complex quantity containing the time factor $e^{i\omega t}$ and the conjugate complex quantity containing the factor $e^{-i\omega t}$. We shall therefore always replace a real quantity by a sum of two conjugate complex quantities in the following way

$$(9) \qquad \boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{u}^*$$

meaning thereby that the factors $e^{i\omega t}$ and $e^{-i\omega t}$ are to be found in the first and second quantity respectively on the right hand side. For the first one of these quantities equ. (1) transforms into

$$(10) \quad \omega^2 \boldsymbol{u} - i\omega \Omega \times \boldsymbol{u} - \sigma g\, u_r \frac{\boldsymbol{r}}{r} = \nabla (p - g\Phi),$$

whereas the additional conditions (3, 4, 5) remain formally unchanged.

By the introduction of (9) in the integral (6) all quantities containing $e^{2i\omega t}$ or $e^{-2i\omega t}$ such as $\boldsymbol{u}^2$ give no contribution to the variational integral. Leaving aside the unimportant factor $\omega^2(t_1 - t_0)$ the variational principle reduces to

$$(11) \quad I = \int [\boldsymbol{u}^* \boldsymbol{u} - \boldsymbol{u}^* (i\boldsymbol{\lambda} \times \boldsymbol{u}) - k^2 u_r^* u_r - \boldsymbol{u}^* \nabla \psi -$$
$$- \boldsymbol{u} \nabla \psi^*]\, dV + \frac{g}{\omega^2} \int \left| \frac{\omega^2}{g} \psi + \Phi \right|^2 df = \text{Extr.},$$

where we have put

$$(12) \qquad \boldsymbol{\lambda} = \frac{2\Omega}{\omega}, \quad k^2 = \frac{\sigma g}{\omega^2}, \quad \psi = \frac{p - g\Phi}{\omega^2}.$$

Giving now arbitrary variations to $\boldsymbol{u}$ and ψ and corresponding variations to $\boldsymbol{u}^*$ and ψ^*, we can put the variations of I due to the variation of $\boldsymbol{u}$ and $\boldsymbol{u}^*$ or ψ and ψ^* separately equal to zero since the resulting variations are complex conjugate quantities. For instance by the variation $\delta \boldsymbol{u}^*$ we get the equation

$$(13) \qquad \boldsymbol{u} - i\boldsymbol{\lambda} \times \boldsymbol{u} - k^2 u_r \frac{\boldsymbol{r}}{r} = \nabla \psi,$$

which is equivalent to (10). By the variation $\delta\psi^*$ using

$$(14) \qquad - \boldsymbol{u} \nabla (\delta \psi^*) = \operatorname{div} \boldsymbol{u} \cdot \delta \psi^* - \operatorname{div} (\boldsymbol{u} \delta \psi^*)$$

we find by the aid of Gauss' theorem the conditions $\operatorname{div} \boldsymbol{u} = 0$ and $u_n = 0$ at solid boundaries, whereas at the sea surface we have

$$(15) \qquad u_r = \frac{\omega^2}{g} \psi + \Phi,$$

which is equivalent to (5).

Passing to spherical polar coordinates $x = r \sin\vartheta \cos\varphi$, $y = r \sin\vartheta \sin\varphi$, $z = r \cos\vartheta$, we shall make intermediate use also of cylindrical coordinates

$\varrho = \sqrt{x^2+y^2}$, φ, z and corresponding displacement components

$$(16) \qquad u_z = u_r \cos\vartheta - u_\vartheta \sin\vartheta,$$
$$u_\varrho = u_x \cos\varphi + u_y \sin\varphi = u_r \sin\vartheta + u_\vartheta \cos\vartheta.$$

If then the z-axis is taken in the direction of $\boldsymbol{\Omega}$ and $\boldsymbol{\lambda}$ we have

$$(17) \qquad -\boldsymbol{u}^* (i\boldsymbol{\lambda} \times \boldsymbol{u}) = i\boldsymbol{\lambda} (\boldsymbol{u}^* \times \boldsymbol{u}) = = i\lambda (u_x^* u_y - u_y^* u_x) = i\lambda (u_\varrho^* u_\varphi - u_\varphi^* u_\varrho)$$

and

$$(18) \qquad |u_\varphi - i\lambda u_\varrho|^2 = u_\varphi^* u_\varphi + + i\lambda (u_\varrho^* u_\varphi - u_\varphi^* u_\varrho) + \lambda^2 u_\varrho^* u_\varrho.$$

The variational principle (11) may then be written

$$(19) \qquad I = \int \Big[(1-k^2)\, u_r^* u_r + u_\vartheta^* u_\vartheta - \lambda^2 u_\varrho^* u_\varrho + + |u_\varphi - i\lambda u_\varrho|^2 - \boldsymbol{u}^* \nabla \psi - \boldsymbol{u} \nabla \psi^* \Big] dV + + \frac{g}{\omega^2} \int_{r=a} \left| \frac{\omega^2}{g} \psi + \Phi \right|^2 df = \text{Extr.},$$

where

$$(20) \qquad \boldsymbol{u}^* \nabla \psi = u_r^* \frac{\partial \psi}{\partial r} + u_\vartheta^* \frac{1}{r} \frac{\partial \psi}{\partial \vartheta} + u_\varphi^* \frac{1}{r \sin\vartheta} \frac{\partial \psi}{\partial \varphi}.$$

