PREFACE

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ON HORIZONTAL MOTION IN A ROTATING FLUID

BY

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1. The Induced Circulation and the Induced Vorticity.

In a system of coordinates rotating with the earth the equation of motion for an ideal fluid in horizontal motion assumes the form

$$(1.1) \qquad \frac{D\boldsymbol{v}}{dt} = -s \nabla_H p - 2\, \boldsymbol{\Omega}_z \times \boldsymbol{v},\ v_z = 0.$$

$\frac{D\boldsymbol{v}}{dt}$ is the horizontal component of the individual time derivative of the velocity $\boldsymbol{v}$, i. e. the horizontal acceleration of a fluid particle, $-\nabla_H p$ is the horizontal component of the pressure gradient, $\boldsymbol{\Omega}_z$ the vertical component of the earth's angular velocity and v_z the vertical velocity here assumed to vanish.

The circulation of the acceleration around a closed curve is given by

$$(1,2) \qquad \oint \frac{D\boldsymbol{v}}{dt} \cdot \delta \boldsymbol{r} = N(p, s) - \oint 2\, \boldsymbol{\Omega}_z \times \boldsymbol{v} \cdot \delta \boldsymbol{r},$$

where $\delta \boldsymbol{r}$ is a line element of the curve and $N(p, s)$ the number of isobaric-isosteric solenoids embraced by the curve.

Disregarding the effect of baroclinicity, i. e. neglecting the solenoid term, our equation reduces to

$$(1,3) \qquad \oint \frac{D\boldsymbol{v}}{dt} \cdot \delta \boldsymbol{r} = -\oint 2\, \boldsymbol{\Omega}_z \times \boldsymbol{v} \cdot \delta \boldsymbol{r}.$$

The solenoid term $N(s, p)$ will vanish when:

1. The fluid is autobarotropic, i. e. when the equation of state is the same for all particles and of the form $s = s(p)$; or

2. The equation of state is of the form $s = s(p, \vartheta)$ where ϑ is the temperature, the mode of motion however so that isobars and isotherms coincide (barotropy).

The left-hand side of our last equation, i. e. the circulation of acceleration, can by a well known procedure be transformed to denote the variation per unit time of the velocity circulation C around the material curve coinciding at the moment with the considered closed curve. Hence we obtain:

$$(1.4) \qquad \frac{dC}{dt} = -\oint 2\, \boldsymbol{\Omega}_z \times \boldsymbol{v} \cdot \delta \boldsymbol{r}.$$

This variation of velocity circulation is entirely due to the earth's rotation. We shall call it the *induced circulation* on account of its similarity with the induced electric current in a closed conductor in motion relative to a magnetic field.

This similarity will appear clearly in our developements below.

If the closed line is a stream-line, then $\boldsymbol{v}$ and $\delta \boldsymbol{r}$ are parallel. Hence it follows that

$$(1,5) \qquad \frac{dC}{dt} = \oint \frac{D\boldsymbol{v}}{dt} \cdot \delta \boldsymbol{r} = 0$$

for a closed horizontal stream-line in the barotropic atmosphere. This equation shows that we can never in the barotropic atmosphere have horizontal oscillations within cells (for instance rectangular cells) and we can have no standing oscillations. A standing oscillation can always be split up into two waves propagating in opposite directions, the positive and negative directions being equivalent. This is not the case for horisontal motion in our system. A wave can only propagate in *one* direction.

Transforming by Stoke's theorem the contour integral on the right-hand side of equation (1,3) into a surface integral, we obtain

$$(1,6) \qquad \frac{dC}{dt} = -2 \int_{\Sigma} (\boldsymbol{v} \cdot \nabla \Omega_z + \Omega_z \nabla \cdot \boldsymbol{v}) \delta\sigma,$$

where Σ is the area in the horizontal plane enclosed by the considered curve. Considering the effect of horizontal divergence, we find the well known result that positive horizontal divergence will produce anticyclonic circulation while negative horizontal divergence or horizontal convergence will produce cyclonic circulation. In the following we shall neglect the divergence term and consider the fluid as incompressible. Equation (1,5) then assumes the form

$$\frac{dC}{dt} = -2 \int_{\Sigma} (\boldsymbol{v} \cdot \nabla \Omega_z)\, \delta\sigma.$$

Since Ω_z is a function of the latitude φ only, this equation can further be written

$$(1,7) \qquad \frac{dC}{dt} = -\frac{2}{a} \int_{\Sigma} \boldsymbol{v}_{\varphi} \frac{d\Omega_z}{d\varphi}\, \delta\sigma,$$

where a is the radius of the earth, and v_{φ} the northward velocity.

This equation clearly reveals the physical principle underlying the "effect of the variation of the Coriolis' parameter". If, namely, a plane closed curve consisting of fluid particles, for instance contained in a closed tube of infinitesimal cross-section, rotates relative to the earth, so that the component Ω_n of the earth's angular velocity along the perpendicular to the tube (the plane of the tube) has an increase $\frac{d\Omega_n}{dt}$ per unit time, then the velocity circulation in the tube will increase at a rate

$$\frac{dC}{dt} = -2 \frac{d\Omega_n}{dt} \Sigma,$$

where Σ is the area enclosed by the tube. This is the induced circulation in an incompressible fluid, which is completely analogous to the induced electric current in a closed conductor moving in a magnetic field so that the number of magnetic solenoids enclosed by the conductor varies with time. The formal conformity between the two cases will appear if we consider the flux of the constant vector $2\,\boldsymbol{\Omega}$. If N be the total flux encircled by our curve, then

$$(1,8) \qquad \frac{dC}{dt} = -\frac{dN}{dt} \text{ or } C = -N + \text{const.}$$

That this induced circulation must appear is easily understood. Assume for instance that the plane of our tube at a given moment is parallel to the earth's axis ($N = 0$), and that there is no relative circulation in the tube at that moment (the constant in the last of eq. (1,8) represents the circulation in the absolute motion). Now let the plane of our tube at another instant be perpendicular to the earth's axis. If we still had no relative circulation in the tube, we would in the absolute motion have a circulation on account of the rotation of the earth and with a direction given by that rotation. But such an absolute circulation would contradict the law of conservation of circulation in absolute motion (Kelvins law). To avoid this absolute circulation we must therefore have in the tube a relative anticyclonic circulation. This is the induced circulation.

A horizontal curve which is being transported horizontally northwards or southwards will rotate around an axis perpendicular to $\boldsymbol{\Omega}$. The total flux encircled by the curve will therefore vary. Hence the induced circulation. It is easily verified that the right-hand side of eq. (1,7) gives just the variation per unit time of the total flux encircled by the curve arising from the northward velocity v_{φ} of its individual points.

From eq. (1,6) or (1,7) we find for the induced vorticity per unit time corresponding to the induced circulation per unit time

$$(1,9) \qquad \frac{D\zeta}{dt} = -\frac{2}{a} \frac{d\Omega_z}{d\varphi} v_{\varphi} = -\beta v_{\varphi},$$

where ζ is the relative vorticity. We have introduced the common notation

$$(1,10) \qquad \beta = \frac{2}{a} \frac{d\Omega_z}{d\varphi} = \frac{2\,\Omega \cos\varphi}{a}$$

for the "variation of the Coriolis' parameter".

