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ON THE FRONTOGENESIS AND CYCLOGENESIS IN THE ATMOSPHERE

PART I.
ON THE STABILITY OF THE STATIONARY CIRCULAR VORTEX

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LIST OF NOTATIONS

x, y, z — cartesian co-ordinates along axes directed eastwards, northwards and vertically upwards, respectively.

$\boldsymbol{i}, \boldsymbol{k}, \boldsymbol{j}$ — unit vectors directed eastwards, northwards and vertically upwards, respectively.

∇ — in Chapter I, three-dimensional Hamilton operator.

∇_3 — three-dimensional Hamilton operator.

∇ — in Ghapters II and III, meridional Hamilton operator.

$\delta \boldsymbol{r} = \delta r \boldsymbol{r}_I$ — line element.

$\delta \boldsymbol{F} = \delta F \boldsymbol{n}$ — surface element.

t — time.

v — in Chapter I, three-dimensional velocity.

$\boldsymbol{v}_3 = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}$ — three-dimensional velocity.

$\boldsymbol{v} = v_y \boldsymbol{j} + v_z \boldsymbol{k}$ — in Chapter II and III, meridional velocity.

$V_x \boldsymbol{i}$ — zonal velocity in the stationary, circular vortex.

C — velocity circulation.

M — the mass of a certain fluid quantity.

dM — mass element.

K — kinetic energy.

p, P — pressure of a fluid particle.

q, Q — density of a fluid particle.

s, S — specific volume of a fluid particle.

T — temperature of a fluid particle.

ϑ, Θ — potential temperature of a fluid particle.

$-\nabla \varphi$ — external force per unit mass.

$\boldsymbol{g}$ — acceleration of gravity.

$\boldsymbol{\Omega}$ — angular velocity of the earth.

Ω — scalar value of Ω.

$\boldsymbol{N}$ — defined on p. 10.

$\boldsymbol{R} = R \boldsymbol{R}_I$ — distance from the axis of the earth to a fluid particle.

W — heat.

c_v, c_p — specific heat of air at constant volume and constant pressure, respectively.

R — gas—constant.

Γ — coefficient of piezotropy by adiabatic processes.

$\boldsymbol{v}_C$ — solenoidal velocity.

$\boldsymbol{v}_D$ — ascendental velocity.

ψ — stream function.

α — velocity potential.

M_Q — tensor defined on p. 11.

$\boldsymbol{j}_Q, \boldsymbol{k}_Q$ — principal axes of M_Q.

$m^2{}_Q, n^2{}_Q$ — defined by:

$$M_Q = m^2{}_Q \boldsymbol{j}_Q \boldsymbol{j}_Q + n^2{}_Q \boldsymbol{k}_Q \boldsymbol{k}_Q.$$

M_Θ — tensor defined on p. 15.

$\boldsymbol{j}_\Theta, \boldsymbol{k}_\Theta$ — principal axes of M_Θ.

$m^2{}_\Theta, n^2{}_\Theta$ — defined by: $M_\Theta = m^2{}_\Theta \boldsymbol{j}_\Theta \boldsymbol{j}_\Theta + n^2{}_\Theta \boldsymbol{k}_\Theta \boldsymbol{k}_\Theta$.

η, ζ — meridional cartesian co-ordinates along $\boldsymbol{j}_Q, \boldsymbol{k}_Q$ or along $\boldsymbol{j}_\Theta, \boldsymbol{k}_\Theta$.

ON THE FRONTOGENESIS AND CYCLOGENESIS IN THE ATMOSPHERE

PART I.

ON THE STABILITY OF THE STATIONARY CIRCULAR VORTEX

BY

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With 4 Figures in the Text.

INTRODUCTION

By the Norwegian school of meteorologists it is assumed that the existence of fronts is always needed for the formation of the cyclones of middle latitudes. This view seems to be theoretically well founded. Firstly, the formation of cyclones can undoubtedly be interpreted as a consequence of an instability of the underlying fundamental field of motion. Secondly, if the atmosphere may as a first approximation be identified with a barotropic stationary circular vortex, which is inertially, as well as statically, stable, there will be no other destabilizing factors than the wellknown dynamical instability connected with the wind shear at the fronts. [See for instance *Godske* (1936)].

In recent years, however, it has been shown by several investigators, first of all by *E. Høiland* (1938, 1941) that if also the baroclinity of the atmosphere is taken into consideration, the stability properties will essentially change from those in the barotropic vortex. The stability criteria for the baroclinic stationary circular vortex by symmetric perturbations were first examined by *Helmholtz* (1888). Later they were found by *H. Solberg* (1936), by Høiland in the papers referred to above, by *E. Kleinschmidt* (1941) and by *E. Ertel* (1941).

In Kleinschmidt's paper and in a paper by *R. Fjørtoft* (1942) the stability criteria under atmospheric conditions were found which showed that instability could actually exist in the atmosphere independently of contingent fronts, if the baroclinity is great enough and the statical stability not too large. The stability criteria were given in Fjørtoft's paper in such a form that it was very natural to ask whether the cyclones of middle latitudes were developing as a consequence of the instability in question.

As early as in 1933, the Russian investigator *P. Moltschanow* (1933) pointed out the possibility of explaining the formation of the cyclones as due to an instability connected with the baroclinity of the atmosphere; but he did not succeed in arriving at precise stability criteria. Later, *P. Raethjen* (1941) expressed a similar opinion as to the explanation of the cyclone formation. In a quite different way I have earlier shown by arguments from the theory of advection that just the baroclinity of the atmosphere outside the fronts must be of essential importance to the development of the cyclones. (Fjørtoft 1942.)[1])

The main purpose of this work is to examine more thoroughly how frontogenesis and cyclogenesis are related to the instability which the air masses may possess owing to their baroclinity. Earlier, P. Raethjen (1939) has tried

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1) This appeared first as an examination paper at the University of Oslo 1940.

to explain the frontogenesis dynamically as due to a convective instability in a baroclinic field of motion. His examinations, however, are not in full accordance with the precise stability criteria.

This work appears in two parts. In the present, first part, I have tried to give the theoretical examinations of the stability properties of the stationary circular vortex a more solid foundation. In the second part, the application to the atmosphere will be the main purpose.

In Chapter I the equations of motion of an inviscid fluid are written down. The influence of the friction and the diffusion will be considered in the second part. In the case of a compressible fluid we have in Part I limited the considerations to adiabatic processes. A remarkable analogy as to the circulation theorems, between the incompressible and adiabatic compressible fluid, has been shown in Chapter I. In Chapter II, the stability criteria have been found by an energetic method of consideration. This energetic method is also useful in connection with the discussion of the kinematics of the motion which is given in the last chapter.

Chapter I.

THE HYDRODYNAMICAL EQUATIONS OF AN INVISCID FLUID

1. The incompressible fluid.

The movements of the fluid particles are in this case determined by

the equation of motion in the form of Euler

$$(1,1) \qquad q\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p - q\nabla\varphi - q\boldsymbol{v}\cdot\nabla\boldsymbol{v}$$

the equation of continuity

$$(1,2) \qquad \frac{\partial q}{\partial t} = -\boldsymbol{v}\cdot\nabla q$$

the condition of incompressibility

$$(1,3) \qquad \nabla\cdot\boldsymbol{v} = 0$$

and certain initial and boundary conditions.

In the above equations $\boldsymbol{v}$ is the velocity, q the density p the pressure of a fluid particle, whereas $-\nabla\varphi$ is an external force per unit mass.

As to the boundary conditions, we shall for the sake of argument consider a rigid closed wall at rest. Denoting with $\boldsymbol{n}$ the unit vector normal to the boundary surface, we obtain

the kinematic boundary condition

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0$$

as a consequence, partly of the geometrical restraint of the walls and partly of the condition of incompressibility. The kinematic boundary condition having to be identically satisfied, we obtain particularly

$$(1,4) \qquad q\frac{\partial \boldsymbol{v}}{\partial t}\cdot\boldsymbol{n} = 0$$

at the boundary.

As to the initial conditions, we must assume the hydrodynamical variables to be known functions of the spatial variables at the beginning of the time. We shall now show that the pressure gradient $-\nabla p$ is uniquely determined by the distribution of velocity and density and the boundary conditions. The equation of motion (1,1) can be written as follows

$$(1,5) \qquad \nabla p = -q\frac{\partial \boldsymbol{v}}{\partial t} - q\nabla\varphi - q\boldsymbol{v}\cdot\nabla\boldsymbol{v}.$$

Performing the scalar multiplication $\nabla\cdot$ on each term of this equation, we obtain

$$\nabla^2 p - \frac{\nabla q}{q}\cdot\nabla p = -q\nabla\cdot\boldsymbol{v}\cdot\nabla\boldsymbol{v} - q\nabla\cdot\frac{\partial \boldsymbol{v}}{\partial t}.$$

Owing to the incompressibility of the fluid, this equation is reduced to

$$(1,6) \qquad \nabla^2 p - \frac{\nabla q}{q}\cdot\nabla p = -q\nabla\cdot\boldsymbol{v}\cdot\nabla\boldsymbol{v}.$$

If we substitute into the kinematic boundary condition (1,4) for $q\frac{\partial \boldsymbol{v}}{\partial t}$ the expression obtained from (1,1) we obtain

the dynamic boundary condition

$$(1,7) \qquad \nabla p\cdot\boldsymbol{n} = -q\,[\nabla\varphi + \boldsymbol{v}\cdot\nabla\boldsymbol{v}]\cdot\boldsymbol{n}.$$

The equations (1,6) and (1,7) determine the pressure with the exception of a spatially constant function. As a particular consequence of this, the intial distribution only of velocity and density, can be arbitrarily chosen.

From the hydrodynamical equations of motion we may derive other useful equations. Of great importance to the applications is the circulation theorem of Bjerknes. We arrive at this theorem if we refer the equation of motion (1,1) to unit mass:

$$\frac{\partial \boldsymbol{v}}{\partial t} = -s\nabla p - \nabla\varphi - \boldsymbol{v}\cdot\nabla\boldsymbol{v}$$

and then take the circulation of each term in this equation round an arbitrary closed curve L which is at rest. Then we obtain

$$\oint_L \frac{\partial \boldsymbol{v}}{\partial t} \cdot \delta \boldsymbol{r} = -\oint_L s \nabla p \cdot \delta \boldsymbol{r} - \oint_L \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \delta \boldsymbol{r}.$$

According to the theorem of Stokes, the first right-hand line integral can be transformed into a surface integral where it is integrated vectorially over an arbitrary surface $\boldsymbol{F}$ bounded by the curve L. Doing so, we obtain, using the notation C for the velocity circulation $\oint \boldsymbol{v} \cdot \delta \boldsymbol{r}$,

$$\frac{\partial C}{\partial t} = -\int_F \nabla s \times \nabla p \cdot \delta \boldsymbol{F} - \oint_L \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \delta \boldsymbol{r}.$$

In the applications of the circulation theorem it is nearly always necessary to obtain an expression for the acceleration $\frac{\partial^2 C}{\partial t^2}$ of the velocity circulation. This is especially the case when the theorem is used in stability examinations. In this way the circulation theorem has been used by *V. Bjerknes* (1937), but first of all by *E. Høiland* (1938, 1941) in an examination of the stability properties of several fluid motions. The expression for $\frac{\partial^2 C}{\partial t^2}$ is most easily arrived at by derivating locally with respect to time the terms in the equation of motion (1,1). Then we obtain

$$q \frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \frac{\partial q}{\partial t} s \nabla p = -\nabla \frac{\partial p}{\partial t} - q \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v}.$$

Here we introduce from the equation of continuity $\frac{\partial q}{\partial t} = -\boldsymbol{v} \cdot \nabla q$ and then take the circulation of each term of the equation round an arbitrary closed curve L. The resulting equation is

$$(1{,}8) \quad \oint_L q \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \delta \boldsymbol{r} = \oint_L q \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) \cdot \delta \boldsymbol{r}$$

According to the theorem of Stokes, this equation can be written

$$\int_F \left[\nabla q \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} + q \nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right] \cdot \delta \boldsymbol{F} =$$
$$\int_F \nabla q \times \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) \cdot \delta \boldsymbol{F} +$$
$$+ \int_F q \nabla \times \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) \cdot \delta \boldsymbol{F}.$$

By division by q on both sides of this equation we arrive at

$$\int_F \nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \delta \boldsymbol{F} = \int_F \nabla \times \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) \cdot \delta \boldsymbol{F}$$
$$+ \int_F \frac{\nabla q}{q} \times \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{\partial^2 \boldsymbol{v}}{\partial t} \right) \cdot \delta \boldsymbol{F}.$$

Again as a consequence of the theorem of Stokes this equation can be written

$$(1{,}9) \quad \frac{\partial^2 C}{\partial t^2} = \oint_L \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \delta \boldsymbol{r} =$$
$$\oint_L \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) \cdot \delta \boldsymbol{r} +$$
$$+ \int_F \frac{\nabla q}{q} \times \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right) \cdot \delta \boldsymbol{F}.$$

The corresponding vorticity equation is

$$(1{,}9') \quad \frac{\partial^2}{\partial t^2} \nabla \times \boldsymbol{v} = \nabla \times \frac{\partial^2 v}{\partial t^2} =$$
$$\nabla \times \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) +$$
$$+ \frac{\nabla q}{q} \times \left(\boldsymbol{v} \cdot \frac{\nabla q}{q} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right).$$

The above equations will be referred to below as the equations for the acceleration of circulation and vorticity, respectively. In the next section the existence of equations for a compressible fluid will be proved which show a remarkable analogy to the above equations.