In the foregoing paper (l. c.) a special coordinate transformation was introduced consisting in a very moderate alteration of the z-coordinate. We write

$$(21) \qquad x = X, \quad y = Y, \quad u = \sqrt{1-\varepsilon^2}\, Z,$$
$$u_x = U_X, \ u_y = U_Y, \ u_z = \sqrt{1-\varepsilon^2}\, U_Z,$$

maintaining thereby the invariant form of $\boldsymbol{u}^* \nabla \psi$. Passing to the corresponding polar coordinates we must therefore have corresponding to equ. (20)

$$(22) \qquad \boldsymbol{u}^* \nabla \psi = U_R^* \frac{\partial \psi}{\partial R} + U_\theta^* \frac{1}{R} \frac{\partial \psi}{\partial \theta} + U_\Phi^* \frac{1}{R \sin\theta} \frac{\partial \psi}{\partial \Phi}.$$

For the quantity ε^2 we shall take

$$(23) \qquad \varepsilon^2 = \frac{\lambda^2}{1-k^2} = \frac{(2\Omega)^2}{\omega^2 - \sigma g} \sim - \frac{(2\Omega)^2}{\sigma g} \sim - \frac{1}{1000}.$$

In the case of homogeneous water $k^2 = 0$, $\sigma g = 0$, the corresponding special value is $\varepsilon^2 = \lambda^2$. For natural mean stability of sea water, equ. (2), ε^2 is a very small quantity and almost independent of the frequency ω.

By this transformation all quantities in the integral (19) with the exception of $u_r^* u_r$ and $u_\vartheta^* u_\vartheta$ are formally unchanged. We have

$$(24) \qquad r^2 = R^2 - \varepsilon^2 Z^2 = R^2 (1 - \varepsilon^2 \cos^2\theta)$$

and, since $r \cos\vartheta = \sqrt{1-\varepsilon^2}\, R \cos\theta$, $r \sin\vartheta = R \sin\theta$,

$$(25) \qquad \cos^2\vartheta = \frac{(1-\varepsilon^2)\cos^2\theta}{1-\varepsilon^2 \text{co}^2\theta}, \quad \sin^2\vartheta = \frac{\sin^2\theta}{1-\varepsilon^2\cos^2\theta}.$$

Therefore, using the relatien $u_\vartheta = u_\varrho \cos\vartheta - u_z \sin\vartheta$ we find

$$(26) \qquad u_\vartheta^* u_\vartheta = \frac{1-\varepsilon^2}{1-\varepsilon^2\cos^2\theta} U_\theta^* U_\theta.$$

Further since

$$(27) \qquad \boldsymbol{u}^* \boldsymbol{u} = \boldsymbol{U}^* \boldsymbol{U} - \varepsilon^2 U_Z^* U_Z$$

we have

$$(28) \qquad u_r^* u_r = U_R^* U_R + \frac{\varepsilon^2 \sin^2\theta}{1-\varepsilon^2\cos^2\theta} U_\theta^* U_\theta - \varepsilon^2 U_Z^* U_Z.$$

From (26) and (28) we get

$$(29) \qquad (1-k^2)\, u_r^* u_r + u_\vartheta^* u_\vartheta - \lambda^2 u_\varrho^* u_\varrho = = (1-k^2)\, U_R^* U_R + \left[\frac{\lambda^2 \sin^2\theta}{1-\varepsilon^2\cos^2\theta} + + \frac{1-\varepsilon^2}{1-\varepsilon^2\cos^2\theta} \right] U_\theta^* U_\theta - \lambda^2 (U_Z^* U_Z + U_\varrho^* U_\varrho) = = (1-k^2)(1-\varepsilon^2)\, U_R^* U_R + + \frac{(1-\varepsilon^2)(1-\lambda^2\cos^2\theta)}{1-\varepsilon^2\cos^2\theta} U_\theta^* U_\theta.$$

The variational principle (19) may now be written

$$(30) \qquad I = \int \Big\{ (1-k^2)(1-\varepsilon^2)\, U_R^* U_R + + \frac{(1-\varepsilon^2)(1-\lambda^2\cos^2\theta)}{1-\varepsilon^2\cos^2\theta} U_\theta^* U_\theta + |U_\Phi - i\lambda U_\varrho|^2 - - \boldsymbol{U}^* \nabla \psi - \boldsymbol{U} \nabla \psi^* \Big\} dV + \frac{g}{\omega^2} \int_{r=a} \left| \frac{\omega^2}{g} \psi + \Phi \right|^2 df = \text{Extr.},$$

where, corresponding to (11), $U_\varrho = U_R \sin\theta + U_\theta \cos\theta$.

We shall use both this formulation and the formulation (19) in discussing the problem. We shall further from now on put $r = a$ in all equations, since the depth of an ocean is never greater than about 1/1000 of the earth's radius.

From the variational principle (19) we find by variation of u_r^*, u_ϑ^* and u_φ^*

$$(31) \qquad (1-k^2)\, u_r + i\lambda \sin\vartheta\, u_\varphi = \frac{\partial \psi}{\partial r},$$
$$u_\vartheta + i\lambda \cos\vartheta\, u_\varphi = \frac{1}{a} \frac{\partial \psi}{\partial \vartheta},$$
$$- i\lambda (\sin\vartheta u_r + \cos\vartheta u_\vartheta) + u_\varphi = \frac{1}{a \sin\vartheta} \frac{\partial \psi}{\partial \varphi}$$

which may be written

(32) $(1-k^2)\, u_r = \dfrac{\partial \psi}{\partial r} - i\lambda \sin\vartheta u_\varphi,$

$$u_\vartheta = \frac{1}{r}\frac{\partial \psi}{\partial \vartheta} - i\lambda \cos\vartheta u_\varphi,$$

$$u_\varphi = \frac{\dfrac{i\lambda \sin\vartheta}{1-k^2}\dfrac{\partial\psi}{\partial r} + \dfrac{i\lambda\cos\vartheta}{a}\dfrac{\partial\psi}{\partial\vartheta} + \dfrac{1}{a\sin\vartheta}\dfrac{\partial\psi}{\partial\varphi}}{1-\varepsilon^2\sin^2\vartheta - \lambda^2\cos^2\vartheta}.$$