Eq. (1,9) can be integrated individually. We obtain

$$(1,11) \qquad \zeta = \zeta_a - 2\,\Omega \sin\varphi.$$

where ζ_a is the absolute vorticity of the considered particle.

2. On the Fundamental Effect of the Induced Vorticity on the Motion in Stationary Cyclones and Anticyclones.

It appears from the discussion in the preceding section, explicitely from formula (1,9) or from (1,11), that the induced vorticity will produce an increase of vorticity for a southward moving particle while a decrease of vorticity is obtained for a northward moving particle.

Suppose that we in a region have a stationary system of closed stream-lines, Fig. 1,

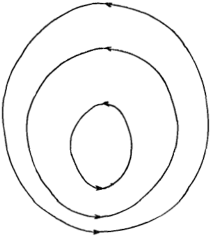

Fig. 1.

with for instance a cyclonic circulation along the stream-lines, and suppose also that the vorticity is cyclonic within the whole region. A particle moving along a stream-line will on account of the induced vorticity have the smallest relative vorticity at the northernmost point of the stream-line and its largest vorticity at the southernmost point of the stream-line. Thus at the southern part of the region we must have a greater cyclonic vorticity than at its northern part. If the curvature of the stream-lines at their southernmost and northernmost points are not much different, the difference in relative cyclonic vorticity at the two points will give rise to a greater cyclonic shear at the southern point than at the northern point. From the centre of the cyclone the velocity will therefore increase more rapidly towards south than towards north, and we obtain a stream-line picture as that drawn in Fig. 1. *A stationary cyclone will have a motion asymmetrical in the south-north direction with the most intense motion south of its centre, where we have westerly winds.*

If the relative vorticity is of about the magnitude observed in large scale motion in the atmosphere, a fluid particle which is transported from a point sufficiently far south of the centre to a corresponding point north of the centre, may by the influence of the induced decrease of vorticity, have lost its entire cyclonic vorticity on arriving at the northern point. If the stream-lines are closed in this region, the vorticity at the northern point must then be zero, and if we have still more closed stream-lines, the vorticity must be anticyclonic north of the point where we had zero vorticity. Assuming the curvature of the closed stream-lines to be cyclonic, the shear must become anticyclonic at a point south of the point with zero vorticity. Thus the velocity must begin to decrease at some point south of the point with zero vorticity. Passing northwards from the point with no shear, the velocity must decrease and at last become zero. Where the velocity becomes zero we have a hyperbolic point with a stream-line intersecting itself, and forming a bow which represents the last closed stream-line, Fig. 2 (in the diagrams cyclones are denoted by C). North of

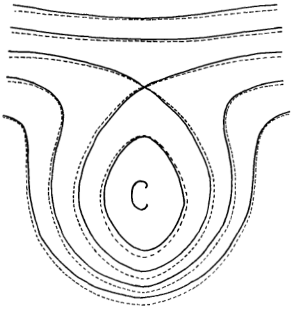

Fig. 2

the hyperbolic point we again get westerly winds. If the hyperbolic point coincides with the point where the vorticity is zero, the stream-line will intersect itself at right angle. If the hyperbolic point is above that point, i. e. if the vorticity at the hyperbolic point is anticyclonic, the angle opening towards north (or south) must be the greatest, while if the hyperbolic point is below, i. e. if the vorticity at the hyperbolic point is cyclonic, the angle towards east (or west) will be the greatest.

The motion within and around a stationary anticyclone can be analysed in the same manner as for a cyclone. We arrive at the result: *A stationary anticyclone will have a motion asymmetrical in the south-north direction with the most intense motion north of its centre, where we have westerly winds,* Fig. 3 (in the diagrams anticyclones are denoted by A). Thus, both for a cyclone and for

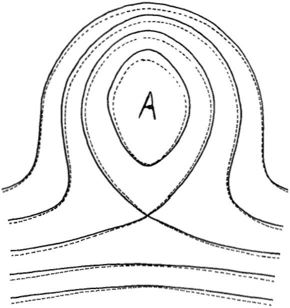

Fig. 3.

an anticyclone the westerly winds will be stronger than the easterly winds. Further: The hyperbolic point for an anticyclone will be situated south of the centre, Fig. 3. South of the hyperbolic point we again have westerly winds. If the vorticity is zero at the hyperbolic point, the stream-line branches through the point will intersect at a right angle. If the vorticity is cyclonic at the hyperbolic point, the angle opening towards north (or south) must be the greatest, while if the vorticity is anticyclonic at the point, the angle opening towards east (or west) must be the greatest.

We have considered cyclones and anticyclones with hyperbolic points above (north of) and below (south of) the centre respectively. Another arrangement which is also a possible stationary motion is to have the hyperbolic points to the left and to the right of the centre. We will then get stream-line patterns as shown in Fig. 4. The asymmetry of the motion will again correspond to stronger westerlies than easterlies within the cyclones (anticyclones). Along a latitude circle we will have a row of only cyclones or only anticyclones. The cyclones divide between an easterly flow north of the row and a westerly south of it, while the anticyclones divide between a westerly flow north of the row and an easterly south of it. The stream-line patterns in Fig. 4 originate from the same circumstances as do the "cat's-eye" patterns studied by Lord Kelvin [1]

The isobar patterns in the stationary cyclones and anticyclones will be similar to the stream-line pattern. The centre and the hyperbolic points will coinside in the two patterns. On a stream-line the pressure must be lowest where the velocity is greatest to give the required accelerations along the stream-line. Hence, to secure stationary conditions the isobar pattern must have a less pronounced asymmetry than the stream-line pattern in a cyclone while in an anticyclone the stream-line pattern will have the least pronounced asymmetry. (See Figs. 2 and 3 where the dotted lines represent the isobars).

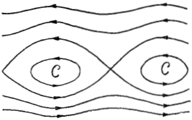

Fig. 4 A.
Cyclones

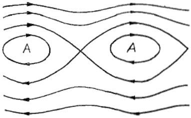

Fig 4 B.
Anticyclones.

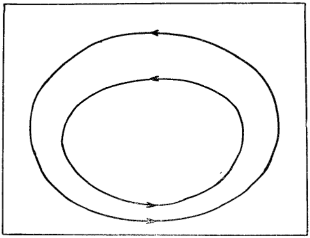

Fig. 5 A.

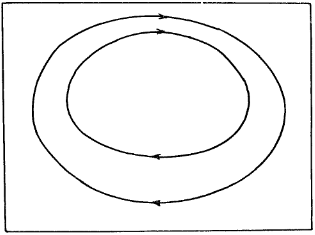

Fig. 5 B.