2. The compressible adiabatic fluid.

The movements of the particles of a compressible inviscid fluid are determined by

the equation of motion in the form of Euler

$$(1{,}10) \quad q \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p - q \nabla \varphi - q \boldsymbol{v} \cdot \nabla \boldsymbol{v}$$

the equation of continuity

$$(1{,}11) \quad \frac{\partial q}{\partial t} = -\boldsymbol{v} \cdot \nabla q - q \nabla \cdot \boldsymbol{v}$$

the gas-equation

$$(1{,}12) \quad p = RqT$$

the first theorem of thermodynamics

$$(1{,}13)\qquad q\frac{dW}{dt} = c_v q\frac{dT}{dt} + p\nabla\cdot\boldsymbol{v}$$

and certain initial- and boundary conditions together with a sufficient number of equations determining the supply of heat $q\frac{dW}{dt}$ per unit time and volume. In the gas-equation R is the gas-constant, and T the temperature. With the assumption of dryadiabatic processes the last equation is reduced to

$$(1{,}13')\qquad 0 = c_v q\frac{dT}{dt} + p\nabla\cdot\boldsymbol{v}.$$

Substituting in this equation for T the expression obtained from the gas-equation, we obtain

$$(1{,}14)\qquad \frac{dp}{dt} = -\frac{q}{\Gamma}\nabla\cdot\boldsymbol{v}$$

where

$$(1{,}15)\qquad \Gamma = \frac{c_v}{c_p}\cdot\frac{1}{RT}$$

c_v and c_p are the specific heats at constant volume and constant pressure, respectively. Having

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \boldsymbol{v}\cdot\nabla p$$

equation (1,14) can be written

$$(1{,}14')\qquad \frac{\partial p}{\partial t} = -\boldsymbol{v}\cdot\nabla p - \frac{q}{\Gamma}\nabla\cdot\boldsymbol{v}.$$

This equation has a form quite analogous to the equationof continuity (1,11) and will be referred to below as

the equation of pressure tendency.

According to the equation of continuity we have

$$-q\nabla\cdot\boldsymbol{v} = \frac{dq}{dt}.$$

Substituting here for $-q\nabla\cdot\boldsymbol{v}$ the expression on the left-hand side of (1,14), we obtain

$$\frac{dq}{dt} = \Gamma\frac{dp}{dt}.$$

This is the so-called equation of piezotropy and Γ is the coefficient of piezotropy by dryadiabatic processes. We shall now characterize the air particles by the temperatures that they are assuming if they are brought adiabatically under constant pressure. This potential temperature ϑ will obey the following equations

$$(1{,}16)\qquad \frac{\partial\vartheta}{\partial t} = -\boldsymbol{v}\cdot\nabla\vartheta$$

and

$$(1{,}17)\qquad \frac{\nabla q - \Gamma\nabla p}{q} = -\frac{\nabla\vartheta}{\vartheta}$$

the first of which expresses the conservation of potential temperature. The equation of continuity can now be written in the form

$$(1{,}11')\qquad \frac{\partial q}{\partial t} = q\boldsymbol{v}\cdot\frac{\nabla\vartheta}{\vartheta} + \Gamma\frac{\partial p}{\partial t}.$$

This equation is arrived at if we substitute for $-q\nabla\cdot v$ in (1,11) the expression $\Gamma\frac{\partial p}{\partial t} + \Gamma\boldsymbol{v}\cdot\nabla p$ obtained from the equation of pressure tendency (1,14').

The circulation theorem of Bjerknes assumes the same form for the compressible as for the incompressible fluid. It is now a remarkable fact that equations for the acceleration of circulation and vorticity exist for the compressible fluid in a form quite analogous to that in the incompressible one. In order to derive these equations, we derivate equation (1,10) locally with respect to time, obtaining

$$q\frac{\partial^2\boldsymbol{v}}{\partial t^2} + \frac{\partial q}{\partial t}s\nabla p = -q\frac{\partial}{\partial t}\boldsymbol{v}\cdot\nabla\boldsymbol{v}.$$

In this equation we introduce from the equation of continuity (1,11') $\frac{\partial q}{\partial t} = q\boldsymbol{v}\cdot\frac{\nabla\vartheta}{\vartheta} + \Gamma\frac{\partial p}{\partial t}$, and obtain then

$$(1{,}18)\qquad \nabla\frac{\partial p}{\partial t} + q\frac{\partial^2\boldsymbol{v}}{\partial t^2} =$$
$$= q\left[\boldsymbol{v}\cdot\frac{-\nabla\vartheta}{\vartheta}s\nabla p - \frac{\partial}{\partial t}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right] + \Gamma s\frac{\partial p}{\partial t}\nabla p.$$

Suppose the vector $\boldsymbol{A}$, to be defined by

$$(1{,}19)\qquad \boldsymbol{A} = \boldsymbol{v}\cdot\frac{-\nabla\vartheta}{\vartheta}s\nabla p - \frac{\partial}{\partial t}\boldsymbol{v}\cdot\nabla\boldsymbol{v}$$

Eq. (1,18) can then be written

$$(1{,}20)\qquad \nabla\frac{\partial p}{\partial t} + q\frac{\partial^2\boldsymbol{v}}{\partial t^2} = q\boldsymbol{A} + \Gamma s\frac{\partial p}{\partial t}\nabla p.$$

Taking the curl of each term of this equation we arrive at

$$\nabla\times q\frac{\partial^2\boldsymbol{v}}{\partial t^2} = \nabla\times q\boldsymbol{A} + \nabla\frac{\partial p}{\partial t}\times\Gamma s\nabla p + \frac{\partial p}{\partial t}\nabla\times\Gamma s\nabla p.$$

According to (1,15) we have $\Gamma s = \frac{c_v}{c_p} \cdot \frac{1}{p}$ and therefore

$$\nabla \times \Gamma s \nabla p = \frac{c_v}{c_p} \nabla \times \nabla \ln p = 0.$$

Consequently, the above equation is reduced to

$$\nabla \times q \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = \nabla \times q \boldsymbol{A} + \nabla \frac{\partial p}{\partial t} \times \Gamma s \nabla p.$$

In this equation we introduce for $\nabla \frac{\partial p}{\partial t}$ the expression given by (1,20), thus obtaining

$$\nabla \times q \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = \Gamma \nabla p \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \nabla \times q \boldsymbol{A} - \Gamma \nabla p \times \boldsymbol{A}.$$

Now we can write

$$\nabla \times q \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = q \nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \nabla q \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2}$$

and

$$\nabla \times q \boldsymbol{A} = q \nabla \times \boldsymbol{A} + \nabla q \times \boldsymbol{A}.$$

According to these identities the last equation can be written

$$q \nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = q \nabla \times \boldsymbol{A} + (\nabla q - \Gamma \nabla p) \times \left(\boldsymbol{A} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right).$$

Dividing this equation by q, we obtain

$$(1{,}21) \quad \nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = \nabla \times \boldsymbol{A} + \frac{\nabla q - \Gamma \nabla p}{q} \times \left(\boldsymbol{A} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right).$$

According to (1,17) we have

$$\frac{\nabla q - \Gamma \nabla p}{q} = \frac{-\nabla \vartheta}{\vartheta}$$

(1,21) can therefore be written

$$\nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = \nabla \times \boldsymbol{A} + \left(\frac{-\nabla \vartheta}{\vartheta} \times \left(\boldsymbol{A} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right) \right).$$

We now take the surface integrals of each term of the last equation over an arbitrary vectorial surface $\boldsymbol{F}$. According to the theorem of Stokes we can transform the first and second integral to line integrals. Doing so, we obtain

$$\oint_L \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \delta \boldsymbol{r} = \oint_L \boldsymbol{A} \cdot \delta \boldsymbol{r} + \int_F \frac{-\nabla \vartheta}{\vartheta} \times \left(\boldsymbol{A} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right) \cdot \delta \boldsymbol{F}$$

where L symbolizes the curve bounding the surface $\boldsymbol{F}$. In the last two equations we substitute again for $\boldsymbol{A}$ the right-hand side of eq. (1,19) and arrive in this way at the following two equations

$$(1{,}22) \quad \frac{\partial^2 C}{\partial t^2} = \oint_L \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \delta \boldsymbol{r} =$$

$$= \oint_L \left[\boldsymbol{v} \cdot \frac{-\nabla \vartheta}{\vartheta} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right] \cdot \delta \boldsymbol{r} +$$

$$+ \int_F \frac{-\nabla \vartheta}{\vartheta} \times \left[\boldsymbol{v} \cdot \frac{-\nabla \vartheta}{\vartheta} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \right] \cdot \delta \boldsymbol{F}$$

and

$$(1{,}22') \quad \frac{\partial^2}{\partial t^2} \nabla \times \boldsymbol{v} = \nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} =$$

$$= \nabla \times \left[\boldsymbol{v} \cdot \frac{-\nabla \vartheta}{\vartheta} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right] +$$

$$+ \frac{-\nabla \vartheta}{\vartheta} \times \left[\boldsymbol{v} \cdot \frac{-\nabla \vartheta}{\vartheta} s \nabla p - \frac{\partial}{\partial t} \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{\partial^2 \boldsymbol{v}}{\partial t^2} . \right]$$

It should be noted that these equations for the acceleration of circulation and vorticity of a compressible fluid develop from the corresponding ones in the incompressible fluid simply by changing $\frac{\nabla q}{q}$ to $\frac{-\nabla \vartheta}{\vartheta}$.

Chapter II.

THE STABILITY OF A STATIONARY CIRCULAR VORTEX BY SYMMETRIC PERTURBATIONS

A.

The stationary circular vortex.

We shall here refer the motion to a coordinate system which rotates with the rotational velocity Ω of the earth and let x, y, z denote cartesian co-ordinates directed eastwards, horizontally northwards and vertically upwards respectively, whereas $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ shall be unit vectors directed in the positive directions of the axes of the co-ordinate system. $\boldsymbol{v}_3$, $\boldsymbol{v}$, ∇_3 and ∇ are defined by:

$$\boldsymbol{v}_3 = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}; \qquad \boldsymbol{v} = v_y \boldsymbol{j} + v_z \boldsymbol{k}$$

$$\nabla_3 = \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z}; \quad \nabla = \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z}.$$

In this relative motion the equation of motion assumes the form

$$(2{,}1) \quad q \frac{\partial \boldsymbol{v}_3}{\partial t} = -\nabla_3 p + q \boldsymbol{g} - q 2 \Omega \times \boldsymbol{v}_3 - q \boldsymbol{v}_3 \cdot \nabla_3 \boldsymbol{v}_3$$

$\boldsymbol{g}$ is the apparent force of gravitation determined by

$$\boldsymbol{g} = -\nabla_3 \varphi + \Omega^2 \boldsymbol{R}$$

where $\boldsymbol{R}$ is the vector distance from the axis of rotation of the earth to a fluid particle.

$\boldsymbol{g}$ obeys the condition

$$\nabla \times \boldsymbol{g} = 0.$$

We shall now consider the motion of the earth's atmosphere and hydrospere, identifying these, as a first approximation, with a stationary circular vortex. By definition, the fluid particles then have to move concentrically round the axis of the earth without changing their speed, and the pressure and the density must be symmetrically distributed round the same axis. Since evidently no heat transformations are possible in such a motion, it is necessary to suppose adiabatic processes in the compressible vortex. If this condition, however, is fulfilled, all other necessary conditions will be satisfied for this system of motion if the meridional distribution of the hydrodynamical variables satisfies the meridional component of the equation of motion. On account of the presupposed stationarity, this must now be written

$$0 = -\nabla p + q\boldsymbol{g} - q2\Omega \times v_x \boldsymbol{i} + \frac{v_x^2}{R^2}\boldsymbol{R}$$

Corresponding to the conditions in the hydrospere and atmospere of the earth, the last term in the above equation can be neglected as compared with the others. In order to distinguish the motion of the stationary circular vortex from the motion of perturbation which we shall next consider, we denote the variables in the former one with capital letters. The above equation can now be written with good approximation

$$(2,2) \qquad 0 = -\nabla P + Q\boldsymbol{g} - Q2\Omega \times V_x \boldsymbol{i},$$

It will prove usefull below to introduce the notation $\boldsymbol{N}$ for the vector defined by:

$$(2,3) \quad \boldsymbol{N} = \left(\frac{\partial V_x}{\partial y} - 2\Omega_z\right)\boldsymbol{j} + \left(\frac{\partial V_x}{\partial z} + 2\Omega_y\right)\boldsymbol{k}.$$

We take the curl, $\nabla\times$, of each term of equation (2,2). Then we obtain the relation

$$(2,4) \qquad \frac{1}{Q}\nabla Q \times S\nabla P + \boldsymbol{N} \times 2\Omega \boldsymbol{R}_I = 0.$$

$\boldsymbol{R}_I$ is the unit vector directed normal to, and outwards from, the axis of the earth. In the case of a compressible fluid this equation may also be written by introducing the potential temperature

$$(2,4') \qquad -\frac{1}{\Theta}\nabla \Theta \times S\nabla P + \boldsymbol{N} \times 2\Omega \boldsymbol{R}_I = 0.$$

The equations (2,4) will be referred to below as the conditions for stationarity in the circular vortex.