On the other hand by variation of ψ^* we get, remembering that $dV = a^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$,

(33) $$\operatorname{div} \boldsymbol{u} = \frac{\partial u_r}{\partial r} + \frac{1}{a\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\sin\vartheta u_\vartheta + \frac{\partial u_\varphi}{\partial\varphi}\right] = 0.$$

From these equations we see that the function ψ has a singularity on the circles of latitude

(34) $$1-\varepsilon^2\sin^2\vartheta - \lambda^2\cos^2\vartheta = 0,$$

provided $\lambda > 1$. Now the partial differential equation (33) has an infinity of solutions of the form

(35) $$\psi = e^{n(r-a)}\, \Psi_{nm}(\vartheta) e^{im\varphi},$$

n and m being arbitrary numbers. Inserting this expression into (33) we get a differential equation for the function Ψ_{nm} with a regular singularity in $\cos\vartheta = \pm (1-\varepsilon^2)/(\lambda^2-\varepsilon^2)$. Independently of n and m we find the indicial equation $\varrho(\varrho-2) = 0$, corresponding to the asymptotic solutions

(36) $\Psi_{nm} = (1-\varepsilon^2\sin^2\vartheta - \lambda^2\cos^2\vartheta)^2$
and $\Psi_{nm} = \log(1-\varepsilon^2\sin^2\vartheta - \lambda^2\cos^2\vartheta)$

near to the singularity. The second solution must be ruled out because according to (32), the first solution only gives finite displacement amplitudes. Every solution ψ then always contains the factor $(1-\varepsilon^2\sin^2\vartheta-\lambda^2\cos^2\vartheta)^2$.

The occurrence of a singularity in the domain of integration necessitates a modification of our variational problem. This modification consists in dividing the integration domain into two parts

(37) $$1-\varepsilon^2\sin^2\vartheta - \lambda^2\cos^2\vartheta \gtrless 0,$$

where at the dividing boundary we must now add the condition that ψ remains finite. On account of (36), this means $\psi = 0$. Thus we are forced to use only the first form of (36).

We proceed to discuss first the case of homogeneous water, corresponding to $\varepsilon^2 = \lambda^2$ or

(38) $$u_\varphi = \frac{i\lambda\sin\vartheta\dfrac{\partial\psi}{\partial r} + i\lambda\cos\vartheta\dfrac{1}{r}\dfrac{\partial\psi}{\partial\vartheta} + \dfrac{1}{r\sin\vartheta}\dfrac{\partial\psi}{\partial\varphi}}{1-\lambda^2}.$$

In the special case of $\lambda\to 1$, $\omega\to 2\Omega$, since all displacement components have to remain finite, ψ must satisfy the first order differential equation

(39) $$i\sin\vartheta\frac{\partial\psi}{\partial r} + i\cos\vartheta\frac{1}{r}\frac{\partial\psi}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial\psi}{\partial\varphi} = ie^{-i\varphi}\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right) = 0$$

with the general solution

(40) $\psi = f(r\sin\vartheta\, e^{-i\varphi},\ r\cos\vartheta) = f(x-iy,\ z).$

This result may more easily be obtained from equ. (13) in Cartesian coordinates putting $k^2 = 0$ and $\lambda = 1$. We then have

(41) $$u_x + iu_y = \frac{\partial\psi}{\partial x},\quad u_y - iu_x = \frac{\partial\psi}{\partial y},$$

or $$\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y} = 0$$

as in (39).

It is obvious that with solutions of the form (40) there is no possibility of satisfying boundary conditions such as for instance $u_r = 0$ at a flat sea bottom $r = a - h$. It is true that the dominating components of tidal forces have frequencies

(42) $\omega = 2\Omega - 2\omega_M$ and $\omega = 2\Omega - 2\omega_S$,

ω_M and ω_S being orbital frequencies of the moon and the sun relative to the earth, and these frequencies are not in exact resonance with the double rotational frequency of the earth 2Ω. However, analyzing the Legendre function $P_2(\cos\theta)$ in the development of the moon or sun potential, we may for a body of orbital frequency ω_1, split up $P_2(\cos\theta)$ in the following way

(43) $$\begin{aligned} P_2(\cos\theta) = {}& P_2(\cos\vartheta)\left[\frac{3}{4}\sin^2\alpha - \frac{1}{2} - \frac{3}{4}\sin^2\alpha\cos 2\omega_1 t\right] + \\ &+ P_2^1(\cos\vartheta)\frac{\sin\alpha}{2}\Big[-\cos^2\frac{\alpha}{2}\sin(\varphi+\Omega t - 2\omega_1 t) + \\ &+ \cos\alpha\sin(\varphi+\Omega t) + \sin^2\frac{\alpha}{2}\sin(\varphi+\Omega t + 2\omega_1 t)\Big] + \\ &+ \frac{1}{4}P_2^2(\cos\vartheta)\Big[\cos^4\frac{\alpha}{2}\cos(2\varphi+2\Omega t - 2\omega_1 t) + \\ &+ \frac{\sin^2\alpha}{2}\cos(2\varphi+2\Omega t) + \\ &+ \sin^4\frac{\alpha}{2}\cos(2\varphi+2\Omega t+2\omega_1 t)\Big].\end{aligned}$$

Here θ is the angle between the celestial body and a point ϑ, φ on the earth's surface as seen from the centre of the earth, whereas α is the inclination of the orbital plane to the equator plane of the earth.

From this expression we see that beside the frequencies (42) we have also tidal forces of the exact resonance frequency $\omega = 2\Omega$, though with smaller amplitude, together with components of frequencies $\omega = 2\Omega + 2\omega_M$ and $\omega = 2\Omega + 2\omega_S$, however with practically negligible amplitudes. The case of perfectly homogeneous water must therefore be regarded as an unallowable idealization, and from now on we shall consider only the actual case of stable stratification of sea water.