If the fluid is contained within for instance a rectangular basin we will get a stream-line pattern as drawn in Fig. 5 A when the vorticity is cyclonic within the basin, and as that drawn in Fig. 5 B when the vorticity is anticyclonic. The problem to find analytic solutions for stationary motions in such basins may turn out to be a very difficult one on account of the asymmetric conditions at the western and eastern boundaries. This asymmetry prevents the ordinary analytic continuations of the solution giving east of and west of the basin motion in cells identic with the considered basin with alternately cyclonic and anticyclonic circulation. The asymmetry found here for stationary motion in rectangular basins seems to contradict the results found by Stommel [2] and Munk[1]) who find an asymmetry in the west—east direction with most intense motion at the western part. There is, however, no discrepancies between their results and ours. Stommel and Munk consider a forced circulation with friction and disregard the field accelerations, while we are considering a "free" circulation and are taking into account the field accelerations. A case more analogous to the Stommel-Munk case will be to consider the result of the following produced circulation in the basin. Consider the fluid at rest in the basin being subject to accelerations which after a short interval of time render to every fluid particle the same positive relative vorticity. Then we will get approximately a symmetric stream-line pattern with a cylonic circulation along the stream-lines. At the western part of the basin we will have a motion towards the south. Therefore the cyclonic vorticity here will increase. At the eastern part, where the motion is directed towards the north, the cyclonic vorticity will decrease. The result is a more intense motion at the western part than at the eastern part of the basin. As is easily seen, by the same reasoning, we will obtain the same asymmetry if we start with an anticyclonic vorticity. The results for a "free" stationary circulation in a rectangular basin given above do suggest, however, that also the exact equation for the case considered by Stommel and Munk will lead to a slight asymmetry in the north—south direction.

[1]) In a paper not yet published

3. On Permanent Cyclones and Anticyclones.

Above we have discussed stationary cyclones and anticyclones, where stream-lines and trajectories coincide. We shall now discuss the case when the stream-line pattern has a propagation relative to the fluid, and we limit our considerations to the case when the hyperbolic point is above (cyclone) or below (anticyclone) the centre. Let us for instance assume that a stream-line pattern with some closed stream-lines is propagating towards east with a velocity c, and that the pattern apart from this propagation is unchanged. We shall denote this mode of motion a permanent motion to keep it distinct from the stationary motion. In a coordinate system following the pattern we will then again have a stationary motion with the asymmetry properties pointed out above. Now to get the stream-line pattern in a coordinate system following the earth in its rotation we must add an easterly flow (a westerly wind) with a velocity c. This velocity will intensify the westerly field of motion, and weaken the easterly field of motion. The points with velocity zero in the coordinate system following the pattern will now both of them be situated in the westerly part of the flow in a coordinate system following the earth. Hence, to an observer at rest relative to the earth the region of westerly winds will appear larger, and the region of easterly winds smaller, than to an observer following the pattern. From the considerations above, some interesting results may be drawn.

1. Since the motion becomes stationary in the coordinate system following the pattern, the stream-lines will also represent the paths or trajectories of the particles. If then in this coordinate system closed stream-lines exist, the mass within the outermost closed stream-line must be conserved.

2. If a cyclone with closed stream-lines is propagated with unchanged pattern towards the east with a moderate velocity (with moderate velocity is meant a velocity less than say half of the greatest velocity in the mean flow), it will have more closed stream-lines when observed from the coordinate system following the cyclone than when observed from the rotating earth.[1]) Thus a greater region around the center of the cyclone than that within the outermost closed stream-line observed from the rotating earth will consist of the same airmasses which follow the cyclone in its motion. If for instance the temperature (assumed also steady in the accompanying coordinate system) may be considered as a quantity which is conserved, then the region with closed isotherms will be greater than the region with closed stream-lines. (This is, of course, in our model exactly fullfilled for the lines of constant absolute vorticity).

[1]) For large scale motions we will in general have small accelerations, and therefore approximately geostrophic wind in both systems of reference. This is possible, in spite of the quite different stream-line patterns in the two systems, since we have different horizontal surfaces, and therefore also different isobar patterns in the horizontal surfaces in the two cases.

3. A cyclone with constant area when considered from the earth, will cover a larger area in a system moving with the cyclone, the higher its speed of propagation towards the east (under the above defined limit). One should therefore expect that cyclones, which cover a very large area when considered from the earth, should have the greatest chance to rest or move west-ward.

4. A cyclone propagated towards the west with unchanged stream-lines, will be of smaller extent than the pattern shown in a coordinate system following the earth. If the temperature is a conservative property, then the region of closed isotherms should also be of smaller extent than the region of observed closed stream-lines. Thus cyclones going towards the west should, ceteris paribus, be of greater extent than those going towards the east. For a sufficiently great velocity of propagation towards the west, we will have no closed isotherms.

5. For anticyclones we get results quite analogous to those obtained for cyclones.

6. For an east-going permanent cyclone the centre will in the accompanying coordinate system be situated south of the centre observed from the earth, and the more so the higher the speed of propagation toward the east. If again the temperature may be considered as conserved, we will for east-moving permanent cyclones have a situation as drawn in Fig. 6 A, while for west-going cyclones we will have a situation as drawn in Fig. 6 B. For anticyclones the corresponding situations are drawn in Fig. 6 C and D.

Above we have considered cyclones and anticyclones propagated towards the east or the west with unchanged stream-line pattern. It is easily seen, however, that a slight departure from the stationarity conditions will not alter much the north-south asymmetry. We will still as a general rule obtain that cyclones and anticyclones will have most intense motion where the wind is towards east (westerly wind). And if a cyclone or an anticyclone has a propagation towards the east or the west, with a slow change of the stream-line pattern, we will in a coordinate system following the pattern, have a motion differing not very much from a stationary motion. If in this accompanying coordinate system closed stream-lines exist, the

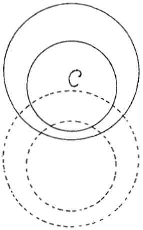

Fig. 6 A. East-moving cyclone.

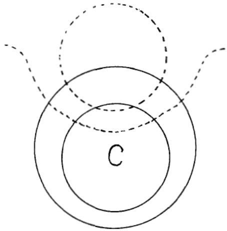

Fig. 6 B. West-moving cyclone.

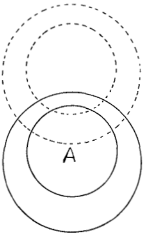

Fig. 6 C. East-moving anticyclone.

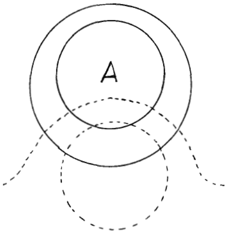

Fig. 6 D. West-moving anticyclone.

Solid curves: stream-lines. Dashed curves: isotherms (or lines of constant absolute vorticity).

air within the outermost closed stream-line must for the most of this region be conserved. All the above results deduced for unchanged stream-line pattern, will therefore as a first approximation be true also for cyclones and anticyclones when the stream-line patterns are changing slowly with time.

We return to the consideration of permanent cyclones and anticyclones. We saw that the air within the *outermost closed stream-line* in the coordinate system following the pattern was conserved, and would accompany the cyclone or anticyclone in its motion. For a cyclone propagated towards the west we also saw that this outermost closed stream-line would enclose a smaller region than the outermost stream-line observed from an observer following the earth in its rotation. If the pattern is propagated towards the west with a velocity equal to or greater than the strongest easterly wind observed in the cyclone or anticyclone, there will exist no closed stream-lines in the accompanying coordinate system. Then no of the air within the cyclone or anticyclone will be conserved. It will be transported towards the east relative to the moving cyclone or anticyclone. For cyclones and anticyclones propagated towards the east we have seen that for moderate velocities of propagation the *outermost* closed stream-line will, ceteris paribus, enclose a greater region the greater the eastward propagation. This will be true as long as the eastward velocity of propagation is smaller than the greatest eastward velocity north of (south of) the hyperbolic point for a cyclone (for an anticyclone). We shall not here discuss the possibility of greater eastward velocities of propagation.