B.

The symmetrically perturbed stationary circular vortex.

1. *The perturbed hydrodynamical equations by symmetric perturbations of the stationary circular vortex.*

We are now going to consider a system of motion having a distribution of the variables which deviates from the distribution in the stationary circular vortex with quantities which are small of the first order. The deviations are supposed to be symmetrically distributed round the axis of rotation of the earth, i. e., they shall all be equal in zonal direction. Neglecting terms which are small of higher order than the first, the hydrodynamical equations can now be written:

The meridional equation of motion

$$(2,5) \quad Q\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + qS\nabla P - Q2\Omega \times V_x \boldsymbol{i}$$

the zonal equation of motion

$$(2,6) \quad \frac{\partial v_x}{\partial t} = -\left(\frac{\partial V_x}{\partial y} - 2\Omega_z\right)v_y - \left(\frac{\partial V_x}{\partial z} + 2\Omega_y\right)v_z = -\boldsymbol{v}\cdot\boldsymbol{N}$$

the equation of continuity in one of the following forms

$$(2,7) \quad \begin{aligned} \frac{\partial q}{\partial t} &= -\boldsymbol{v}\cdot\nabla Q - Q\nabla_3\cdot\boldsymbol{v} \\ \frac{\partial q}{\partial t} &= Q\frac{\nabla\Theta}{\Theta}\cdot v + \Gamma\frac{\partial p}{\partial t} \end{aligned}$$

and

the equation of pressure tendency

$$(2,8) \qquad \frac{\partial p}{\partial t} = -\boldsymbol{v}\cdot\nabla P - \frac{Q}{\Gamma}\nabla_3\cdot\boldsymbol{v}.$$

2. *The stability criteria.*

The stability criteria by symmetric perturbations of the most general baroclinic stationary circular vortex have earlier been found by *H. Solberg* (1936), *E. Høyland* (1938, 1941), *E. Kleinschmidt* (1941), *H. Ertel* (1941) and incompletely by the Russian investigator *Moltshanow* (1933). Of the mentioned investigators, Høiland is the one who has developed the stability criteria most satisfactorily, having based his examinations on the circulation theorem of Bjerknes. In the case of a compressible vortex, however, also this method is not satisfying (Høiland 1941 p. 16). We shall therefore develop the stability criteria by means of an energetic method of consideration. In connection with this, the circulation method also will prove useful when the equations for the acceleration of circulation and vorticity are used in the form developed in Chapter I (1,22).

In Part II it will be shown that a function φ^* exists depending only on the meridional positions of the fluid particles and being connected in the following way with the meridional kinetic energy $K = \int\limits_M \frac{1}{2} v^2 dM$ of a fluid mass M which is symmetrically distributed around the axis of rotation of the earth:

$$K = -\int\limits_M \varphi^* dM + c + W_p = -\Phi^* + c + W_p.$$

Here c is a constant with respect to time. W_p is the work received from the pressure forces at the boundary. Considering especially an isolated fluid having no contact with other fluid masses at its boundary, we obtain

$$(2{,}9) \qquad W_p = 0$$

according to which the above relation is reduced to

$$K = -\Phi^* + c.$$

In Part II it will further be shown that Φ^* satisfies the necessary condition for an extremum

$$(2{,}10) \qquad \delta\Phi^* = 0$$

and that accordingly the criterium for stability can be found if we know the conditions making

$$\Delta\Phi^* = \Phi^*(\boldsymbol{r} + \Delta\boldsymbol{r}) - \Phi^*(\boldsymbol{r}) > 0.$$

Here $\boldsymbol{r}$ denotes the positions of the particles in a stationary circular vortex and $\boldsymbol{r} + \Delta\boldsymbol{r}$ arbitrary positions in the neighbourhood. If the positions $\boldsymbol{r} + \Delta\boldsymbol{r}$ are reached in a real motion after a small time Δt, we obtain according to the above energy equation an increase ΔK in K which is equal to $-\Delta\Phi^*$. Othersides we have for ΔK:

$$\Delta K = \Delta t \left.\frac{dK}{dt}\right|_{t=0} + \frac{(\Delta t)^2}{2}\int\limits_M \left[\left(\frac{dv}{dt}\right)^2 + \boldsymbol{v}\cdot\frac{d^2 v}{dt^2}\right] dM\Bigg]_{t=0} + h(o)$$

$h(o)$ indicating terms which are small of a still higher order. On account of (2,10) this equation is simplified to

$$(2{,}11) \qquad \Delta K = \tfrac{1}{2}(\Delta t)^2 \int\limits_\tau Q\left[\frac{\partial^2 v}{\partial t^2}\cdot v\right]_{t=0} d\tau$$

having neglected terms which are small of a still higher order, and introduced $dM = Qd\tau$, τ being the volume of M.

The expression for $Q\dfrac{\partial^2 \boldsymbol{v}}{\partial t^2}$ is obtained by derivating the meridional equation of motion (2,5) locally with respect to time. This gives

$$Q\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\nabla\frac{\partial p}{\partial t} + \frac{\partial q}{\partial t} S\nabla P - Q\,2\,\boldsymbol{\Omega}\times\frac{\partial v_x}{\partial t}\boldsymbol{i}.$$

If we here for $\dfrac{\partial v_x}{\partial t}$ substitute the expression on the right-hand side of the zonal equation of motion (2,6) and for $\dfrac{\partial q}{\partial t}$ the right-hand side of the first equation (2,7), we obtain

$$Q\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\nabla\frac{\partial p}{\partial t} - Q\boldsymbol{v}\cdot\left[\frac{\nabla Q}{Q} S\nabla P + \boldsymbol{N}2\Omega\boldsymbol{R}_I\right] - Q\nabla_3\cdot\boldsymbol{v}S\nabla P.$$

Below, the tensor within the brackets above will be referred to as M_Q:

$$(2{,}12) \qquad M_Q = \frac{\nabla Q}{Q} S\nabla P + \boldsymbol{N}2\Omega\boldsymbol{R}_I.$$

The last equation can therefore be written

$$Q\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\nabla\frac{\partial p}{\partial t} - Q\boldsymbol{v}\cdot M_Q - Q\nabla_3\cdot\boldsymbol{v}S\nabla P.$$

Carrying out the scalar multiplication $\boldsymbol{v}\cdot$ on the terms of this equation we arrive at

$$(2{,}13) \qquad Q\frac{\partial^2 \boldsymbol{v}}{\partial t^2}\cdot\boldsymbol{v} = -\nabla\frac{\partial p}{\partial t}\cdot\boldsymbol{v} - Q\boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v} - Q\boldsymbol{v}\cdot S\nabla P\nabla_3\cdot\boldsymbol{v}.$$

We can write

$$-\nabla\frac{\partial p}{\partial t}\cdot\boldsymbol{v} = -\nabla_3\cdot\frac{\partial p}{\partial t}\boldsymbol{v} + \frac{\partial p}{\partial t}\nabla_3\cdot\boldsymbol{v}$$

Substituting in the last term by the equation of pressure tendency $\dfrac{\partial p}{\partial t} = -\boldsymbol{v}\cdot\nabla P - \dfrac{Q}{\Gamma}\nabla_3\cdot\boldsymbol{v}$, we obtain

$$-\nabla\frac{\partial p}{\partial t}\cdot\boldsymbol{v} = -\frac{Q}{\Gamma}(\nabla_3\cdot\boldsymbol{v})^2 - Q\boldsymbol{v}\cdot S\nabla P\nabla_3\cdot\boldsymbol{v} - \nabla_3\cdot\frac{\partial p}{\partial t}\boldsymbol{v}.$$

This expression for $-\nabla \frac{\partial p}{\partial t} \cdot \boldsymbol{v}$ substituted into (2,13) gives

$$Q \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \boldsymbol{v} =$$
$$-Q\left[\frac{1}{\Gamma}(\nabla_3 \cdot \boldsymbol{v})^2 + 2\boldsymbol{v} \cdot S \nabla P \nabla_3 \cdot \boldsymbol{v} + \boldsymbol{v} \cdot M_Q \cdot \boldsymbol{v}\right] - \nabla_3 \cdot \frac{\partial p}{\partial t} \boldsymbol{v}.$$

Taking now the volume integrals of each term in this equation over the volume τ, we arrive at

$$(2,14) \quad \int_M \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \boldsymbol{v} dM =$$
$$-\int_M \left[\frac{1}{\Gamma}(\nabla_3 \cdot \boldsymbol{v}) + 2\boldsymbol{v} \cdot S\nabla P \nabla_3 \cdot \boldsymbol{v} + \boldsymbol{v} \cdot M_Q \cdot \boldsymbol{v}\right] dM - \int_F \frac{\partial p}{\partial t} \boldsymbol{v} \cdot \boldsymbol{n} dF$$

having introduced $dM = Q d\tau$ in the first two integrals, and transformed the last integral by means of the theorem of Gauss, F denoting the surface bounding M and $\boldsymbol{n}$ the unit vectors normal to it.

Making use of this equation in eq. (2,11), we obtain

$$(2,15) \quad -\Delta K =$$
$$\frac{(\Delta t)^2}{2} \int_M \left[\frac{1}{\Gamma}(\nabla_3 \cdot \boldsymbol{v})^2 + 2\boldsymbol{v} \cdot S\nabla P \nabla_3 \cdot \boldsymbol{v} + \boldsymbol{v} \cdot M_Q \cdot \boldsymbol{v}\right] dM - \frac{(\Delta t)^2}{2} \int_F \left[\frac{\partial p}{\partial t} \boldsymbol{v} \cdot \boldsymbol{n}\right] \delta F$$

dropping the index $t = 0$.

This equation may be used to obtain the stability criteria when the boundary conditions are known. These must be in accordance with the condition (2,9). Here two cases shall be considered: Either closed walls, which of course according to our earlier assumptions must be symmetrically distributed around the axis of rotation of the earth, or a single wall above which in the incompressible vortex there is a free surface, in the compressible vortex the pressure decreases to zero in finite or infinite height. For the incompressible vortex we assume besides that the density decreases to zero simultaneously with the pressure. With the introduction of the cellular boundaries which actually do not exist in the earth's atmosphere or hydrosphere, we only intend to simplify the calculations in Chapter III. The second case of boundaries is approximately existing at the earth with the surface of the earth as the rigid wall.

In connection with the mentioned boundaries it will be of some interest to consider the corresponding different types of streamlines. Owing to the symmetric property of the perturbations we obtain

$$0 = \nabla_3 \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{v} - \frac{v_y}{R_e} \operatorname{tg} \varphi + \frac{v_z}{R_e}$$

where φ is the latitude, R_e the radius of the earth. If the meridional dimensions of the motion are not too large, the last equation may be written with sufficient accuracy:

$$(2,16) \qquad \nabla \cdot \boldsymbol{v} = 0.$$

Owing to the above relations the streamlines in the meridional motion for the incompressible vortex must be of the cellular closed type in the first case of boundaries, (fig. 1 a). In the compressible vortex the field of the meridional streamlines may be either of the cellular type, of the type of mainly expansion and contraction, (fig. 1 b), or a combination of both types. In the second case of boundaries, all the streamlines, or some of them, may end at the free surface of the incompressible vortex, or at the upper finite or infinite limit of the atmosphere, (fig. 1 c and 1 d).

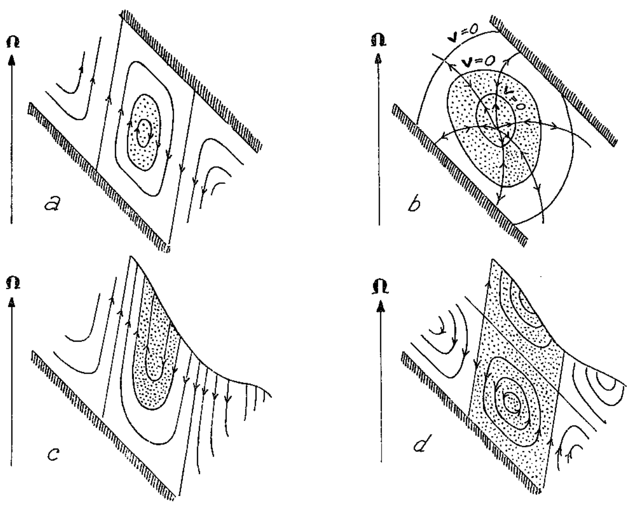

Fig. 1.
Illustration of types of streamlines and boundary conditions.