The expression (43) contains the majority of observed tidal frequencies. A component with frequency approximately equal to $2\Omega - 3\omega_M$ must be attributed to the same term $P_2(\cos\theta)$ if the excentricity of the moon's orbit is taken into account. All components coming from the next term containing $P_3(\cos\theta)$ in the expansion of the potential of the moon are much weaker. The components with frequencies $4\Omega - 4\omega_M = 2(2\Omega - 2\omega_M)$ together with $(2\Omega - 2\omega_M) + (2\Omega - 2\omega_S)$ and $2(2\Omega - 2\omega_M) - (2\Omega - 2\omega_S)$ must certainly be interpreted as combination frequencies due to the non-linearity of the fundamental (non-linearized) hydrodynamic equations. Oscillations of this kind (shallow water components) cannot be accounted for in the present treatment.

Thus from an observational point of view we have very little need for treating oscillations of higher frequencies than $\omega = 3\Omega$, 4Ω, etc. Moreover, in this case with decreasing values of the parameter $\lambda = 2\Omega/\omega < 1$ the equations of motion will rapidly approach the equations corresponding to a non-rotating earth. With higher frequencies, therefore, we meet with no particular theoretical difficulties.

The important tidal frequencies being thus smaller than 2Ω we shall usually have the case of (37) with two separate domains of integration.

Returning now to the variational problem (19) we shall for the sake of brevity denote the surface integral by S. If the displacement components are defined by ψ, then from the equations of motion we have

$$I = \int [-\boldsymbol{u}\nabla\psi^*]\, dV + S = \text{Extr.}, \tag{44}$$

which may easily be obtained from (11) and (13) when multiplying (13) by $\boldsymbol{u}^*$.

Now for the sake of brevity we shall in (32) make the non-important alteration of writing $1 - \lambda^2\cos^2\vartheta$ for the denominator, transforming thereby the singularity from the circles of latitude $1 - \varepsilon^2\sin^2\vartheta - \lambda^2\cos^2\vartheta = 0$ to $1 - \lambda^2\cos^2\vartheta = 0$. Inserting u_r, u_ϑ, u_φ into (44) we find

$$I = -\int\left[\frac{1}{1-k^2}\frac{\partial\psi^*}{\partial r}\frac{\partial\psi}{\partial r} + \frac{1}{a^2}\frac{\partial\psi^*}{\partial\vartheta}\frac{\partial\psi}{\partial\vartheta} + (1-\lambda^2\cos^2\vartheta)\, u_\varphi^* u_\varphi\right] dV + S = \text{Extr.} \tag{45}$$

We shall now write

$$u_\varphi = u_1 + u_2, \tag{46}$$

$$u_1 = \frac{\dfrac{i\lambda\cos\vartheta}{1-k^2}\dfrac{\partial\psi}{\partial r}}{1-\lambda^2\cos^2\vartheta}, \quad u_2 = \frac{1}{a}\,\frac{i\lambda\cos\vartheta\dfrac{\partial\psi}{\partial\vartheta} + \dfrac{1}{\sin\vartheta}\dfrac{\partial\psi}{\partial\varphi}}{1-\lambda^2\cos^2\vartheta},$$

in order to show that u_1 may be dropped against u_2. Putting for brevity

$$I_1 = \frac{1}{k^2-1}\int\frac{\partial\psi^*}{\partial r}\frac{\partial\psi}{\partial r}\, dV, \tag{47}$$

$$I_2 = \int (1-\lambda^2\cos^2\vartheta)\, u_2^* u_2\, dV,$$

we have as to the order of magnitude

$$\int (1-\lambda^2\cos^2\vartheta)\, u_1^* u_1\, dV \sim \frac{I_1}{k^2}. \tag{48}$$

Further, using the inequality of *Schwarz*

$$\int (f^*g + g^*f)\, dV \leq 2\sqrt{\int f^*f dV \cdot \int g^*g dV}, \tag{49}$$

we find

$$\int (1-\lambda^2\cos^2\vartheta)(u_1^*u_2 + u_2^*u_1)\, dV \sim \frac{2}{k}\sqrt{I_1 I_2}. \tag{50}$$

The integral (48) may always be neglected on account of $k^2 \sim 1000$. Then consider the particular case of $I_1 \sim I_2$ corresponding to appreciable variations of ψ in vertical direction, or, what is the same, to higher order internal waves. Then the integral (50) may be neglected against both I_1 and I_2, its value being $k \sim 30$ times smaller than I_1 and I_2. Its actual magnitude is, however, much smaller than can be seen from the inequality (49), due to the approximate orthogonality of u_1 and u_2. This orthogonality is exact for an ocean bounded by two meridians on a non-rotating earth corresponding to $\lambda = 0$. But also apart from this particular reduction of the integral (50) its order of magnitude as given by (50) will justify dropping this part of the integral (45).

Next consider the case of I_1 being small, say of the order of magnitude I_2/k^2, corresponding to the case of only zeroth order internal waves. Then the integral (50) must be of the same order of magnitude and the integrals I_1 and (50) may both be neglected against I_2. Thus in either case we find that the integrals (48) and (50) can be neglected as small quantities.