We saw in the preceding section that the hyperbolic points and the centres would coincide for the stream-line patterns and the isobar patterns in stationary cyclones and anticyclones. This will not be the case when we consider a permanent (propagating) cyclone or anticyclone. As is readily seen the centre and the hyperbolic point in the stream-line pattern in a permanent cyclone propagated towards the east will have an acceleration towards the south. At these points we must therefore have a pressure gradient directed towards the south. From this it follows that the hyperbolic point in the isobar pattern will be situated north of the hyperbolic point in the stream-line pattern while the centre in the isobar pattern will be situated south of the centre in the stream-line pattern. The opposite is true for a permanent cyclone propagated towards the west. Considering a stationary cyclone with the corresponding pressure field, an addition of a southward directed pressure gradient may have as a consequence propagation of the cyclone eastward, while the addition of a northward directed pressure gradient may cause a westward propagation of the cyclone. An anticyclone propagated towards the east will behave in the same manner as a cyclone propagated towards the west, while an anticyclone propagated towards the west will behave in the same manner as a cyclone propagated towards the east. Thus an addition of a south-ward directed pressure gradient to the stationary pressure field may cause the anticyclone to be propagated westward while the addition of a northward directed pressure gradient may lead to a propagation of the anticyclone towards the east.

We have hitherto considered a propagation of permanent cyclones and anticyclones along latitude circles. Since our results concerning the propagation are based on purely kinemathical reasoning, we can of course apply our method for investigating propagation in other directions. For a propagation in the south—north direction we obtain for instance the cases illustrated in Fig. 7 A, B, C and D corresponding to the cases illustrated in Fig. 6 A, B, C and D.

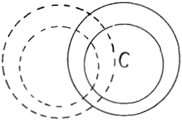

Fig. 7 A.
South-moving cyclone.

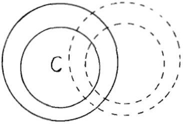

Fig. 7 B.
North-moving cyclone.

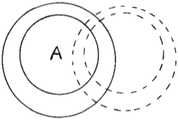

Fig. 7 C.
South-moving anticyclone.

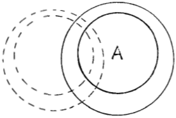

Fig 7 D.
North-moving anticyclone.

Solid curves: stream-lines. Dashed curves: isotherms (or lines of constant absolute voticity).

In all the cases considered small deviations from the geostrophic wind relationship appeared to be necessary to keep the motion stationary or permanent.

We shall in a later section see how this will also appear from an analysis of the impulse given by the Coriolis and pressure forces.

4. Some Exact Stationary Solutions of Equation (1,9).

In this section we shall mainly consider solutions of equation (1,9) representing stationary motions. In a stationary motion the equiscalar curves for the absolute vorticity must coincide with the equiscalar curves for the streamfunction ψ, i. e. we must have

$$\zeta_a = f(\psi). \tag{4,1}$$

Introducing this in eq. (1,11) we obtain:

$$\zeta + 2\,\Omega \sin\varphi = f(\psi),$$

or since we have

$$\zeta = -\nabla^2 \psi,$$

$$\nabla^2 \psi = -f(\psi) + 2\,\Omega \sin\varphi. \tag{4,2}$$

This equation determines all stationary solutions of equation (1,11).

The relationship (4,1) may be chosen quite arbitrarily. We shall in this paper discuss only the case that we have proportionality between absolute vorticity and value of the streamfunction, i. e. we put

$$f(\psi) = k^2\psi. \tag{4,3}$$

Equation (4,2) then assumes the linear form

$$\nabla^2 \psi = -k^2\psi + 2\,\Omega \sin\varphi. \tag{4,4}$$

This equation has the solution

$$\psi = \sum_{m=1}^{n} A_n^m P_n^m(\sin\varphi)\,[\sin m\,\Theta + B_n^m \cos m\,\Theta] + C_n P_n(\sin\varphi) + \frac{2\,\Omega a^2}{a^2k^2-2}\sin\varphi. \tag{4,5}$$

P_n^m is the associated Legendre function of degree n and order m, with

$$n\,(n+1) = a^2k^2. \tag{4,6}$$

Owing to the periodicity of the motion around latitude circles, m must be an integer. To satisfy the conditions at the poles k must be chosen so that n is a positive integer. P_n is the Legendre polynomial of degree n. Θ is the angle of longitude, and A_n^m, B_n^m and C_n are constants.

Before discussing the solution (4,5)[1]) we shall consider equation (4,4) in the approximation that β in equation (1,9) can be treated as a constant, i. e. only the first order term in the Taylor series developement for the induced vorticity is taken into account. Furthermore we neglect the curvature of the earth at the latitude in question so that we can write the equation (4,4) in the form

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{\tan\varphi}{a}\frac{\partial\psi}{\partial y} = -k^2\psi + \beta y, \tag{4,7}$$

where the axis X is directed eastward, the axis Y northward. The last term on the left-hand side of the equation represents a correction due to the fact that in our Cartesian coordinate system a line element parallel to the X-axis will represent a shorter real distance the further north the element is situated.

Equation (4,7) has a solution with $v_y = 0$ for $y = 0$ given by:

$$\psi = A\,e^{\frac{\tan\varphi}{2a}y}\sin\sqrt{(k^2-\mu^2) - \frac{\tan^2\varphi}{4a^2}}\,y\,\sin\mu x$$
$$+ B\,e^{\frac{\tan\varphi}{2a}y}\left[\sin\sqrt{k^2 - \frac{\tan^2\varphi}{4a^2}}\,y + C\cos\sqrt{k^2 - \frac{\tan^2\varphi}{4a^2}}\,y\right] + \frac{\beta}{k^2}y - \frac{\beta}{k^4}\frac{\tan\varphi}{a}.$$

Denoting by $\varkappa$ the wave number in the Y-direction i. e. putting

$$\varkappa^2 = k^2 - \mu^2 - \frac{\tan^2\varphi}{4a^2} \quad\text{or}\quad k^2 = \varkappa^2 + \mu^2 + \frac{\tan^2\varphi}{4a^2}, \tag{4,8}$$

we obtain

$$\psi = A\,e^{\frac{\tan\varphi}{2a}y}\,y\sin\varkappa y\sin\mu x + B\,e^{\frac{\tan\varphi}{2a}y}\,y\,(\sin\sqrt{\varkappa^2+\mu^2}\,y + C\cos\sqrt{\varkappa^2+\mu^2}\,y) + \frac{\beta}{\varkappa^2+\mu^2+\frac{\tan^2\varphi}{4a^2}}y - \frac{\beta}{\left(\varkappa^2+\mu^2+\frac{\tan^2\varphi}{4a^2}\right)^2}\frac{\tan\varphi}{a}. \tag{4,9}$$

For the waves considered we will generally assume

$$\varkappa^2 + \mu^2 > \frac{4}{a^2}.$$

[1]) I had originally written a special paper where I had given the deduction and a detailed discussion of the solution (4,5) and also the corresponding solution for propagating waves. However, the same day that the papers should have been presented for print, Mr. Eliassen drew my attention to a paper by Richard A. Craig in The Journal of Meteorology [3] where the solution for propagating waves was deduced. After that this section has been rewritten to incorporate mainly those results of my discussion which I don't think has been published before. In Mr. Craig's paper the solution for propagating waves corresponding to the solution (4,10) was also deduced.