Having assumed the boundaries as mentioned above we see that the integrand in the last integral in (2,14) or (2,15) vanishes identically except perhaps in the case where M extends to the free surface of an incompressible vortex or to the upper limit of the atmosphere. The mentioned integral may therefore be written

$$-\int_f \frac{\partial p}{\partial t}\, \boldsymbol{v}\cdot\boldsymbol{n}\delta f$$

f denoting the part of the free surface or the upper limit of the atmosphere which bounds the considered mass M.

According to definition we have at the free surface $\frac{dp}{dt}=0$, giving

$$\frac{\partial p}{\partial t}=-\boldsymbol{v}\cdot\nabla P$$

which, owing to (2,2), also may be written

$$\frac{\partial p}{\partial t}=-\boldsymbol{v}\cdot Q[\boldsymbol{g}-2\boldsymbol{\Omega}\times V_x\boldsymbol{i}].$$

Since $\boldsymbol{n}$ is the unit vector normal to the free surface, directed outwards, we obtain

$$\frac{\partial p}{\partial t}\,\boldsymbol{v}\cdot\boldsymbol{n}=-v^2{}_n\frac{\partial P}{\partial n}$$

In the case of a compressible vortex, the relation (1,14) gives

$$\frac{dp}{dt}=\frac{c_p}{c_v}P\nabla_3\cdot\boldsymbol{v}$$

owing to which $\frac{dp}{dt}$ vanishes at the upper limit of the atmosphere, the case that here $\nabla_3\cdot v=\infty$ must be excluded. Consequently, we obtain also at the upper limit of the atmosphere

$$\frac{\partial p}{\partial t}=-v\cdot\nabla P$$

or, by a substitution for ∇P by means of eq. (2,2),

$$\frac{\partial p}{\partial t}=-Qv\cdot[\boldsymbol{g}-2\boldsymbol{\Omega}\times V_x\boldsymbol{i}]$$

which is simplified to

$$\frac{\partial p}{\partial t}=0$$

as Q should be equal to zero. The last integral in (2,14) or (2,15) will therefore vanish in all cases except in the case of a free moving surface in the incompressible vortex, where it is equal to

$$\int v^2{}_n\frac{\partial P}{\partial n}\,\delta f.$$

Accordingly (2,15) may be written:

The incompressible vortex:

$$\text{a}\qquad -\Delta K=\frac{(\Delta t)^2}{2}\left[\int_M \boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v}dM-\int_f v^2{}_n\frac{\partial P}{\partial n}\,\delta f\right]$$

(2,17) (moving free surface)

$$\text{b}\qquad -\Delta K=\frac{(\Delta t)^2}{2}\int_M\left[\boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v}\right]dM$$

(cellular motion)

The compressible vortex:

$$\text{c}\qquad -\Delta K=$$

$$\frac{(\Delta t)^2}{2}\int_M\left[\frac{1}{I}(\nabla_3\cdot\boldsymbol{v})^2+2\boldsymbol{v}\cdot S\nabla P\nabla_3\cdot\boldsymbol{v}+\boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v}\right]dM$$

The stability criteria will now be given by:

1°. If $\Delta K<0$ for any kinematics, the initial equilibrium is stable.

2°. If $\Delta K>0$ for some kinematics, the initial equilibrium is unstable.

If we wish to examine the degree of stability or instability released by a certain system of symmetric perturbations of the stationary circular vortex, we cannot use eq. (2,17) in its original form. The only rational measure for the stability, if a negative stability is identified with instability, is the relative decrease $\frac{-\Delta K}{K}$ of the kinetic energy of the meridional motion. For $\frac{-\Delta K}{K}$ we obtain the expressions:

The incompressible vortex:

$$\text{a}\qquad \frac{-\Delta K}{K}=\frac{(\Delta t)^2\left[\int_M \boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v}dM+\int_f v^2{}_n\frac{\partial P}{\partial n}\,\delta f\right]}{\int_M v^2\cdot dM}$$

(2,18) (moving free surface)

$$\text{b}\qquad \frac{-\Delta K}{K}=\frac{(\Delta t)^2\int_M[\boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v}]\,dM}{\int_M \boldsymbol{v}^2dM}$$

(cellular motion)

The compressible vortex:

$$\text{c}\qquad \frac{-\Delta K}{K}=$$

$$=\frac{\int_M\left[\frac{1}{I}(\nabla_3\cdot\boldsymbol{v})^2+2\boldsymbol{v}\cdot S\nabla P\nabla_3\cdot\boldsymbol{v}+\boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v}\right]dM}{\int_M v^2dM}$$

Supposing the solution for the velocity to have the form

$$v = v_0 \cos \nu t$$

we can introduce

$$\frac{\partial^2 v}{\partial t^2} = -\nu^2 v$$

in eq. (2,14). Doing so, we obtain the frequency formulae:

The incompressible vortex:

$$\text{a}' \quad \nu^2 = \frac{\int_M v \cdot M_Q v dM - \int_f v^2{}_n \frac{\partial P}{\partial n} \delta f}{\int_M v^2 dM}$$

(2,18') (moving free surface)

$$\text{b}' \quad \nu^2 = \frac{\int_M v \cdot M_Q \cdot v dM}{\int_M v^2 dM}$$

(cellular motion).

The compressible vortex:

$$\text{c}' \quad \nu^2 = \frac{\int_M \left[\frac{1}{\Gamma}(\nabla_3 \cdot v)^2 + 2v \cdot S\nabla P \nabla_3 \cdot v + v \cdot M_Q \cdot v\right] dM}{\int_M v^2 dM}$$

The justification of chosing $\frac{-\Delta K}{K}$ as a measure for the stability is thus evident, noting that ν^2 is a direct measure for the stabilizing forces.

Discussing now the stability conditions, we consider first the *incompressible circular vortex.*

Referring the tensor M_Q to its principal axes j_Q and k_Q, we may write

$$M_Q = m^2{}_Q j_Q j_Q + n^2{}_Q k_Q k_Q.$$

If further v_η and v_ζ are the components of v along j_Q and k_Q, respectively, (2,17) a, and (2,17) b may be written

$$\text{a}' \quad \Delta K = -\frac{(\Delta t)^2}{2}\left[\int_M (m^2{}_Q v^2{}_\eta + n^2{}_Q v^2{}_\zeta)\, dM - \int_f Q v^2{}_n \frac{\partial P}{\partial n}\, \delta f\right]$$

(2,17')

$$\text{b}' \quad \Delta K = -\frac{(\Delta t)^2}{2}\int_M [m_Q{}^2 v_\eta{}^2 + n_Q{}^2 v_\zeta{}^2]\, dM$$

M_Q was given by (2,12)

$$M_Q = \frac{1}{Q} \nabla Q S \nabla P + N 2\Omega R_I$$

by means of which we can show tat $m^2{}_Q$ and $n^2{}_Q$ are determined by

$$\text{a} \quad m^2{}_Q + n^2{}_Q = \frac{\nabla Q}{Q} \cdot S\nabla P + N \cdot 2\Omega R_I$$

(2,19)

$$\text{b} \quad m_Q{}^2 \cdot n_Q{}^2 = S\nabla P \times 2\Omega R_I \cdot \frac{1}{Q} \nabla Q \times N$$

By the vector multiplication of the terms in eq. (2,2) by $S\nabla P$ we find

$$S\nabla P \times 2\Omega R_I = g 2\Omega_z i$$

according to which eq. b above may also be written

$$(2,19') \quad \text{b}' \quad m^2{}_Q \cdot n^2{}_Q = \frac{g 2\Omega_z}{Q} \nabla Q \times N \cdot i.$$

The vectors appearing in (2,19) must in addition satisfy the condition of stationarity of the circular vortex (2,4)

$$\frac{1}{Q} \nabla Q \times S\nabla P + N \times 2\Omega R_I = 0.$$

Basing the discussion on eq. (2,19) and (2,4) we may arrive at the final form of the stability criteria expressed by magnitudes determined by the state of the stationary circular vortex. As to this discussion we refer to Høiland (1941). At this place it will be of interest only to determine the signs of $m^2{}_Q$ and $n^2{}_Q$ for the atmosphere when this is as a first approximation identified with a stationary circular vortex. It can then be shown that both signs are positive, mainly as a consequence of the relatively rapid decrease of the density with altitude. In the atmosphere we will therefore always have

$$v \cdot M_Q \cdot v > 0.$$

Now we proceed to derive the stability criteria for the *compressible stationary circular vortex.* We denote the integrand in (2,15) with I. Accordingly we may write

$$(2,20) \quad -\Delta K = \frac{(\Delta t)^2}{2}\int_M \left[\frac{1}{\Gamma}(\nabla_3 \cdot v)^2 + 2v \cdot S\nabla P \nabla_3 \cdot v + v \cdot M^Q \cdot v\right] dM = \tfrac{1}{2}(\Delta t)^2 \int_M I dM.$$

According to this equation, an increase of the kinetic energy of the meridional motion, when this is initially in a state of equilibrium, can be obtained by symmetric perturbations only if I is negative. Before using this to obtain precise stability criteria, we shall draw an interesting conclusion from (2,20) when this is applied to the atmosphere. It was emphasized above that the last term of I always is positive in the atmosphere. Since now the first term of I is always positive too, having a positive factor before the quadratic function of $\nabla_3 \cdot \boldsymbol{v}$, an increase of the energy will be the result, only if the second term of I is negative. This can only then be the case if the air is expanding when it moves towards lower pressure, and is contracting when it moves towards higher pressure. Quite generally it can be shown that the most unstable motions in the atmosphere must be thus characterized, in full accordance with what is normally observed in the atmosphere.

On account of its capability to expand and contract, the air may therefore be less stable than if it should move like an incompressible fluid. The second term of I in (2,20), representing the destabilizing influenze of the compressibility, is a linear function of $\nabla_3 \cdot \boldsymbol{v}$. The first term of I, representing the stabilizing influence of the compressibility, is a quadratic function of $\nabla_3 \cdot \boldsymbol{v}$, and thus limits the destabilizing influenze of the compressibility. In order to arrive at precise stability criteria, we must therefore determine the minimum, $I_{\min}$, of the integrand I. Since I is a quadratic function of $\nabla_3 \cdot \boldsymbol{v}$, $I_{\min}$ is given by

$$I_{\min} = [\boldsymbol{v} \cdot M_Q \cdot \boldsymbol{v} - \Gamma(\boldsymbol{v} \cdot S\nabla P)^2]$$

when

$$\nabla \cdot \boldsymbol{v} = -\frac{\Gamma}{Q} \boldsymbol{v} \cdot S\nabla P$$

i. e. according to (2,8), when

$$\frac{\partial p}{\partial t} = 0.$$

Writing in the expression for $I_{\min}$

$$\Gamma(\boldsymbol{v} \cdot S\nabla P)^2 = \boldsymbol{v} \cdot \frac{\Gamma}{Q} \nabla P S \nabla P \cdot \boldsymbol{v}$$

and, according to (2,12),

$$\boldsymbol{v} \cdot M_Q \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \frac{1}{Q} \nabla Q S \nabla P \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{N} 2\Omega \boldsymbol{R}_I \cdot v$$

we obtain

$$I_{\min} = \boldsymbol{v} \cdot \left[\frac{\nabla Q - \Gamma \nabla P}{Q} S \nabla P + \boldsymbol{N} 2\Omega \boldsymbol{R}_I\right] \cdot \boldsymbol{v}.$$

Owing to the relation (1,17) we have

$$\frac{\nabla Q - \Gamma\nabla P}{Q} = -\frac{\nabla \Theta}{\Theta}.$$

Consequently, the *maximum* of increase of the energy K, will be given by

(2,21)

a $\Delta K_{\text{maks}} =$
$$-\tfrac{1}{2}(\Delta t)^2 \int_M \left[v \cdot \left(\frac{-\nabla\Theta}{\Theta} S\nabla P + \boldsymbol{N} 2\Omega \boldsymbol{R}_I\right) \boldsymbol{v} \cdot \right] dM$$
when

b $$\nabla_3 \cdot \boldsymbol{v} = -\frac{\Gamma}{Q} \boldsymbol{v} \cdot \nabla P.$$

We denote with M_Θ the tensor within the brackets above,

(2,24) $$M_\Theta = -\frac{\nabla\Theta}{\Theta} S\nabla P + \boldsymbol{N} 2\Omega \boldsymbol{R}_I.$$

If M_Θ is referred to its principal axes $\boldsymbol{j}_\Theta$ and $\boldsymbol{k}_\Theta$, we can write

$$M_\Theta = m^2{}_\Theta \boldsymbol{j}_\Theta \boldsymbol{j}_\Theta + n^2{}_\Theta \boldsymbol{k}_\Theta \boldsymbol{k}_\Theta$$

and

(2,21′) $$\Delta K_{\text{maks}} = -\tfrac{1}{2}(\Delta t)^2 \int_M (m^2{}_\Theta v^2{}_\eta + n^2{}_\Theta v^2{}_\zeta] \, dM$$

having also introduced the components v_η and v_ζ of $\boldsymbol{v}$ along $\boldsymbol{j}_\Theta$ and $\boldsymbol{k}_\Theta$, respectively. According to the expressions (2,12) and (2,22), $m^2{}_\Theta$ and $n^2{}_\Theta$ must now be determined by a system of equations which are quite analogous to the corresponding ones (2,19) in the incompressible case. In the latter we have only to change $\frac{\nabla Q}{Q}$ to $\frac{-\nabla\Theta}{\Theta}$. In this way we obtain

(2,23)

a $$m^2{}_\Theta + n^2{}_\Theta = -\frac{\nabla\Theta}{\Theta} \cdot S\nabla P + \boldsymbol{N} \cdot 2\Omega \boldsymbol{R}_I$$

b $$m^2{}_\Theta \cdot n^2{}_\Theta = -\frac{g 2\Omega_z}{\Theta} \nabla\Theta \times \boldsymbol{N} \cdot \boldsymbol{i}.$$

Besides we have the condition for stationarity of the circular vortex (2,4′)

$$-\frac{1}{\Theta} \nabla\Theta \times S\nabla P + \boldsymbol{N} \times 2\Omega \boldsymbol{R}_I = 0.$$

Discussing the stability conditions, three essentially different cases are to be regarded:

A: $m^2{}_\Theta$ and $n^2{}_\Theta$ are both positive. Total stability.