Obviously this simplification of the problem corresponds not only to the dropping of the first term u_1 in u_φ but also to the dropping of the second term of u_r in (32). We thus obtain at the variational principle

$$(51)\quad I = -\int \Big[\frac{1}{1-k^2} \frac{\partial \psi^*}{\partial r} \frac{\partial \psi}{\partial r} + \frac{1}{a^2} \frac{\partial \psi^*}{\partial \vartheta} \frac{\partial \psi}{\partial \vartheta} + + (1-\lambda^2 \cos\vartheta)\, u_\varphi^* u_\varphi \Big] dV + S = \text{Extr.}$$

with

$$(52)\quad u_r = \frac{1}{1-k^2}\frac{\partial \psi}{\partial r}, \quad u_\vartheta = \frac{1}{a}\frac{\partial \psi}{\partial \vartheta} - i\lambda \cos\vartheta\, u_\varphi,$$

$$u_\varphi = \frac{1}{a} \frac{i\lambda \cos\vartheta \frac{\partial \psi}{\partial \vartheta} + \frac{1}{\sin\vartheta}\frac{\partial \psi}{\partial \varphi}}{1-\lambda^2\cos^2\vartheta}.$$

The system of equations (52) may also be written

$$(53)\quad (1-k^2)\, u_r = \frac{\partial \psi}{\partial r},$$

$$u_\vartheta + i\lambda\cos\vartheta\, u_\varphi = \frac{1}{a}\frac{\partial \psi}{\partial \vartheta},$$

$$u_\varphi - i\lambda\cos\vartheta\, u_\vartheta = \frac{1}{a\sin\vartheta}\frac{\partial \psi}{\partial \varphi},$$

or in vectorial form

$$(54)\quad \boldsymbol{u} - i\lambda\cos\vartheta \frac{\boldsymbol{r}}{r} \times \boldsymbol{u} - k u_r \frac{\boldsymbol{r}}{r} = \nabla \psi$$

showing that the horizontal component of $\lambda = 2\Omega/\omega^2$, and consequently the vertical component of the Coriolis' acceleration, may be neglected.

In the case of I_1 being very small compared to I_2, which corresponds to zeroth order internal waves, we may take $\frac{\partial \psi}{\partial r}$ and u_r to be approximately equal to zero and ψ approximately constant. Putting then

$$(55)\quad \Psi = \frac{\omega^2}{g}\psi = \left(\frac{p}{g} - \Phi\right)$$

we have

$$(56)\quad U_r = \Psi + \Phi$$

if U_r means the elevation of the sea surface. Writing further U_ϑ and U_φ for the horizontal displacement components, which are now approximately independent of r, we have from (53) the equations

$$(57)\quad U_\vartheta + i\lambda\cos\vartheta\, U_\varphi = \frac{1}{\omega^2}\frac{1}{a}\frac{\partial(p-g\Phi)}{\partial\vartheta},$$

$$U_\varphi - i\lambda\cos\vartheta\, U_\vartheta = \frac{1}{\omega^2}\frac{1}{a\sin\vartheta}\frac{\partial(p-g\Phi)}{\partial\varphi},$$

corresponding to the time dependent two-dimensional equations of motion

$$(58)\quad \ddot{U}_\vartheta - 2\Omega\cos\vartheta\, \dot{U}_\varphi = -\frac{1}{a}\frac{\partial(p-g\Phi)}{\partial\vartheta},$$

$$\ddot{U}_\varphi + 2\Omega\cos\vartheta\, \dot{U}_\vartheta = -\frac{1}{a\sin\vartheta}\frac{\partial(p-g\Phi)}{\partial\varphi}.$$

Performing the integration in r in the volume integral (51) for a variable depth $h = h(\vartheta, \varphi)$ and dividing by the mean depth h_0 times $(g/\omega^2)^2$, the variational principle may be written

$$(59)\quad I = \int\!\!\int \Big\{ -\frac{h}{h_0}\Big[\frac{\partial \Psi^*}{\partial \vartheta}\frac{\partial \Psi}{\partial \vartheta} + \frac{\left| i\lambda\cos\vartheta \frac{\partial \Psi}{\partial \vartheta} + \frac{1}{\sin\vartheta}\frac{\partial \Psi}{\partial \varphi}\right|^2}{1-\lambda^2\cos^2\vartheta} \Big] + + \frac{\omega^2 a^2}{g h_0} |\Psi + \Phi|^2 \Big\} \sin\vartheta\, d\vartheta\, d\varphi = \text{Extr.}$$

This formulation of the tidal problem is applicable to arbitrary depths and arbitrary boundaries of the sea.

In the foregoing paper (l. c.) a somewhat different method was used to establish the same result. In order to get partly separable equations the coordinate transformations (21) were introduced. From the corresponding variational problem (30) in polar coordinates we find by variation of U_R^*, U_Θ^* and U_Φ^* the following equations

$$(60)\quad (1-k^2)(1-\varepsilon^2)\, U_R + + i\lambda\sin\theta\,(U_\Phi - i\lambda\cos\theta\, U_R - i\lambda\cos\theta\, U_\Theta) = \frac{\partial \psi}{\partial R},$$

$$(1-\varepsilon^2)\frac{1-\lambda^2\cos^2\theta}{1-\varepsilon^2\cos^2\theta} U_\Theta + + i\lambda\cos\theta\,(U_\Phi - i\lambda\cos\theta\, U_R - i\lambda\cos\theta\, U_\Theta) = \frac{1}{R}\frac{\partial \psi}{\partial \theta},$$

$$(U_\Phi - i\lambda\cos\theta\, U_R - i\lambda\cos\theta\, U_\Theta) = \frac{1}{R\sin\theta}\frac{\partial \psi}{\partial \Phi}$$

or, solving in U_R, U_Θ, U_Φ,

$$(61)\quad (1-\varepsilon^2)\, U_R = \frac{1}{1-k^2}\left(\frac{\partial \psi}{\partial R} - \frac{i\lambda}{R}\frac{\partial \psi}{\partial \Phi}\right),$$

$$(1-\varepsilon^2)\, U_\Theta = \frac{1-\varepsilon^2\cos^2\theta}{1-\lambda^2\cos^2\theta}\left(\frac{1}{R}\frac{\partial \psi}{\partial \theta} - \frac{i\lambda\cos\theta}{\sin\theta}\frac{\partial \psi}{\partial \Phi}\right),$$

$$(1-\varepsilon^2)\, U_\Phi = \frac{i\lambda\sin\theta}{1-k^2}\frac{\partial \psi}{\partial R} + + \frac{1-\varepsilon^2\cos^2\theta}{1-\lambda^2\cos^2\theta}\left(\frac{i\lambda\cos\theta}{R}\frac{\partial \psi}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \psi}{\partial \Phi}\right).$$