If then φ is less than about 60° we can with an error of less than about 10 % neglect the terms containing $\tan\varphi$, and obtain

$$(4,10)\quad \psi = A \sin \varkappa y \sin \mu x + B\,(\sin \sqrt{\varkappa^2+\mu^2}\, y + C \cos \sqrt{\varkappa^2+\mu^2}\, y) + \frac{\beta}{\varkappa^2+\mu^2}\, y.$$

The differential equation (4,7) reduces to

$$(4,11)\qquad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -k^2\psi + \beta y,$$

i. e. the "scale-effect" drops out. We shall in the following only discuss cases where this effect may be disregarded.

In the spesial case that the term periodic in x is independent of y, we obtain:

$$(4,12)\quad \psi = A \sin \mu x + B\,(\sin \mu y + C \cos \mu y) + \frac{\beta}{\mu^2}\, y = A \sin \mu x + B' \sin \mu\,(y+\gamma) + \frac{\beta}{\mu^2}\, y$$

where γ is also a constant. For $B' = 0$ this solution reduces to Rossby's well known stationary wave with stream-lines as shown in Fig. 8 A. For B' different from zero we get stream-lines as those shown in Fig. 8 B, where B'

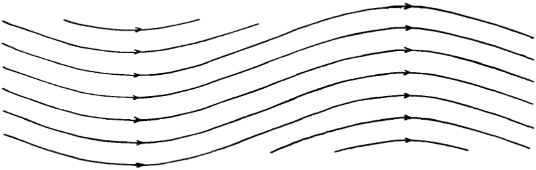

Fig. 8 A.

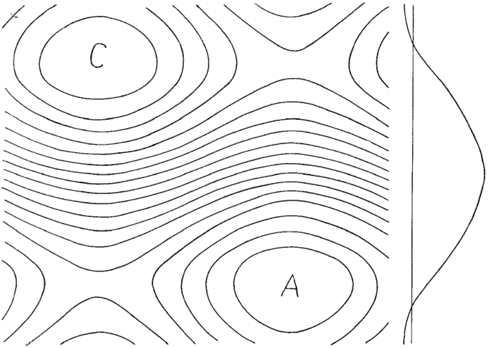

Fig. 8 B.

and γ are chosen so that the velocity profile takes the form shown at the right side of the diagram.

Putting B equal to zero, the solution (4,10) reduces to the solution given by Haurwitz [4]. When $A < \frac{\beta}{\varkappa(\varkappa^2+\mu^2)}$ we get stream-lines as those shown in Fig. 9 A. No closed stream-lines appear. When $A > \frac{\beta}{\varkappa(\varkappa^2+\mu^2)}$, we get stream-lines as those drawn in Fig. 9 B. In the northern part of the layer cyclones appear, in the southern part, at a longitude intermediate between the cyclones, anticyclones appear.

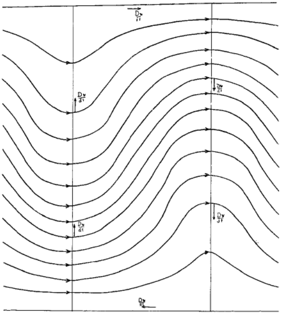

Fig. 9 A.

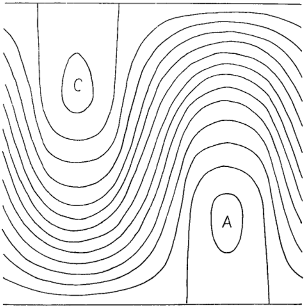

Fig 9 B.

In Fig. 10 are drawn the stream-lines in a case when B and C are different from zero. The corresponding velocity profile is again shown at the right side of the diagram. If we have no rotation, β becomes zero, and the solution (5,10) reduces to:

$$(4,13)\quad \psi = A\sin\varkappa y\sin\mu x + B\left(\sin\sqrt{\varkappa^2+\mu^2}\,y + C\cos\sqrt{\varkappa^2+\mu^2}\,y\right).$$

Putting B equal to zero we refind the solution given by Godske [5]. For B different from zero the solution is the same as that given by Fjørtoft [6]. Fjørtoft, however, developed it for

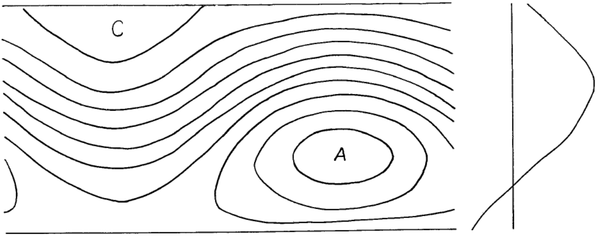

Fig 10.

infinitely small amplitudes of the disturbances of the mean flow, while our deduction shows that it is valid also for finite amplitudes.

All the solutions given above will in another system of reference represent propagating waves. Let us, as an example, consider the case that B equals zero in the expression (4,10). In a coordinate-system where the fluid has no mean motion we will then get the propagating wave given by

(4,14) $$\psi = A \sin \varkappa y \sin \mu (x + ct)$$

with

(4.15) $$c = \frac{\beta}{\mu^2 + \varkappa^2}.$$

Since c is always positive, the wave will propagate towards the west. For a wave with no closed stream-lines this fact is easily deduced from the circulation theorem. Considering the circulation of acceleration around the rectangle drawn in Fig. 9 A, we see that

$$\oint \frac{D\boldsymbol{v}}{dt} \cdot \delta \boldsymbol{r} > 0,$$

if the motion shall be stationary. Introducing this into equation (1,7) we obtain

$$\int \beta v_\varphi \, \delta\sigma < 0.$$

Hence, v_φ must within the considered closed curve be negative, i. e. directed towards south. Consequently the fluid must move towards east or the wave must be propagating westwards relative to the fluid. To get a propagation eastwards it would be necessary that the amplitudes of the stream-lines should have a sufficient increase northwards or southwards. Then at most one rectilinear stream-line would occur in the layer.

We shall now shortly consider the case that all particles have the same absolute vorticity. Then equation (1,11) is valid not only individually, but also spatially. Making the same assumptions as above, we then get

(4,16) $$\nabla^2 \psi = -\zeta_0 + \beta y,$$

where ζ_0 now is a constant.

Equation (4,16) has a solution of the form

$$\psi = X(x) + Y(y)$$

given by

(4,17) $$\psi = -\frac{\eta}{2} x^2 + \frac{\beta}{6} y^3 - \frac{\zeta_0 - \eta}{2} y^2,$$

where η is a constant of integration. The other constants of integration are determined in such a way that v_x and v_y vanish at the point $x = 0$, $y = 0$. The stream-lines are drawn in Fig. 11.