In this case it appears from eq. (2,21′) that the kinetic energy of the meridional motion will decrease whatever are the characteristics of the kinematic of the symmetrically distributed meridional velocities. In this case, therefore, the stationary circular vortex is totally stable for symmetric perturbations,

B: m^2_θ and n^2_θ are both negative. Instability with o n e *kinematical condition. «Total instability».*

In this case it appears from (2,21′) that the kinetic energy of the meridional motion will increase provided that the relation b of (2,21) is fulfilled. As it will always be possible to arrange the initial meridional motion in such a way that this condition is satisfied, the stationary circular vortex will in this case be unstable.

C: m^2_θ and n^2_θ have opposite signs. Instability with t w o *kinematical conditions. Conditional instability.*

Suppose in this case m^2_θ to be negative. Then it appears from (2,21′) that the kinetic energy of the meridional motion will increase, provided that the initial meridional velocities are mainly directed along the η-axis, and the relation b of (2,21) simultaneously is satisfied. Since in the compressible case the two components of $\boldsymbol{v}$ are independent of each other, it will always be possible to get both these kinematical conditions fulfilled, and consequently the stationary circular vortex also in this case is unstable. This case, having a kinematical condition not contained in the case B, will be referred to below as the case of conditional instability, whereas B will be referred to as the case of «total instability».

The condition b of (2,20) is not a necessary one for obtaining instability in the cases B and C, since a contingent increase of the kinetic energy under the assumed conditions, of course also indicates the instability when it is not the maximum possible one. We shall now derive an expression for the increase of the kinetic energy under conditions where the relation b of (2,21) is not necessarily satisfied. In the meridional equation of motion derivated locally with respect to time

$$Q\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\nabla\frac{\partial p}{\partial t} + \frac{\partial q}{\partial t}S\nabla P - Q2\Omega\times\frac{\partial v_x}{\partial t}\boldsymbol{i}$$

we now introduce for $\frac{\partial q}{\partial t}$ the right-hand side of the second equation (2,7) whereas for $\frac{\partial v_x}{\partial t}$ we make the same substitution as on p. 11. Then we arrive at

$$(2,24)\qquad Q\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\nabla\frac{\partial p}{\partial t} - \Gamma\frac{\partial p}{\partial t}S\nabla P - Q\boldsymbol{v}\cdot M_\theta.$$

In the same way as on pp. 11—12 we obtain

$$\int\limits_M \frac{\partial^2 \boldsymbol{v}}{\partial t^2}\cdot\boldsymbol{v}dM = -\int\limits_M \boldsymbol{v}\cdot M_\theta\cdot\boldsymbol{v}dM - \int\limits_M \Gamma S^2\left(\frac{\partial p}{\partial t}\right)^2 dM.$$

The corresponding expression for ΔK is

$$(2,25)\qquad \Delta K =$$

$$-\tfrac{1}{2}(\Delta t)^2\int\limits_M[\boldsymbol{v}\cdot M_\theta\cdot\boldsymbol{v}]\,dM - \tfrac{1}{2}(\Delta t)^2\int\limits_M \Gamma S^2\left(\frac{\partial p}{\partial t}\right)^2 dM$$

It appears from this equation that in the cases B and C, instability will then and only then be the case if the pressure tendencies determined by (2,8)

$$\frac{\partial p}{\partial t} = -\boldsymbol{v}\cdot\nabla P - \frac{Q}{\Gamma}\nabla_3\cdot\boldsymbol{v}$$

have numerical values not so large as to compensate the destabilizing effect of the first term on the right-hand side of the above eq. (2,25).

Applying the relations (2,23) and (2,4′), the stability criteria may be expressed in terms characterized by the state of the stationary circular vortex. As to this we refer again to Høiland (1941). The stability criteria under atmospheric conditions have been derived by E. Kleinschmidt (1941) and *R. Fjørtoft* (1942) and will be subject to a closer examination in the second part of this work. Here we shall only write down the criterium of conditional instability of case C. In this case we should have opposite signs for m^2_θ and n^2_θ, giving the condition

$$m^2_\theta\cdot n^2_\theta < 0.$$

According to (2,22) we have

$$m^2_\theta\cdot n^2_\theta = -\frac{g2\Omega_z}{\Theta}\nabla\Theta\times\boldsymbol{N}\cdot\boldsymbol{i}.$$

Consequently, the stationary circular vortex will be conditionally unstable if

$$(2,26)\qquad \nabla\Theta\times\boldsymbol{N}\cdot\boldsymbol{i} > 0.$$

3. *Stationarity and indifference.*

If the stationary circular vortex is totally indifferent against symmetric perturbations we must particularly have

$$\frac{\partial\vartheta}{\partial t} = 0,\quad \frac{\partial v_x}{\partial t} = 0.$$

Neglecting terms which are small of a higher order than the first, eq. (1,16) may, owing to the symmetry of the perturbations, be written

(2,27)
$$\left\{\begin{array}{l} \frac{\partial \vartheta}{\partial t} = -\boldsymbol{v}\cdot\nabla\Theta \\ \text{Besides we have the zonal eq. of motion} \\ \frac{\partial v_\lambda}{\partial t} = -\boldsymbol{v}\cdot\boldsymbol{N} \end{array}\right.$$

In the stationary meridional motion this system is reduced to

$$(2,27') \qquad \left\{\begin{array}{l} 0 = -\boldsymbol{v}\cdot\nabla\Theta \\ 0 = -\boldsymbol{v}\cdot\boldsymbol{N} \end{array}\right.$$

By a symmetric perturbation of the stationary circular vortex, $\boldsymbol{v}$ cannot identically vanish. Consequently (2,27′) can exist only if $\nabla\Theta$ and $\boldsymbol{N}$ are parallell, i.e. when

$$(2,28) \qquad \nabla\Theta\times\boldsymbol{N}\cdot\boldsymbol{i} = 0$$

including the special case where $\nabla\Theta$, $\boldsymbol{N}$ or both are vanishing. Comparing this result with the preceding section, it appears that we in the above simple way have determined the important intermediate state of the stationary circular vortex which separates the states of

total stability and 'total instability',

total stability and conditional instability,

and

'total instability' and conditional instability.

Chapter III.

ON THE KINEMATICS OF SIMPLE MOTIONS RESULTING FROM SYMMETRIC PERTURBATIONS OF THE STATIONARY CIRCULAR VORTEX

1. The perturbed hydrodynamical equations in question.

In Chapter II we have discussed the stability properties of the stationary circular vortex by symmetric perturbations without much regard to the nature of the corresponding meridional motions. Now we proceed to find simple solutions of the hydrodynamical equations in question, which confirm the stability criteria arrived at in the preceding chapter and which can also be used in the study of the kinematics of the motion.

In the *incompressible* case, $\boldsymbol{v}$ may, owing to relation (2,16), be expressed by a stream function ψ as follows

$$(3,1) \qquad \boldsymbol{v} = \frac{\partial\psi}{\partial z}\boldsymbol{j} - \frac{\partial\psi}{\partial y}\boldsymbol{k}.$$

Eq. (2,12) is reduced to

$$(3,2) \qquad Q\frac{\partial^2\boldsymbol{v}}{\partial t^2} = -\nabla\frac{\partial p}{\partial t} - Q\boldsymbol{v}\cdot M_Q.$$

Performing here on each term the vector multiplication $\nabla\times$, we obtain

$$(3,3) \qquad \nabla\times\frac{\partial^2\boldsymbol{v}}{\partial t^2} + \nabla\times\boldsymbol{v}\cdot M_Q + \frac{\nabla Q}{Q}\times\left[\frac{\partial^2\boldsymbol{v}}{\partial t^2} + \boldsymbol{v}\cdot M_Q\right] = 0$$

which is the equation for the acceleration of vorticity (1,9′) modified to the relative and perturbed meridional motion, now considered. If we substitute into this equation

$$M_Q = m^2{}_Q\boldsymbol{j}_Q\boldsymbol{j}_Q + n^2{}_Q\boldsymbol{k}_Q\boldsymbol{k}_Q$$

and
$$\boldsymbol{v} = \frac{\partial\psi}{\partial\zeta}\boldsymbol{j}_Q - \frac{\partial\psi}{\partial\eta}\boldsymbol{k}_Q.$$

η and ζ being the cartesian co-ordinates along $\boldsymbol{j}_Q$ and $\boldsymbol{k}_Q$, respectively, we obtain a partial differential equation of order four

$$(3,4) \qquad \nabla^2\frac{\partial^2\psi}{\partial t^2} + n^2{}_Q\frac{\partial^2\psi}{\partial\eta^2} + m^2{}_Q\frac{\partial^2\psi}{\partial\zeta^2} + \frac{\frac{\partial Q}{\partial\eta}}{Q}\left(\frac{\partial^3\psi}{\partial\eta\partial t^2} + n^2{}_Q\frac{\partial\psi}{\partial\eta}\right) + \frac{\frac{\partial Q}{\partial\zeta}}{Q}\left(\frac{\partial^3\psi}{\partial\zeta\partial t^2} + m^2{}_Q\frac{\partial\psi}{\partial\zeta}\right) = 0$$

which determines ψ as a function of η, ζ and t when the initial and boundary conditions are known. (It must be remembered that $m^2{}_Q$ and $n^2{}_Q$ are supposed to be spatially constant functions). Afterwards, when $\boldsymbol{v}$ is determined $\frac{\partial p}{\partial t}$ can be found by means of eq. (3,2) above.

In the *compressible* case, the meridional velocities may also in general have divergences $\nabla_3\cdot\boldsymbol{v}$ different from zero. The most general meridional velocity can be written as a sum of two velocities

$$(3,5) \qquad \boldsymbol{v} = \boldsymbol{v}_C + \boldsymbol{v}_D$$

where $\boldsymbol{v}_C$ is a solenoidal vector satisfying the condition

$$\nabla_3\cdot\boldsymbol{v}_C = 0$$

or, by sufficiently small dimensions of the motion

$$\nabla\cdot\boldsymbol{v} = 0$$

and $\boldsymbol{v}_D$ is an ascendental vector satisfying the condition

$$\nabla\times\boldsymbol{v}_D = 0.$$

According to these two conditions we may write

$$v_C = \frac{\partial \psi}{\partial z} j - \frac{\partial \psi}{\partial y} k$$

$$v_D = -\nabla a$$

and

$$v = \frac{\partial \psi}{\partial z} j - \frac{\partial \psi}{\partial y} k - \nabla a.$$

Since v_C in the general case will possess vorticities different from zero, we shall also call this vector the circulatoric part of the meridional velocity.