It is obvious that it makes very little difference if in these equations we put $\varepsilon^2 = 0$ which corresponds to the non-important simplification of (45) and following equations. Secondly, we suppose the result to become very little different if, instead of taking as free boundary surface

$$(62)\quad r = R\sqrt{\frac{1-\varepsilon^2}{1-\varepsilon^2\cos^2\theta}} = a$$

and as solid boundary surface a flat sea bottom $r = a - h$, we take these boundary surfaces to be

$R = a$ and $R = a - h$, the thickness of the water sheet then being the same in either case. This is mathematically equivalent to retaining as boundary surfaces $r = a$ and $r = a - h$ and in return supposing the equations (61) to be valid in the original coordinate system r, ϑ and φ. Adopting this view we may replace all capital letters with small ones and write

$$(63) \qquad u_r = \frac{1}{1-k^2}\left[\frac{\partial\psi}{\partial r} - \frac{i\lambda}{a}\frac{\partial\psi}{\partial\varphi}\right],$$

$$u_\vartheta = \frac{1}{1-\lambda^2\cos^2\vartheta}\left[\frac{1}{a}\frac{\partial\psi}{\partial\vartheta} - \frac{i\lambda\cos\vartheta}{a\sin\vartheta}\frac{\partial\psi}{\partial\varphi}\right],$$

$$u_\varphi = \frac{i\lambda\sin\vartheta}{1-k^2}\frac{\partial\psi}{\partial r} + \frac{1}{1-\lambda^2\cos^2\vartheta}\left[\frac{i\lambda\cos\vartheta}{a}\frac{\partial\psi}{\partial\vartheta} + \frac{1}{a\sin\vartheta}\frac{\partial\psi}{\partial\varphi}\right],$$

corresponding to the system of equations

$$(64) \qquad (1-k^2)\,u_r + \lambda^2\cos\vartheta\sin\vartheta\,u_\vartheta + i\lambda\sin\vartheta\,u_\varphi = \frac{\partial\psi}{\partial r},$$

$$\lambda^2\cos\vartheta\sin\vartheta\,u_r + u_\vartheta + i\lambda\cos\vartheta\,u_\varphi = \frac{1}{a}\frac{\partial\psi}{\partial\vartheta},$$

$$-i\lambda\sin\vartheta\,u_r - i\lambda\cos\vartheta\,u_\vartheta + u_\varphi = \frac{1}{a\sin\vartheta}\frac{\partial\psi}{\partial\varphi}.$$

Comparing this system with the system of equations (31) there is but little difference. In the first and second equation we have the terms $\lambda^2\cos\vartheta\sin\vartheta\,u_\vartheta$ and $\lambda^2\cos\vartheta\sin\vartheta\,u_r$ respectively. We might also have changed the system (31) into the above system at once by the following argument: If in the first equation we are allowed to drop the term $i\lambda\sin\vartheta\,u_\vartheta$ asunimportant, this must also be allowed with $\lambda^2\cos\vartheta\sin\vartheta\,u_\vartheta$; and if in the third equation $-i\lambda\sin\vartheta\,u_r$ is unimportant, so is also the term $\lambda^2\cos\vartheta\sin\vartheta\,u_r$ in the second equation.

Adopting the equations (63) and (64) the corresponding variational problem for the function ψ turns into

$$(65) \qquad I = -\int\left\{\frac{1}{1-k^2}\left[a^2\frac{\partial\psi^*}{\partial r}\frac{\partial\psi}{\partial r} + i\lambda a\left(\frac{\partial\psi^*}{\partial\varphi}\frac{\partial\psi}{\partial r} - \frac{\partial\psi^*}{\partial r}\frac{\partial\psi}{\partial\varphi}\right)\right] + \frac{\partial\psi^*}{\partial\vartheta}\frac{\partial\psi}{\partial\vartheta} + \frac{\left|i\lambda\cos\vartheta\frac{\partial\psi}{\partial\vartheta} + \frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial\varphi}\right|^2}{1-\lambda^2\cos^2\vartheta}\right\}dr\sin\vartheta\,d\vartheta\,d\varphi + \frac{g}{\omega^2}\int\left|\frac{\omega^2}{g}\psi + \Phi\right|^2\sin\vartheta\,d\vartheta\,d\varphi = \text{Extr.}$$

The non-separability is now due to the second term of (65). Dropping first this term we may solve the problem in the following way.

Consider the two-dimensional eigen-value problem

$$(66) \qquad A = \int\left\{\frac{\partial\Psi^*}{\partial\vartheta}\frac{\partial\Psi}{\partial\vartheta} + \frac{\left|i\lambda\cos\vartheta\frac{\partial\Psi}{\partial\vartheta} + \frac{1}{\sin\vartheta}\frac{\partial\Psi}{\partial\varphi}\right|^2}{1-\lambda^2\cos^2\vartheta}\right\}\sin\vartheta\,d\vartheta\,d\varphi = \text{Extr.}$$

with the additional condition

$$(67) \qquad B = \int\Psi^*\Psi\sin\vartheta\,d\vartheta\,d\varphi = 1.$$

Writing

$$(68) \qquad \Psi = \sum_n c_n\,\Psi_n$$

we have

$$(69) \qquad A = \sum_{mn} c_m^*\,c_n\,A_{mn},\quad B = \sum_{mn} c_m^*\,c_n\,B_{mn}$$

in which

$$(70) \qquad B_{mn} = \int\Psi_m^*\,\Psi_n\sin\vartheta\,d\vartheta\,d\varphi$$

and where corresponding expressions for A_{mn} may easily be found from (66).