The corresponding pressure is given by

(4,18) $$p = p_0 + q(\zeta_0 - \eta + 2\Omega_{z_0}) \frac{\eta}{2} x^2 + \frac{\beta}{6}(2\zeta_0 - 3\eta - 2\Omega_{z_0}) y^3 + \frac{\zeta_0 - \eta}{2}(\eta + 2\Omega_{z_0}) y^2$$

where we have dropped a term $-\frac{\beta^2}{6} y^4$. From this formula it appears that the wind will be approximately geostrophic if the relative vorticity is small compared to $2\Omega_{z_0}$.

Equation (4,16), which in the considered case is valid also for non-stationary motions, is fulfilled also for

(4,19) $$\psi = -\frac{\eta}{2}(x - ct)^2 + \frac{\beta}{6} y^3 - \frac{\zeta_0 - \eta}{2} y^2,$$

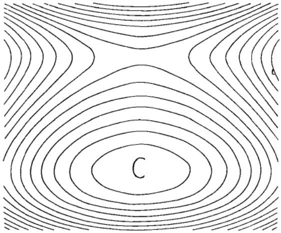

Fig. 11 A

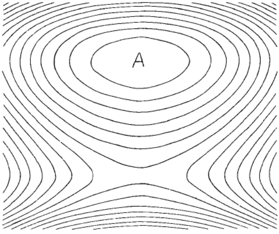

Fig 11 B.

giving a propagation of the stream-lines with an arbitrary velocity c. The velocity of propagation c may be positive or negative, giving eastwards as well as westwards propagation. This is connected with the fact that our stream-line pattern does not fulfill the conditions stated above (p. 16) for waves with purely westward propagation.

Corresponding to the solution (4,19) we get the pressure (4,18) with the addition of a term given by

$$c\,\eta\,y.$$

Thus, an additional pressure gradient towards the north corresponds to a westward propagation, while an additional pressure gradient towards the south corresponds to an eastward propagation, in accordance with what we deduced in section 3.

The stream functions

$$(4,20)\quad \psi = A \sinh \mu y \sin \mu x + \frac{\beta}{6} y^3 - \frac{\zeta_0}{2} y^2$$

and

$$(4,21)\quad \psi = A \sinh \mu y \sin \mu (x - ct) + \frac{\beta}{6} y^3 - \frac{\zeta_0}{2} y^2,$$

will also fullfill equation (4,16). The corresponding stream-line pattern is drawn in Fig. 12.

The solutions given above for constant absolute vorticity cannot exist as single solutions in all space since the velocity will increase infinitely with increasing y (for the solutions (4,17) and (4,19) also for increasing x). They can therefore only claim to represent a solution in a region limited in the south-north direction and for the solutions (4'17) and (4,19) also limited in the east-west direction. Outside the boundaries enclosing the limited region, other conditions must prevail. The vorticity may, for instance, undergo a sudden change at the boundaries.

We now pass to the discussion of the solution (4,5). By a suitable choice of zero meridian $\Theta = 0$, the solution for a definite value of m may be brought to the form

$$(4,22)\quad \psi = A_n^m P_n^m(\sin\varphi) \sin m\,\Theta + C_n P_n(\sin\varphi) + \frac{2\,\Omega\,a^2}{n\,(n+1)-2} \sin\varphi.$$

We shall discuss this solution separately in the case that the motion is observed in a coordinate system with no rotation, $\Omega = 0$, and in the case that the coordinate system has a rotation, $\Omega \neq 0$.

a) $\Omega = 0$. The solution (2,8) then reduces to

$$(4,23)\quad \psi = A_n^m P_n^m(\sin\varphi) \sin m\,\Theta + C_n P_n(\sin\varphi).$$

In the spesial case that

$$C_n = 0,$$

we obtain

$$(4,24)\quad \psi = A_n^m P_n^m(\sin\varphi) \sin m\,\Theta,$$

giving stationary circulations within cells formed

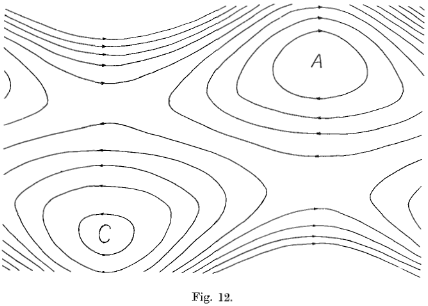

Fig. 12.

by latitude circles determined from the solutions of the equation

$$(4,25)\qquad P_n^m(\sin\varphi) = 0,$$

and longitude circles determined by

$$(4,26)\qquad \sin m\,\Theta = 0.$$

The equation (4,25) has $n - m$ solutions between 1 and -1, while the function on the left-hand side of equation (4,26) has $2m$ zero points around a latitude circle. Thus the solution (4,24) gives

$$(4,27)\qquad i = 2m\,(n - m + 1)$$

cells on the sphere. Fig. 13 illustrates the cellular motions when $n = 5$, $m = 2$. (In the diagrams we use the stereographic polar projection.) If $n - m$ is an odd number, the equator is a stream-line.

The associated Legendre functions up to degree 5 are:

$$
(4,28)\qquad
\begin{aligned}
&P_1^1 = \cos\varphi,\\
&P_2^1 = 3\sin\varphi\cos\varphi,\quad P_2^2 = -3\,(\sin^2\varphi - 1),\\
&P_3^1 = \tfrac{3}{2}\cos\varphi\,(5\sin^2\varphi - 1),\\
&P_3^2 = -15\sin\varphi\,(\sin^2\varphi - 1),\\
&P_3^3 = -15\cos\varphi\,(\sin^2\varphi - 1),\\
&P_4^1 = \tfrac{5}{2}\sin\varphi\cos\varphi\,(7\sin^2\varphi - 3),\\
&P_4^2 = -\tfrac{15}{2}\,(7\sin^4\varphi - 8\sin^2\varphi + 1),\\
&P_4^3 = -105\sin\varphi\cos\varphi\,(\sin^2\varphi - 1),\\
&P_4^4 = 105\,(\sin^2\varphi - 1)^2,\\
&P_5^1 = \tfrac{15}{8}\cos\varphi\,(21\sin^4\varphi - 14\sin^2\varphi + 1),\\
&P_5^2 = -\tfrac{105}{2}\sin\varphi\,(3\sin^4\varphi - 4\sin^2\varphi + 1),\\
&P_5^3 = -\tfrac{105}{2}\cos\varphi\,(9\sin^4\varphi - 10\sin^2\varphi + 1),\\
&P_5^4 = 945\sin\varphi\,(\sin^2\varphi - 1)^2,\\
&P_5^5 = 945\cos\varphi\,(\sin^2\varphi - 1)^2.
\end{aligned}
$$

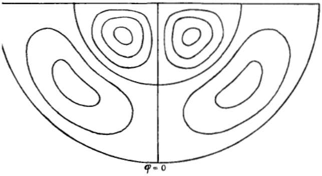

Fig. 13.