Evidently, we need in the general compressible case two equations in order to determine the two components of v, or the two functions ψ and a defined above. The first equation is the equation for the acceleration of vorticity which, according to the remark at the end of Chapter I, is simply obtained by changing $\frac{\nabla Q}{Q}$ in eq. (3,3) to $-\frac{\nabla \Theta}{\Theta}$. Then particulary M_Q given by (2,12), transforms to M_0 given by (2,22). Hence we obtain

$$(3,6)\quad \nabla \times \frac{\partial^2 v}{\partial t^2} + \nabla \times v \cdot M_0 - \\ - \frac{\nabla \Theta}{\Theta} \times \left[\frac{\partial^2 v}{\partial t^2} + v \cdot M_0\right] = 0.$$

The other equation is arrived at if we for instance in one of the component equations of (2,24) introduce for $\frac{\partial p}{\partial t}$ the expression in the pressure tendency equation (2,8). Doing so for the y-component we obtain

$$(3,7)\quad \frac{\partial}{\partial y}\left(v \cdot \nabla P + \frac{Q}{\Gamma} \nabla_3 \cdot v\right) - \Gamma \left(v \cdot \nabla P + \\ + \frac{Q}{\Gamma} \nabla \cdot v\right) S \frac{\partial P}{\partial y} - Q \frac{\partial^2 v_y}{\partial t^2} - Q v \cdot M_0 \cdot j = 0.$$

It is easily seen that it will be much more difficult to find solutions in the general compressible case than in the incompressible one. Owing to the compressibility, namely, we will now meet with several kinds of motion which are excluded in the incompressible case. In order to obtain a survey of the different possibilities, equation (2,18) c will prove useful. By the introduction of v_D into this equation we may write it

$$(3,8)\quad \frac{-\Delta K}{K} = \\ = \frac{(\Delta t)^2 \int_M \left[\left(\frac{1}{\Gamma}(\nabla_3 \cdot v_D)^2 + 2v \cdot S\nabla P \nabla_3 \cdot v_D + + v \cdot M_Q \cdot v\right) : v^2\right] v^2\, dM}{\int_M v^2\, dM}$$

According to this formula we obtain as measure of the stability

$$(3,8)\quad \frac{-\Delta K}{K} = \\ = (\Delta t)^2 \left[\frac{\frac{1}{\Gamma}(\nabla_3 \cdot v_D)^2 + 2v \cdot S\nabla P \nabla_3 \cdot v_D + v \cdot M_Q \cdot v}{v^2}\right]$$

having put outside the integral sign an average value of the expression within the brackets. We now let the last expression determine the stability for an arbitrary system of perturbations and consider in comparison with that a perturbed motion with velocities v' in which all distances are multiplied by the factor L, and where $v'_D = av_D$, $v' = bv$. With these assumptions we intend in the most simple way to study the influence on the stability from a change in the dimensions of the perturbed motion and in the ratio $\frac{v_D}{v}$ which is now multiplied by the factor $\frac{a}{b}$. Denoting this with r, we obtain as measure of the stability in the new motion

$$(3,8')\quad \frac{-\Delta K}{K} = \\ = (\Delta t)^2 \left[\frac{\frac{1}{\Gamma}(\nabla_3 \cdot v_D)^2 \left(\frac{r}{L}\right)^2 + 2v \cdot S\nabla P \nabla_3 \cdot v_D \left(\frac{r}{L}\right) + v \cdot M_Q \cdot v}{v^2}\right]$$

From this formula it is seen that if the dimensions of the motion are steadily decreased and r does not simultaneously tend to zero, we must by sufficiently small dimensions, obtain

$$\frac{-\Delta K}{K} = (\Delta t)^2 \frac{1}{\Gamma} \frac{\overline{\nabla_3 \cdot v_D}}{v^2} \left(\frac{r}{L}\right)^2$$

according to which stable conditions are now existing. The corresponding phenomena of motion are those of sound. On the other hand, if the dimensions are steadily increased and r does not simultaneously increase infinitely, we obtain by sufficiently large dimensions

$$\frac{-\Delta K}{K} = \frac{\boldsymbol{v}\cdot M_Q \cdot \boldsymbol{v}}{\boldsymbol{v}^2}$$

According to what was said in Chapter II. $\boldsymbol{v}\cdot M_Q\cdot\boldsymbol{v}$ is positive in the atmosphere, and a less stable motion will generally be obtained if the fluid makes use of its capability to expand and contract. Neither of the two motions considered above, will accordingly be the less stable one. However, it is seen from (3,8′) that the dominance of the stabilizing influence from the first term by very small dimensions of the motion, can be fully compensated if r is sufficiently small. Noting the meaning of the symbol r, we can formulate the above results as follows:

1°. The less stable or most unstable motions, or more generally, motions which are not similar to that of the sound, will if the dimensions of the meridional motion are sufficiently small, be kinematically characterized by the circulatoric part $\boldsymbol{v}_C$ of the meridional velocity, $\boldsymbol{v}_D$ being small compared with v.

In connection with 1° above it is of interest to notice that the large-scaled motions in the atmosphere obey the condition of incompressibility

$$\nabla_3 \cdot \boldsymbol{v}_3 = 0$$

with a relative error of about 10 %. In the atmosphere there will, owing to the variability of the stability conditions, be a rather well-defined limited room for the unstable motions. Consequently, we can probably explain by arguments similar to those above, why the motions in the atmosphere generally approximately satisfy the condition of incompressibility. This will be subject to a more thorough examination in the second part of this work.

Below, we shall exclude motions which are essentially sound motions. Further, we shall assume the dimensions of the meridional motion to be so small that, according to 1° above, $\boldsymbol{v}_D$ can be supposed to be small compared with $\boldsymbol{v}_C$. The equation of vorticity formation (3,6) will then be approximately satisfied for $\boldsymbol{v} = \boldsymbol{v}_C$, and may consequently, if we also introduce

$$M_O = m_O^2 \boldsymbol{j}_O \boldsymbol{j}_O + n^2_O \boldsymbol{k}_O \boldsymbol{k}_O\,;\; \boldsymbol{v}_C = \frac{\partial\psi}{\partial\zeta}\boldsymbol{j}_O - \frac{\partial\psi}{\partial\eta}\boldsymbol{k}_O$$

be written

$$(3,9)\quad \nabla^2\frac{\partial^2\psi}{\partial t^2} + n^2_O\frac{\partial^2\psi}{\partial\eta^2} + m^2_O\frac{\partial^2\psi}{\partial\zeta^2} - \frac{\frac{\partial\Theta}{\partial\eta}}{\Theta}\left(\frac{\partial_2\psi}{\partial\eta\partial t^2} + n^2_O\frac{\partial\psi}{\partial\eta}\right) - \frac{\frac{\partial\Theta}{\partial\xi}}{\Theta}\left(\frac{\partial^2\psi}{\partial\zeta\partial t^2} + m^2_O\frac{\partial\psi}{\partial\zeta}\right) = 0.$$

When solutions of this equation are found which satisfy the initial and boundary conditions, $\frac{\partial p}{\partial t}$ can afterwards be found from eq. (2,24) which in the case under consideration can be approximately written

$$(3,10)\quad \nabla\frac{\partial p}{\partial t} - \Gamma\frac{\partial p}{\partial t}S\nabla P = -Q\boldsymbol{v}_C\cdot M_O - Q\frac{\partial^2\boldsymbol{v}_C}{\partial t^2}.$$

Then we can find the *dynamical* important divergences $\nabla\cdot\boldsymbol{v}_D$ from the equation of pressure tendency which under the present conditions can be approximately written

$$(3,11)\nabla\cdot\boldsymbol{v}_D = -\nabla^2\alpha = -\frac{\Gamma}{Q}\frac{\partial p}{\partial t} - \frac{\Gamma}{Q}\boldsymbol{v}_C\cdot\nabla P.$$

By means of this equation, which can be solved with respect to the potential α, $\boldsymbol{v}_D$ may be determined. From the analytical expressions for $\boldsymbol{v}_C$ and $\boldsymbol{v}_D$ it will then be possible to establish more precise kinematic conditions for the above approximate method of solution, which was based on the assumption that $\boldsymbol{v}_D$ should be small compared with $\boldsymbol{v}_C$. This will be done for the atmosphere in the second part of this work.

If we compare eq. (3,4) with (3,9), the equations are seen to be quite analogous. The coefficients in the compressible case are simply obtained from those in the incompressible one by changing $\frac{\nabla Q}{Q}$ to $\frac{-\nabla\Theta}{\Theta}$. These equations are therefore both solved when solutions of the differential equation

$$\nabla^2\frac{\partial^2\psi}{\partial t^2} + n^2\frac{\partial^2\psi}{\partial\eta^2} + m^2\frac{\partial^2\psi}{\partial\zeta^2} +$$

$$(3,12) + b\left(\frac{\partial^2\psi}{\partial\eta\partial t^2} + n^2\frac{\partial\psi}{\partial\eta}\right) + C\left(\frac{\partial^3\psi}{\partial\zeta\partial t^2} + m^2\frac{\partial\psi}{\partial\zeta}\right) = 0$$

are found. In order to obtain the solutions in the incompressible and compressible cases, we have in the solutions of (3,12) to substitute for m^2, n^2, b and c, the expressions they are assuming in these cases, respectively. It must be noted that the choice of co-ordinates is not the same in the two cases, these having directions along the principal axes of M_Q in the former and along those of the tensor M_θ in the latter case.

2. Solutions of eq. (3,12).

We are now going to solve eq. (3,12) by the method of separation of the variables. We assume a solution of the form

$$\psi = \psi_0 \cos \nu t + \frac{1}{\nu^*} \psi_0^{(1)} \sin \nu^* t \tag{3,13}$$

where ψ_0, $\psi_0^{(1)}$, ν and ν^* are independent of time. The corresponding initial conditions are

$$\psi_{t=0} = \psi_0, \quad \frac{\partial \psi}{\partial t_{t=0}} = \psi_0^{(1)}.$$

By the introduction of the above expression for ψ into (3.12) it will be seen that this equation will be satisfied if and only if

$$\text{(3,14)}\quad \begin{aligned} &\text{a} \quad (m^2 - \nu^2)\frac{\partial^2 \psi_0}{\partial \eta^2} + (m^2 - \nu^2)\frac{\partial^2 \psi_0}{\partial \zeta^2} + \\ &\qquad + b(n^2 - \nu^2)\frac{\partial \psi_0}{\partial \eta} + c(m^2 - \nu^2)\frac{\partial \psi_0}{\partial \zeta} = 0 \\ &\text{b} \quad (n^2 - \nu^{*2})\frac{\partial^2 \psi_0^{(1)}}{\partial \eta^2} + (m^2 - \nu^{*2})\frac{\partial^2 \psi_0^{(1)}}{\partial \zeta^2} + \\ &\qquad + b(n^2 - \nu^{*2})\frac{\partial^2 \psi_0^{(1)}}{\partial \eta} + c(m^2 - \nu^{*2})\frac{\partial \psi_0^{(1)}}{\partial \zeta} = 0. \end{aligned}$$

It is seen that the functions ψ_0 and $\psi_0^{(1)}$ have to satisfy analogous differential equations and boundary conditions. It will be sufficient for our purpose to find one of them, for instance ψ_0, corresponding to the initial conditions $\psi_{t=0} = \psi_0$ and $\frac{\partial \psi}{\partial t_{t=0}} = 0$. The solution (3,13) assumes now the form

$$\psi = \psi_0 \cos \nu t. \tag{3,15}$$

As boundary for the meridional motion we imagine a zonal tube having a parallelogram as projection in the meridional planes. The position of this parallelogram is, as indicated in fig. 2, determined by the angles α_2 and α_3 which two neighbour sides of the parallelogram form with the η — and ζ axis, respectively, and by the pieces of lines, H_2 and H_3, which the same sides cut off from the respective axes. The positive directions for α_2 and α_3 are as indicated in fig. 2. It is now immediately seen that the boundary condition $\psi = \text{const}$ at the sides of

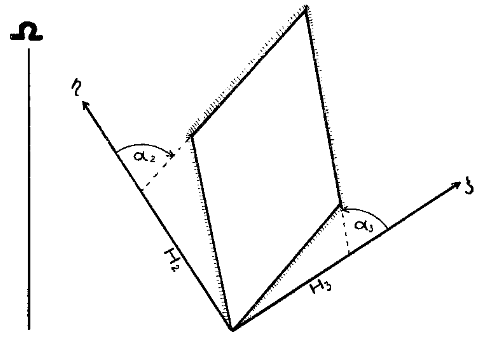

Fig. 2. Illustration of a boundary having a parallelogram as projection in the meridional planes.

the parallelogram, is satisfied if we assume for ψ_0 the expression determined by

$$\text{(3,16)}\quad \psi_0 = \varepsilon e^{\varkappa_2 \eta + \varkappa_3 \zeta} \sin \frac{k\pi(\eta - \operatorname{cotg} \alpha_2 \zeta)}{H_2} \cdot \sin \frac{l\pi(\zeta - \operatorname{cotg} \alpha_3 \eta)}{H_3}$$

k and l denoting arbitrary integers, and ε a constant small of the first order. By the introduction of this expression for ψ_0 into eq. (3,14) we shall find that this will be satisfied then and only then if

$$\text{(3,17)}\quad \begin{aligned} &\text{a} \begin{cases} (n^2 - \nu^2)\left[\varkappa_2^2 - \pi^2\left(\dfrac{k^2}{H^2_2} + \dfrac{l^2 \operatorname{cotg}^2 \alpha_3}{H_3^2}\right)\right] + \\ + (m^2 - \nu^2)\left[\varkappa_3^2 - \pi^2\left(\dfrac{k^2}{H^2_3} + \dfrac{k^2 \operatorname{cotg}^2 \alpha_2}{H_2^2}\right)\right] \\ + b(n^2 - \nu^2)\varkappa_2 + c(m^2 - \nu^2)\varkappa_3 = 0 \end{cases} \\ &\text{b} \begin{cases} 2(n^2 - \nu^2)\varkappa_2 - 2(m^2 - \nu^2)\varkappa_3 \operatorname{cotg} \alpha_2 \\ + b(n^2 - \nu^2) - c(m^2 - \nu^2) \operatorname{cotg} \alpha_2 = 0 \end{cases} \\ &\text{c} \begin{cases} -2(n^2 - \nu^2)\varkappa_2 \operatorname{cotg} \alpha_3 + 2(m^2 - \nu^2)\varkappa_3 \\ - b(n^2 - \nu^2) \operatorname{cotg} \alpha_3 + c(m^2 - \nu^2) = 0 \end{cases} \\ &\text{d} \quad (n^2 - \nu^2) \operatorname{cotg} \alpha_3 + (m^2 - \nu^2) \operatorname{cotg} \alpha_2 = 0 \end{aligned}$$

From the eq. b and c above we find

$$\text{(3,18)}\quad \begin{aligned} \varkappa_2 &= -\frac{b}{2} \\ \varkappa_3 &= -\frac{c}{2} \end{aligned}$$

Introducing these expressions for $\varkappa_2$ and $\varkappa_3$ into eq. a, we obtain the frequency formula

$$(3,19)\qquad \nu^2 = \frac{\left[\frac{b^2}{4} + \pi^2\left(\frac{k^2}{H_2^2} + \frac{l^2 \operatorname{cotg}^2 \alpha_3}{H_3^2}\right)\right] n^2 + \left[\frac{c^2}{4} + \pi^2\left(\frac{l^2}{H_3^2} + \frac{k^2 \operatorname{cotg}^2 \alpha_2}{H_2^2}\right)\right] m^2}{\frac{b^2}{4} + \frac{c^2}{4} + \pi^2\left(\frac{k^2}{H_2^2} + \frac{l^2}{H_3^2} + \frac{k^2 \operatorname{cotg}^2 \alpha_2}{H_2^2} + \frac{l^2 \operatorname{cotg}^2 \alpha_2}{H_3^2}\right)}$$

The stability criteria of Chapter II p. 16 are immediately seen to be confirmed by this frequency formula:

A: m^2 and n^2 are both positive. Total stability.