Assuming the functions Ψ_n to be eigen-functions of the problem and A_n the corresponding eigen-values, then we first have $A_{nn} = A_n$, $B_{nn} = 1$. Secondly, putting all coefficients of (69) except for instance c_n and c_m equal to zero, we find that if A_n and A_m are stationary values of the integral (66) they must be roots of the equation

$$(71) \qquad \begin{vmatrix} A_n - A & A_{nm} - B_{nm}A \\ A_{nm}^* - B_{nm}^*A & A_m - A \end{vmatrix} = 0.$$

This is possible only if $A_{nm} - B_{nm}\,A = 0$ for both $A = A_n$ and $A = A_m$, which means that $A_{nm} = 0$ and $B_{nm} = 0$ or

$$(72) \qquad A_{nm} = \delta_{nm}A_n,\quad B_{nm} = \delta_{nm}.$$

Writing now

$$(73) \qquad \psi = \sum_n R_n(r)\,\Psi_n(\vartheta,\varphi)$$

the volume integral in (65), apart from the second term, turns into

$$(74) \qquad I' = -\sum_n\int\left[\frac{a^2}{1-k^2}\frac{dR_n^*}{dr}\frac{dR_n}{dr} + A_n\,R_n^*R_n\right]dr$$

Then the functions R_n must satisfy the differential equations

$$(75) \qquad \frac{a^2}{k^2-1}\frac{d^2R_n}{dr^2} + \gamma_n{}^2\,R_n = 0,\quad \gamma_n{}^2 = A_n,$$

of which the solutions

$$(76) \qquad R_n = a_n\cos\left(\gamma_n\sqrt{k^2-1}\,\frac{r-a+h}{a}\right)$$

satisfy the boundary condition $\dfrac{dR_n}{dr} = 0$ at a flat sea bottom $r = a - h$.

In order to solve the problem (65) we shall write

$$(77)\quad \psi = \sum_n \left\{ R_n \Psi_n + \right. \\ \left. + \frac{i\lambda a_n}{\gamma_n \sqrt{k^2-1}} \sin\left(\gamma_n \sqrt{k^2-1}\,\frac{r-a+h}{a}\right) \frac{\partial \Psi_n}{\partial \varphi} \right\}$$

from which we find

$$(78)\quad (k^2-1)\, u_r = \frac{\partial \psi}{\partial r} - \frac{i\lambda}{a}\frac{\partial \psi}{\partial a} = \\ = \sum R'_n \left[\Psi_n - \frac{\lambda^2}{\gamma_n^2 (k^2-1)} \frac{\partial^2 \Psi_n}{\partial \varphi^2} \right]$$

showing that the boundary condition for u_r at $r = a - h$ is satisfied. On the other hand we see that the second term of this expression, being $k^2 \sim 1000$ times smaller than the first one, may be neglected against the first term.

As to the second term of (77) we have to consider two different cases $\gamma_m \ll k$ and $\gamma_n \sim k$. In the first case the cosine function of R_n may be taken equal to unity, whereas the sine function of R'_n may be replaced by its argument. We readily find the second term to be $h/a = 1/1000$ times the first term. In the second case, since $k^2 \frac{h}{a} \sim 1$ the sine and cosine functions are both of magnitude from zero to unity. We therefore have to compare the coefficients of sine and cosine getting the ratio $1/k^2 \sim h/a$ as before.

Thus in the variational integral (65) we may neglect this second term of (77) and (78), its role being only to adjust the radial displacement component to its given boundary condition at the bottom of the sea. The first two terms in the integrand of (65) may from (78) be written

$$a^2 (1-k^2)\, r_r^* u_r - \frac{\lambda^2}{1-k^2} \frac{\partial \psi^*}{\partial \varphi} \frac{\partial \psi}{\partial \varphi}$$

the last term being of a negligible order of magnitude. Thus we see that the variational problem (65) is equivalent to the simplified problem

$$(79)\quad I = -\int \left\{ \frac{a^2}{1-k^2} \frac{d\psi^*}{\partial r} \frac{\partial \psi}{\partial r} + \frac{\partial \psi^*}{\partial \vartheta} \frac{\partial \psi}{\partial \vartheta} + \right. \\ \left. + \frac{\left| i\lambda \cos\vartheta \frac{\partial \psi}{\partial \vartheta} + \frac{1}{\sin\vartheta}\frac{\partial \psi}{\partial \varphi} \right|^2}{1-\lambda^2 \cos^2\vartheta} \right\} dr \sin\vartheta\, d\vartheta\, d\varphi + \\ + \frac{g}{\omega^2} \int \left| \frac{\omega^2}{g} \psi + \Phi \right|^2 \sin\vartheta\, d\vartheta\, d\varphi = \text{Extr.}$$

where for the displacement components we have the modified equations

$$(80)\quad u_r = \frac{1}{1-k^2}\frac{\partial \psi}{\partial r}, \quad u_\vartheta = \frac{1}{a}\frac{\partial \psi}{\partial \vartheta} - i\lambda \cos\vartheta\, u_\varphi,$$

$$u_\varphi = \frac{1}{a} \frac{i\lambda \cos\vartheta \frac{\partial \psi}{\partial \vartheta} + \frac{1}{\sin\vartheta}\frac{\partial \psi}{\partial \varphi}}{1-\lambda^2\cos^2\vartheta}$$

as in (52) and (53).