The velocity components are given by

$$
(4,29)\qquad
\begin{aligned}
v_\Theta &= \frac{1}{a}\frac{\partial\psi}{\partial\varphi} = \frac{A_n^m}{a}\frac{d}{d\varphi}P_n^m(\sin\varphi)\sin m\,\Theta,\\
v_\varphi &= -\frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\Theta} = -\frac{mA_n^m}{a\cos\varphi}P_n^m(\sin\varphi)\cos m\Theta.
\end{aligned}
$$

It is easily seen that for $m > 1$ v_Θ and v_φ will vanish at the poles, i. e. for $\varphi = \pm\frac{\pi}{2}$. But for $m = 1$ we will get finite velocities at the poles. For $n = 1$ and $m = 1$ we obtain for instance:

$$(4,30)\qquad v_\Theta = -\frac{A_1^1}{a}\sin\varphi\sin\Theta,\quad v_\varphi = -\frac{A_1^1}{a}\cos\Theta,$$

giving at the poles

$$(4,31)\qquad v_{\Theta_{\varphi=\pm\frac{\pi}{2}}} = \mp\frac{A_1^1}{a}\sin\Theta,\quad v_{\varphi_{\varphi=\pm\frac{\pi}{2}}} = -\frac{A_1^1}{a}\cos\Theta.$$

The finite velocities at the poles for $m = 1$ is possible, because in this case the poles do not form an edge for the cellular motion since the "length" of the cells is equal to π. For $m > 1$ the poles form edges for the cells, and v_Θ and v_φ must therefore nec essarily vanish at these points.

When C_n is different from zero we have the solution (4,22). The Legendre polynomials up to degree 5 are

$$
(4.32)\qquad
\begin{aligned}
&P_1 = \sin\varphi,\quad P_2 = \tfrac{1}{2}\,(3\sin^2\varphi - 1),\\
&P_3 = \tfrac{1}{2}\sin\varphi\,(5\sin^2\varphi - 3),\\
&P_4 = \tfrac{1}{8}\,(35\sin^4\varphi - 30\sin^2\varphi + 3),\\
&P_5 = \tfrac{1}{8}\sin\varphi\,(63\sin^4\varphi - 70\sin^2\varphi + 15).
\end{aligned}
$$

Assuming for instance $n = 3$, we get the following solutions:

$$
(4,33)\qquad
\begin{aligned}
\psi_3^1 &= \tfrac{3}{2}A_3^1\cos\varphi\,(5\sin^2\varphi - 1)\sin\Theta + \frac{C_3}{2}\sin\varphi\,(5\sin^2\varphi - 3),\\
\psi_3^2 &= -15\,A_3^2\sin\varphi\,(\sin^2\varphi - 1)\sin 2\,\Theta + \frac{C_3}{2}\sin\varphi\,(5\sin^2\varphi - 3),\\
\psi_3^3 &= -15\,A_3^3\cos\varphi\,(\sin^2\varphi - 1)\sin 3\,\Theta + \frac{C_3}{2}\sin\varphi\,(5\sin^2\varphi - 3).
\end{aligned}
$$

If the coefficient of $\sin m\,\Theta$ is zero on a latitude circle $\varphi = \varphi_0$, this latitude circle will be a stream-line. The number of zeros of P_n^m between $\sin\varphi = -1$ and $\sin\varphi = 1$ therefore determines the number of stream-lines coinciding with a latitude circle (circular stream-lines) between the poles. The function ψ_3^1 gives two (for $\varphi = \pm \text{arc}\sin\sqrt{\frac{1}{5}} = \pm 26{,}^\circ 6$), the function ψ_3^2 one (the equator) and the function ψ_3^3 no circular stream-lines. The stream-lines corresponding to the first and last of the solutions (4,33) are drawn in fig. 14 A and B. Where the velocity in the mean flow is zero, i. e. for $\varphi = \pm 26{,}^\circ 6$ we get cats-eyes. In the two last cases, cats-eyes resembling those studied by Lord Kelvin [1] and in the first case "cats-eyes" resembling those studied by G. I. Taylor [7]. In this last case the velocity in the mean flow is zero where the velocity v_φ is zero. From the general expression for the velocity components, viz.

$$(4{,}34)\quad \begin{aligned} v_\Theta &= \frac{A_n^m}{a}\frac{d}{d\varphi}P_n^m(\sin\varphi)\sin m\,\Theta + \frac{C_n}{a}P_n^1, \\ v_\varphi &= -\frac{m\,A_n^m}{a\cos\varphi}P_n^m(\sin\varphi)\cos m\,\Theta, \end{aligned}$$

we see that for $m = 1$, we will always get rows of "cats-eyes" of the Taylor type, the number

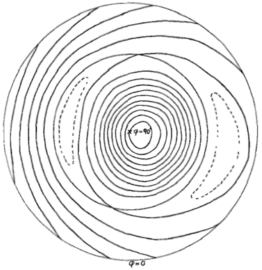

Fig. 14 A.

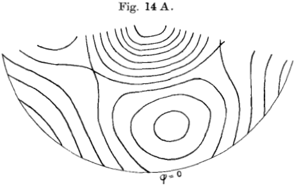

Fig. 14 B.

of such rows being equal to $n-1$. For $m>1$ we will also get $n-1$ rows of cats-eyes at the latitudes where P_n^1 is zero, now, however, not of the Taylor type.

b) $\Omega \neq 0$. When Ω is different from zero, we have the solution (4,22). This solution has a singularity for $n = 1$ (the last term becomes infinite) unless in this case $\Omega = 0$. (This singularity is due to the fact that a propagating permanent wave corresponding to the solution (4,22) will for $n = 1$ have a velocity of propagation equal to $-\Omega$.) Thus for $n = 1$ we must return to the solution (4,23). In the following we therefore assume

$$n > 1.$$

If C_n is put equal to zero, the solution (4,22) reduces to

$$(4{,}35)\quad \psi = A_n^m P_n^m(\sin\varphi)\sin m\,\Theta + \frac{2\,\Omega\,a^2}{n\,(n+1)-2}\sin\varphi.$$

The relative mean flow is now a constant rotation with the angular velocity

$$(4{,}36)\qquad \omega = \frac{2\,\Omega}{n\,(n+1)-2},$$

so that the mean flow in absolute motion is a rotation with the angular velocity

$$(4{,}37)\quad \Omega_0 = \Omega + \omega = \Omega\,\frac{n\,(n+1)}{n\,(n+1)-2}.$$

The relative angular velocity for n up to 10 is given in the following table:

n	2	3	4	5	6	7	8	9	10
ω	$\frac{\Omega}{2}$	$\frac{\Omega}{5}$	$\frac{\Omega}{9}$	$\frac{\Omega}{14}$	$\frac{\Omega}{20}$	$\frac{\Omega}{27}$	$\frac{\Omega}{35}$	$\frac{\Omega}{44}$	$\frac{\Omega}{54}$

In Fig. 15 A and B we have drawn the stream-lines (one wave length) on the northern (or southern) hemisphere for the case that $n = 5$, $m = 3$, i. e.

$$\psi = -\tfrac{105}{8}A_5^3\cos\varphi\,(9\sin^4\varphi - 10\sin^2\varphi + 1)\sin 3\Theta + \frac{\Omega a^2}{14}\sin\varphi.$$

Fig. 15 A corresponds to a relatively small value of the constant A_5^3. Fig. 15 B corresponds to a relatively great value of the constant A_5^3.