According to the frequency formula above, ν^2 will in this case be positive whatever the values of $\frac{k}{H_2}$, $\frac{l}{H_3}$, α_2, α_3, b and c. The solution (3,15) is in this case representing an oscillation with the frequency ν along an invariable system of streamlines determined by the function ψ_0 of (3,16).

B: m^2 and n^2 are both negative. 'Total instability'.

According to the frequency formula, ν^2 will be negative in this case. Consequently, ν will be imaginary and the solution (3,15) can now be written

$$\psi = \psi_0 \cosh i\nu t = \tfrac{1}{2}\,\psi_0\,(e^{i\nu t} + e^{-i\nu t}).$$

Accordingly, there will in this case be a tendency to generate circulations along an unvariable system of streamlines determined by (3,16). The instability is of an exponential type, $\frac{2\pi}{i\nu}$ being the 'time of flight'. (V. Bjerknes 1938 p. 276).

C: m^2 and n^2 have opposite signs. Conditional instability.

If we suppose in this case m^2 to be negative, it is seen from the frequency formula that ν^2 will be negative if

$$\frac{\frac{b^2}{4} + \pi^2\left(\frac{k^2}{H_2^2} + \frac{l^2 \operatorname{cotg}^2 \alpha_3}{H_3^2}\right)}{\frac{c^2}{4} + \pi^2\left(\frac{l^2}{H_3^2} + \frac{k^2 \operatorname{cotg}^2 \alpha_2}{H_2^2}\right)} < \frac{-m^2}{n^2}.$$

This case corresponds to the case C of conditional instability on p. 16.

Eq. d of (3,17) restricts the boundary conditions essentially. Without this condition being satisfied, no simple solutions of the form (3,13) will be possible. *E. Høiland* (1938 Chapter IV) is the first one who has discussed more thoroughly the kinematic boundary conditions which must be fulfilled if solutions of (3,12) having the form (3,16) shall exist. The examinations of E. Høiland is limited to a special case of (3,12) corresponding to the case of rotational stability in a homogeneous, incompressible vortex. In this case, the constants of eq. (3,12) are determined by

$$m^2 = 0,\ n^2 = 4\Omega^2$$
$$b = 0,\quad c = 0.$$

It is not difficult to extend his results to the general equation (3,12). When we are now about to do so, it is in order to examine whether results which are important to motions in the atmosphere then can be obtained.

From formula (3,19) it is seen that the value ν^2 will always be intermediate those of m^2 and n^2. Consequently, $m^2 - \nu^2$ and $n^2 - \nu^2$ will have opposite signs. In connection with this, $\operatorname{cotg}\alpha_2$ and $\operatorname{cotg}\alpha_3$, will as a consequence of eq. d of (3,17) have equal signs. Having, as indicated also in fig. 2, chosen the positive directions for α_2 and α_3 from the η — to the ζ axis and from the ζ — to the η axis, respectively, it will be understood that no rectangular boundary is possible when α_2 and α_3 are different from zero. On the other hand, eq. d of (3,17) will be satisfied if $\alpha_2 = \alpha_3 = 0$, corresponding to a rectangular boundary. Consequently, there will be one and only one rectangular boundary satisfying the condition d of (3,17), the sides of which are parallel to the axes of co-ordinates η and ζ, i.e. parallel to the principal axes of the tensors M_Q and M_0 in the incompressible and compressible case, respectively.

From eq. d of (3,17) it is found that

$$\frac{\operatorname{cotg}\alpha_3}{\operatorname{cotg}\alpha_2} = -\frac{m^2 - \nu^2}{n^2 - \nu^2}.$$

If we here for $-\frac{m^2 - \nu^2}{n^2 - \nu^2}$ substitute the expression obtained by means of eq. a of (3,17), we arrive at

$$(3,20)\qquad \frac{\operatorname{cotg}\alpha_3}{\operatorname{cotg}\alpha_2} = \frac{\frac{b^2}{4} + \frac{\pi^2 k^2}{H_2^2}\,(1 + \operatorname{cotg}\alpha_2 \cdot \operatorname{cotg}\alpha_3)}{\frac{c^2}{4} + \frac{\pi^2 l^2}{H_3^2}\,(1 + \operatorname{cotg}\alpha_2 \cdot \operatorname{cotg}\alpha_3)}.$$

The expression for the solution ψ will now according to (3,15), (3,16) and (3,18) be given by

(3,21) $\psi =$

$$\varepsilon e^{-\frac{1}{2}(b\eta+c\zeta)} \cdot \sin \frac{k\pi\,(\eta - \operatorname{cotg} \alpha_2 \zeta)}{H_2} \sin \frac{l\pi(\zeta - \operatorname{cotg} \alpha_3 \eta)}{H_3} \cos \nu t.$$

From (3,20) above it is understood that the integers k and l cannot be arbitrarily chosen except in the case of a rectangular boundary where α_2 and α_3 are equal to zero. Let us in the latter case suppose H_2 and H_3 to be so small that the function $e^{-\frac{1}{2}(b\eta+c\zeta)}$ in the solution (3,21) can be neglected. Then we obtain the approximate solution

$$(3,21') \qquad \psi = \varepsilon \sin \frac{k\pi\eta}{H_2} \sin \frac{l\pi\zeta}{H_3} \cos \nu t.$$

As now k and l can be arbitrarily chosen, we may obtain the solution corresponding to the initial conditions

$$\psi_{t=0} = \psi_0\,, \quad \frac{\partial \psi}{\partial t}_{t=0} = 0$$

where ψ_0 now is an arbitrary function vanishing at the boundary, by the series

$$\psi = \sum_{k=1,\,l=1}^{\infty} \varepsilon_{k,l} \sin \frac{k\pi\eta}{H_2} \sin \frac{l\pi\zeta}{H_3} \cos \nu_{k,l} t$$

where the coefficients are determined by the Fourier development of the function ψ_0:

$$\psi_0 = \sum_{k=1,\,l=1}^{\infty} \varepsilon_{k,l} \sin \frac{k\pi\eta}{H_2} \sin \frac{l\pi\zeta}{H_3}.$$

The system of streamlines given by (3,16) consists of $k \cdot l$ similar cellular motions. In the case of only one singular cell we have $k = l = 1$, according to which the condition (3,20) must now be written

$$(3,22) \quad \frac{\operatorname{cotg} \alpha_3}{\operatorname{cotg} \alpha_2} = \frac{\frac{b^2}{4} + \frac{\pi^2}{H_2^2}\,(1 + \operatorname{cotg} \alpha_2 \cdot \operatorname{cotg} \alpha_3)}{\frac{c^2}{4} + \frac{\pi^2}{H_3^2}\,(1 + \operatorname{cotg} \alpha_2 \cdot \operatorname{cotg} \alpha_3)}.$$

3. Elementary frequency relations.

In the preceding section we have found simple solutions of the equations in question of a trigonometric or exponential time dependency and satisfying the boundary conditions given by a parallelogram in the meridional planes. We will now more generally suppose that the meridional projection of the rigid boundary is a closed curve which, without necessarily being a parallelogram, shall be consistent with a solution of the form

$$(3,23) \qquad \boldsymbol{v} = \boldsymbol{v}_0 \cos \nu t.$$

In the *incompressible* case, the frequency ν is determined by the formula (2,18′) b′ p. 14.

$$\nu^2 = \frac{\int_M \boldsymbol{v} \cdot M_Q \cdot \boldsymbol{v} dM}{\int_M v^2 dM}$$

which by the introduction of

$$M_Q = m^2{}_Q \boldsymbol{j}_Q \boldsymbol{j}_Q + n^2{}_Q \boldsymbol{k}_Q \boldsymbol{k}_Q\,; \; \boldsymbol{v} = v_\eta \boldsymbol{j}_Q + v_\zeta \boldsymbol{k}_Q$$

can be written

$$(3,24) \qquad \nu^2 = m^2{}_Q \frac{\int_M v^2{}_\eta dM}{\int_M v^2 dM} + n^2{}_Q \frac{\int_M v^2{}_\zeta dM}{\int_M v^2 dM}$$

Assuming $n^2{}_Q > m^2{}_Q$, it appears from the last formula that ν^2 must satisfy the following relation

$$(3,25) \qquad m^2{}_Q \leqq \nu^2 \leqq n^2{}_Q.$$

This relation can also be found by means of the circulatoric method of Høiland.

In the *compressible* case we can find the lower limit for ν^2 by means of formula (2,18′) c′ on p. 14. According to the developments on p. 15, the lover limit of the integrand in this formula will be given by

$$\boldsymbol{v} \cdot M_\theta \cdot \boldsymbol{v}$$

where M_θ is the tensor defined by eq. (2,22). Consequently, we must always have

$$\nu^2 \gtreqless \frac{\int_M \boldsymbol{v} \cdot M_\theta \cdot \boldsymbol{v} dM}{\int_M v^2 dM}.$$

Writing in this formula

$$M_\theta = m^2{}_\theta \boldsymbol{j}_\theta \boldsymbol{j}_\theta + n^2{}_\theta \boldsymbol{k}_\theta \boldsymbol{k}_\theta\,; \; \boldsymbol{v} = v_\eta \boldsymbol{j}_\theta + v_\zeta \boldsymbol{k}_\theta$$

we obtain

$$\nu^2 \gtreqless m^2{}_\theta \frac{\int_M v^2{}_\eta dM}{\int_M v^2 dM} + n^2{}_\theta \frac{\int_M v^2{}_\zeta dM}{\int_M v^2 dM}$$

and, with the assumption $n^2_0 > m^2_0$

(3,26) $$\nu^2 \gtreqless m^2_0$$

On the other hand, it appears from formula (2,18′) c′ that no *upper* limit for ν^2 exists as long as we may consider the fluid as continuous. These high frequent motions are those of the sound, and the frequencies will be determined by

$$\nu^2 = \frac{\int\limits_M \frac{1}{\Gamma}(\nabla_3 \cdot \boldsymbol{v})^2 dM}{\int\limits_M v^2 dM}.$$

Excluding, however, motions similar to that of the sound, we can also in the compressible case obtain an expression for the *upper* limit of ν^2. In order to arrive at this expression, we start with the equation for the acceleration of vorticity (3,6)

$$\nabla \times \frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \nabla \times \boldsymbol{v} \cdot M_0 + \frac{-\nabla \Theta}{\Theta} \times \left[\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \boldsymbol{v} \cdot M_0\right] = 0$$

from which, by an application of the theorem of Stokes, the corresponding equation for the acceleration of circulation is obtained:

(3,27) $$\oint\limits_L \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \cdot \delta \boldsymbol{r} = -\int\limits_L \boldsymbol{v} \cdot M_0 \cdot \delta \boldsymbol{r} + \int\limits_F \frac{-\nabla \Theta}{\Theta} \times \left[\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \boldsymbol{v} \cdot M_0\right] \cdot \boldsymbol{i} \delta F.$$

F is an arbitrary surface in the meridional plane L its bounding curve. Substituting here

$$\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\nu^2 \boldsymbol{v}$$

we obtain the frequency formula

(3,28) $$\nu^2 = \frac{\oint\limits_L \boldsymbol{v} \cdot M_0 \cdot \delta \boldsymbol{r} + \int\limits_F \frac{-\nabla \Theta}{\Theta} \times \boldsymbol{v} \cdot M_0 \cdot \boldsymbol{i} \delta F}{\oint\limits_L \boldsymbol{v} \cdot \delta \boldsymbol{r} + \int\limits_F \frac{-\nabla \Theta}{\Theta} \times \boldsymbol{v} \cdot \boldsymbol{i} \delta F}$$

We now suppose that the streamlines of $\boldsymbol{v}$ also in the compressible case are closed curves along which the velocity circulates in the same direction. This condition must, with a closed boundary for the motion, be satisfied for the circulatoric part $\boldsymbol{v}_C$ of the meridional volocity $\boldsymbol{v}$, but not for the ascendental part $\boldsymbol{v}_D$. If now $\boldsymbol{v}_C$ is great compared with $\boldsymbol{v}_D$, then the condition in question will also be satisfied for $\boldsymbol{v}$, and therefore especially for the motions we are searching for (cfr. 1° pp. 18—19).