Putting

$$(81)\quad \Phi = \sum_n b_n \Psi_n, \quad b_n = \int \Psi_n^* \Phi \sin\vartheta\, d\vartheta\, d\varphi$$

and taking into account the boundary condition at the sea surface (equ. 15)

$$(82)\quad u_r = \frac{\omega^2}{g}\psi + \Phi$$

we find the boundary condition for R

$$(83)\quad \frac{1}{1-k^2}\frac{dR_n}{dr} = \frac{\omega^2}{g} R_n + b_n,$$

or from (77)

$$(84)\quad a_n \frac{\gamma_n}{a\sqrt{k^2-1}} \sin x_n = a_n \frac{\omega^2}{g} \cos x_n + b_n,$$

$$x_n = \gamma_n \sqrt{k^2-1}\,\frac{h}{a},$$

$$a_n \frac{\gamma_n}{a\sqrt{k^2-1}} \sin x_n = \frac{b_n}{1 - \frac{\omega^2 a^2}{\gamma_n^2 gh} x_n \operatorname{cotg} x_n} = \\ = b_n + \frac{b_n}{\frac{\gamma_n^2 gh}{\omega^2 a^2}\frac{\operatorname{tg} x_n}{x_n} - 1}.$$

At the sea surface we then have, writing U_r for u_r at $r = a$,

$$(85)\quad U_r = \sum_n \frac{1}{1-k^2}\frac{dR_n}{dr}\Psi_n = \sum_n a_n \frac{\gamma_n}{a\sqrt{k^2-1}} \sin x_n \Psi_n = \\ = \Phi + \sum_n \frac{b_n \Psi_n}{\frac{\gamma_n^2 gh}{\omega^2 a^2}\frac{\operatorname{tg} x_n}{x_n} - 1}.$$

For small arguments x_n this is equivalent to

$$(86)\quad U_r = \Phi + \sum_n \frac{b_n \Psi_n}{\frac{A_n gh}{\omega^2 a^2} - 1}$$

corresponding to the solution of the two-dimensional problem (59) for a flat sea bottom.

Only if in the neighbourhood of $x_n = \pi, 2\pi, 3\pi, \ldots$ one of the denominators of the expression (85) is almost equal to zero can internal waves of higher orders be excited with measureable amplitudes. We shall estimate the probability for first order internal waves corresponding to $x_n \sim \pi$ to be excited, in which case we claim one of the denominators of (85) to be smaller than the expansion coefficient b_n of (81).

Taking the most favourable case of a rather *wide* and *deep* sea and the *highest* tidal frequencies we have

$$(87) \qquad \overline{(\gamma_{n+1}-\gamma_n)} \sim 1, \ \frac{h}{a} \sim \frac{1}{1000},$$
$$\omega^2 \sim (2\Omega)^2 \sim 2 \cdot 10^{-8}.$$

Then from (85) we find the denominator to be approximately

$$(88) \qquad \frac{\gamma_n^2}{10} \frac{\operatorname{tg} x_n}{x_n} - 1, \ x_n \sim \frac{\gamma_n}{30}.$$

Further, since γ_n corresponds approximately to the numbers of zeros of the eigen-function Ψ_n, we must have $b_n \sim 1/\gamma_n$. We shall now evaluate the interval $\Delta\gamma_n$ in which the denominator is smaller than b_n. Since all quantities except $\operatorname{tg} x_n$ have only slow variations we have

$$(89) \qquad \frac{\gamma_n}{x_n} \Delta \operatorname{tg} x_n \sim \frac{\gamma_n}{x_n} \Delta x_n =$$
$$= \Delta \gamma_n \sim \frac{10}{\gamma_n^2} \sim \frac{1}{100\, x_n^2} \sim \frac{1}{1000}$$

or

$$(90) \qquad \Delta \gamma_n \sim \frac{1}{1000} \overline{(\gamma_{n+1}-\gamma_n)}.$$

Thus we find that the probability for the first order internal waves to be excited is of the order of magnitude 1 : 1000. In practice this probability may be said to be infinitely small and we may therefore safely assume only zeroth order internal waves. For these waves the formulation (59) is valid.

Thus we see there is no difficulty in considering also an ocean of variable depth $h(\vartheta, \varphi)$. We need only make use of the eigen-functions of the variational problem

$$(91) \qquad A = \int \frac{h}{h_0} \left[\frac{\partial \Psi^*}{\partial \vartheta} \frac{\partial \Psi}{\partial \vartheta} + \right.$$
$$\left. + \frac{\left| i\lambda \cos\vartheta \dfrac{\partial \Psi}{\partial \vartheta} + \dfrac{1}{\sin\vartheta} \dfrac{\partial \Psi}{\partial \varphi} \right|^2}{1 - \lambda^2 \cos^2\vartheta} \right] \sin\vartheta \, d\vartheta \, d\varphi = \text{Extr.},$$
$$B = \int \Psi^* \Psi \sin\vartheta \, d\vartheta \, d\varphi = 1.$$

Then taking

$$(92) \qquad \Psi = \sum_n c_n \Psi_n, \ U_r = \Phi + \Psi,$$

we have from (59)

$$(93) \qquad \sum_n \left[\left(-A_n + \frac{\omega^2 a^2}{g h_0} \right) c_n^* c_n + \right.$$
$$\left. + \frac{\omega^2 a^2}{g h_0} (c_n^* b_n + c_n b_n^* + \text{const.}) \right] = \text{Extr.},$$

from which by variation of c_n^* we find

$$(94) \qquad \left(-A_n + \frac{\omega^2 a^2}{g h_0} \right) c_n + \frac{\omega^2 a^2}{g h_0} b_n = 0, \ c_n = \frac{b_n}{\dfrac{A_n g h_0}{a^2 \omega^2} - 1},$$

corresponding to equ. (86). Thus the validity of the two-dimensional Laplacian tidal theory is fully established for real oceans.

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