Into (3,28) we substitute

$$\boldsymbol{v} = v \boldsymbol{r}_I, \ \delta \boldsymbol{r} + \delta r \boldsymbol{r}_I$$

and then take the circulation for the terms of this equation round a closed streamline bounding a surface of such a small area that the frequency, with sufficient accuracy is determined by

$$\nu^2 = \frac{\oint \boldsymbol{r}_I \cdot M_0 \cdot \boldsymbol{r}_I v \delta r}{\oint v \delta r}.$$

According to the above presuppositions, v is of equal sign at all places of the streamline along which it is integrated. We may therefore put an average value of $\boldsymbol{r}_I \cdot M_0 \cdot \boldsymbol{r}_I$ outside the integral sign in the above formula. Doing so, we obtain

$$\nu^2 = \overline{\boldsymbol{r}_I \cdot M_0 \cdot \boldsymbol{r}_I}.$$

Consequently, ν^2 will be intermediate the extreme values of $\boldsymbol{r}_I \cdot M_0 \cdot \boldsymbol{r}_I$. Having

$$\boldsymbol{r}_I \cdot M_0 \cdot \boldsymbol{r}_I = m^2_0 r^2_{I\eta} + n^2_0 r^2_{I\zeta}$$

we see that the extreme values of $\boldsymbol{r}_I \cdot M_0 \cdot \boldsymbol{r}_I$ will be equal to m^2_0 and n^2_0. Hence we obtain the relation for the frequency

(3,29) $$m^2_0 \leqq \nu^2 \leqq n^2_0.$$

Having assumed $n^2_0 > m^2_0$ this relation is quite analogous to that in the incompressible case, (3,25).

4. The general frequency formula for a quadrangular boundary.

Let us below assume $k = l = 1$ in the solution (3,21), and further H_2 and H_3 to be so small that we in this solution can neglect the function $e^{-\frac{1}{2}(b\eta + c\zeta)}$, and in the frequency formula (3,19) and in eq. (3,22), the terms containing the magnitudes b and c. Then we obtain approximately the solution

(3,30) $$\psi = \varepsilon \sin \frac{\pi(\eta - \operatorname{cotg} \alpha_2 \zeta)}{H_2} \sin \frac{\pi(\zeta - \operatorname{cotg} \alpha_3 \eta)}{H_3}$$

the frequency formula

(3,31) $$\nu^2 = \frac{(H^2_2 + H^2_3 \operatorname{cotg}^2 \alpha_2) m^2 + (H^2_3 + H^2_2 \operatorname{cotg}^2 \alpha_3) n^2}{H^2_2 + H^2_3 + H^2_3 \operatorname{cotg}^2 \alpha_2 + H^2_2 \operatorname{cotg}^2 \alpha_3}$$

and the condition

$$H^2_2 \operatorname{cotg}\alpha_3 - H^2_3 \operatorname{cotg}\alpha_2 = 0 \tag{3,32}$$

which must be satisfied at the boundary. The function ψ of (3,30) will now be the solution of the differential equation

$$\nabla^2 \frac{\partial^2 \psi}{\partial t^2} + n^2 \frac{\partial^2 \psi}{\partial \eta^2} + m^2 \frac{\partial^2 \psi}{\partial \zeta^2} = 0 \tag{3,33}$$

having as boundary a parallelogram satisfying the condition (3,32), and a frequency determined by the frequency formula (3,31). By more general boundary conditions, solutions of (3,33) will also be solutions of (3,12) provided that the dimensions of the motion are sufficiently small. Let us now suppose a square to be the meridional projection of the boundary for the motion. The conditions, which this square must satisfy if solutions of (3,33) shall exist having the form

$$\psi = \psi_0 \cos \nu t, \tag{3,34}$$

can be found together with the corresponding frequencies, by a method quite identical to that used by Høiland (1938 p. 51) in his examination of the earlier mentioned (p. 21) special case of eq. (3,33). The above function will be the solution of eq. (3,33) if and only if

$$(n^2 - \nu^2) \frac{\partial^2 \psi_0}{\partial \eta^2} + (m^2 - \nu^2) \frac{\partial^2 \psi_0}{\partial \zeta^2} = 0. \tag{3,35}$$

If the co-ordinate system (η, ζ) is turned an angle β, the last equation, if referred to the new system (η', ζ'), must be written

$$\begin{aligned} & 2\,(m^2 + n^2 - 2\nu^2) \sin\beta \cos\beta \frac{\partial v_{\eta}'}{\partial \eta'} \\ & +[(m^2 - \nu^2) \sin^2\beta + (n^2 - \nu^2) \cos^2\beta] \frac{\partial v_{\zeta}'}{\partial \eta'} \\ & +[(m^2 - \nu^2) \cos^2\beta + (n^2 - \nu^2) \sin^2\beta] \frac{\partial v_{\eta}'}{\partial \zeta'} = 0 \end{aligned} \tag{3,36}$$

having introduced

$$\frac{\partial \psi_0}{\partial \zeta'} = v_{\eta'}, \quad \frac{\partial \psi_0}{\partial \eta'} = - v_{\zeta'}.$$

According to the relations (3,25) and (3,29) of the preceding chapter we have always

$$(m^2 - \nu^2)\,(n^2 - \nu^2) < 0.$$

According to this, there will exist two angles β_1 and β_2 satisfying the equation

$$(m^2 - \nu^2) \cos^2\beta + (n^2 - \nu^2) \sin^2\beta = 0. \tag{3,37}$$

Let us now suppose that η' and ζ' are co-ordinates in a system which appears if (η, ζ) is turned an angle β_1 or β_2. Transformed to these co-ordinates, eq. (3,36) will, according to (3,37), be reduced to

$$\begin{aligned} & 2\,(m^2 + n^2 - 2\nu^2) \sin\beta \cos\beta \frac{\partial v_{\eta}'}{\partial \eta'} \\ & + [(m^2 - \nu^2) \sin\beta + (n^2 - \nu^2) \cos^2\beta] \frac{\partial v_{\zeta}'}{\partial \eta'} = 0. \end{aligned}$$

By an integration of this equation we obtain

$$\boldsymbol{B} \cdot \boldsymbol{v} = f(\zeta') \tag{3,38}$$

having introduced the notation $\boldsymbol{B}$ defined by

$$\begin{aligned} & 2\,(m^2 + n^2 - 2\nu^2) \sin\beta \cos\beta\, \boldsymbol{j}_{\eta'} + \\ & + [(m^2 - \nu^2) \sin^2\beta + (n^2 - \nu^2) \cos^2\beta]\, \boldsymbol{k}_{\zeta'}. \end{aligned}$$

The right-hand side of (3,38) will depend only upon ζ' and $\boldsymbol{B}$ must in the general case of $m^2 + n^2$ be different from zero.

Let us first consider the straight line

$$\zeta' = c_1$$

passing through the edge $P_2 : (b_1, c_1)$ of the square $P_1 P_2 P_3 P_4$ of fig. 3. The corresponding

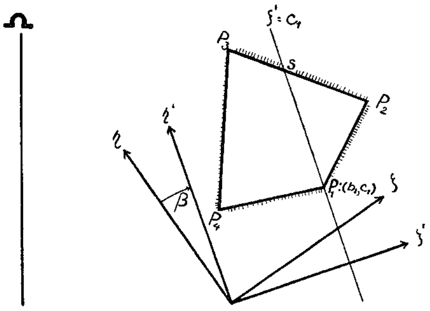

Fig. 3. Illustration of a boundary inconsistent with a solution of the form $\psi = \psi_0 \cos \nu t$.

constant function $f(c_1)$ of (3,38) must vanish because this equation has to be satisfied by $\boldsymbol{v}\,(b_1, c_1) = 0$. We may therefore write

$$\boldsymbol{B} \cdot \boldsymbol{v}\,(\eta', c_1) = 0. \tag{3,39}$$

Suppose now the straight line $\zeta' = c_1$ not to be a diagonal of the square, and denote with S the intersecting point between $\zeta' = c_1$ and the side P_2P_3 opposite to the edge P_1. According to (3,39) we must have

$$\boldsymbol{B} \cdot \boldsymbol{v}(S) = 0.$$

This relation must now also be valid if S denotes an arbitrary point of the side P_2P_3, $\boldsymbol{B}$ being a constant vector and all velocities at the side P_2P_3 having equal directions. In the same way we obtain that

$$\boldsymbol{B} \cdot \boldsymbol{v}(S') = 0$$

when S' here denotes an arbitrary point of the opposite side P_4P_1. For all values of ζ' in consideration we must therefore have $f(\zeta') = 0$, and consequently, (3,38) is reduced to

$$\boldsymbol{B} \cdot \boldsymbol{v} = 0. \tag{3,40}$$

According to this relation, however, the streamlines must be parallel straight lines which is inconsistent with the condition

$$\nabla \cdot \boldsymbol{v} = 0 \text{ (in the compressible case } \nabla \cdot \boldsymbol{v}_C = 0)$$

and the boundary conditions, except in the case $\boldsymbol{v} = 0$. Consequently, the diagonals of the square must be parallel to the directions determined by the angles β_1 and β_2 which satisfy eq. (3,37). According to this equation, we have

$$\beta_1 + \beta_2 = \pi.$$

The diagonals of the square must therefore be symmetric to the η — and ζ axis. Remembering, that the co-ordinates have been chosen along the principal axes of the tensors M_Q and M_Θ in the incompressible and compressible case, respectively, we may form the result obtained, as follows:

1°. The differential equation (3,33) can by a quadrangular boundary have solutions of the form

$$\psi = \psi_0 \cos \nu t$$

only if the diagonals of the square are symmetric to the principal axes of

$$M_Q = \frac{\nabla Q}{Q} S \nabla P + \boldsymbol{N} 2 \Omega \boldsymbol{R}_I$$

in the incompressible case, and to the principal axes of

$$M_\Theta = \frac{-\nabla \Theta}{\Theta} S \nabla P + \boldsymbol{N} 2 \Omega \boldsymbol{R}_I$$

in the compressible case.

It can be shown that condition 1° also is sufficient. The frequencies or 'times of flight' are determined by (3,37) from which we find the general frequency formula

$$\nu^2 = m^2 \cos^2 \beta + n^2 \sin^2 \beta. \tag{3,41}$$

As an interesting consequence of this formula, we observe that all squares corresponding to a given value for β will have equal frequencies or 'times of flight'. In fig. 4 we have as an illustration of this result drawn to squares corresponding to precisely the same frequency.

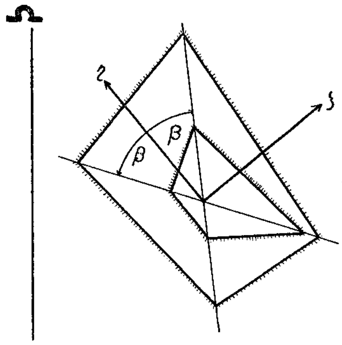

Fig. 4. Illustration of two boundaries consistent with a solution of the form $\psi = \psi_0 \cos \nu t$ and corresponding to equal frequencies.

Evidently ν in the stable case is a measure of the stability, and $i\nu$ in the unstable case a measure of the instability of the motions. If therefore m^2 is the algebraically smaller one of m^2 and n^2, it is seen from the frequency formula above that the less stable or most unstable motions will be characterized by an angle β_1 ($\beta_1 < \beta_2$) which is as small as possible, i.e., for motions which are directed as much as possible in the direction along which the vector component of M_Q or M_Θ has its algebraically smallest scalar value.

If required, we may by simple geometrical considerations confirm the frequency formula (3,31) and the condition (3,32). Here we shall only consider the case with a rectangular boundary. Evidently, the diagonals of the rectangle can only then be symmetric to the co-ordinate axes η and ζ, if the sides of the rectangle are parallel to the co-ordinate axes. Let us suppose the sides to have lengths H_2 and H_3 along the η — and ζ axis, respectively. Then we must have

$$\cos^2 \beta = \frac{H^2{}_2}{H^2{}_2 + H^2{}_3}, \quad \sin^2 \beta = \frac{H^2{}_3}{H^2{}_2 + H^2{}_3}$$

which, if introduced into the frequency formula (3,41), gives

$$\nu^2 = \frac{H^2{}_2 m^2 + H^2{}_3 n^2}{H^2{}_2 + H^2{}_3}$$

in accordance with the frequency formula of (3,32) when α is equal to zero.

In applying to the atmosphere the results just obtained, we have to remember that one of the sides of the square must coincide with the surface of the earth. It will therefore be important to determine the direction of the principal axes of M_0 relative to earth. As will be shown in the second part of this work, this is easily done by means of the expression (2,20) defining the tensor M_0.

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