Fig. 15 A.

Fig 15 B.

The number of circular stream-lines (stream-lines coinciding with latitude circles) is

$$n - m = 2.$$

We notice that the smaller m, i. e. the greater the wave-length, the more circular stream-lines are obtained.

If C_n is different from zero, we have the solution (4,22). Now, if the mean motion shall be symmetric with regard to the equator, n must be an odd number. The possible values for ω up to $n = 9$ is therefore

n	3	5	7	9
ω	$\frac{\Omega}{5}$	$\frac{\Omega}{14}$	$\frac{\Omega}{27}$	$\frac{\Omega}{44}$

The zonal velocities v_{0_0} at the earth's equator corresponding to these values of ω are directed towards the east and have the magnitudes

n	3	5	7	9	
v_{0_0}	93	33	17	10	m/sek

The mean flow of the atmosphere has in general two points at each hemisphere where the mean zonal velocity vanishes. The value of n giving a mean flow of that character is

$$n = 5.$$

Further, to get easterlies at equator and westerlies at temperate latitudes, the constant C_n must be negative. Putting C_n equal to $64a/3$, we get an easterly wind at the equator of strength 7 m sek^{-1}, and the solution

$$\psi = A_5^m P_5^m(\sin\varphi)\sin m\Theta$$
$$-\frac{8a}{3}\sin\varphi\,(63\sin^4\varphi - 70\sin^2\varphi + 15) + \frac{\Omega a^2}{14}\sin\varphi.$$

Assuming no transport across the equator, i. e. assuming $v_\varphi = 0$ for $\varphi = 0$, m must be set equal to either 2 or 4. $m = 2$ gives only two wave-lengths around the latitude circles. Since we have always more than two quasi-stationary highs at low latitudes, we put

$$m = 4,$$

and obtain the solution:

$$(4{,}38)\quad \psi = 945\,A_5^4\sin\varphi\,(\sin^2\varphi - 1)^2\sin 4\Theta$$
$$-\frac{8a}{3}\sin\varphi\,(63\sin^4\varphi - 70\sin^2\varphi + 15) + \frac{\Omega a^2}{14}\sin\varphi.$$

The velocity profile for the mean flow corresponding to this solution is shown in Fig. 16. At the equator we have as mentioned above a westward velocity of magnitude 7 m sek^{-1}. At about $\varphi = 7°$ and $\varphi = 55°$ we have no mean zonal velocities. At about $\varphi = 33°$ we have maximum velocity towards the east (westerly wind) amounting to 71 m sek^{-1} and at about $\varphi = 72°$ a maximum

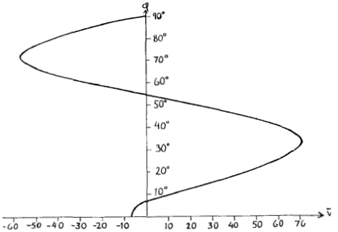

Fig. 16.

velocity towards the west (easterly wind) amounting to 57 m sek^{-1}. We see that by assuming weak easterlies at the equator, the westerlies around 30° latitude in the upper part of the troposphere are qualitatively correctly reproduced, but in addition we get strong easterlies around the poles, and these strong easterly winds do not fit with observations. A smaller value of the constant C_n giving westerlies at equator would perhaps fit better to the observation in the upper part of the troposphere. Our model is, however, so simple, involving so many assumptions which we know are not approximately fulfilled in the atmosphere that one should not expect a too close agreement with observations.

In Fig. 17 is shown the stream-line pattern corresponding to the solution (4,38). At either side of the equator we get four cells of high pressure (anticyclones), the centres of which, with our choise of the constant A_5^4, will be situated at about $\pm 15°$ latitude. Further we get on each hemisphere four cells of low pressure (cyclones) with centres at about $\pm 50°$ atitudes. Qualitatively this corresponds to what is observed for the mean average motion in the atmosphere.

The strong westerlies at middle latitudes and the strong easterlies at high latitudes are caused by the large value of the last term in equation (4,22) for $n = 5$. If by the action of friction between earth and air this term is essentially reduced, i. e. if the last term in equation (4,4) is essentially reduced what would very likely be a consequence of the frictional effect, the westerlies and easterlies would be correspondingly reduced. Then our velocity profile would give a fair representation of the zonal circulation observed at the surface of the earth. In this layer our assumption of horizontal motion should also be almost exactly fulfilled. Of course, also in the free atmosphere

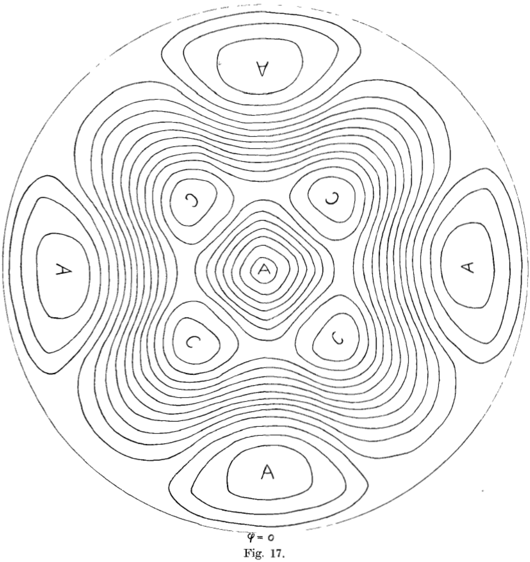

Fig. 17.

a great turbulent friction must be acting which would in due time destroy the motion. To compensate this effect some sort of "instability" arising from the absorption and emission of heat must occur. If the turbulent friction had not occurred, this instability would produce an intensification of cyclones and anticyclones. It is very probable that such an instability effect will always be connected with a change of phase with latitude of the waves (compare section 6). Thus we may consider as one primary effect of the instability which is necessary to compensate the effect of turbulent friction, a change in phase of the waves. This phase-change will in general alter the longitudinal position of the anticyclones relative to the cyclones, so that the anticyclones will no longer be situated on longitudes midway between the cyclones.

5. Consequences of the Impulse Theorem.

The impulse theorem for horizontal motion of a limited part of the fluid of unit thickness (in the direction perpendicular to the XY planes) takes the form:

$$\frac{d\boldsymbol{I}}{dt} = -q\left[\boldsymbol{i}\iint_{\Sigma} 2\,\Omega_z \frac{\partial\psi}{\partial x}\,\delta x\,\delta y + \boldsymbol{j}\iint_{\Sigma} 2\,\Omega_z \frac{\partial\psi}{\partial y}\,\delta x\,\delta y\right] - \boldsymbol{i}\oint_L p\delta y + \boldsymbol{j}\oint_L p\,\delta x,$$

where $\boldsymbol{I}$ is the impulse, Σ is the horizontal area of the considered part of the fluid, L the closed curve enclosing the area Σ, and $\boldsymbol{i}$ and $\boldsymbol{j}$ the unit vectors along the X- and Y-axes respectively. The expression for $\frac{d\boldsymbol{I}}{dt}$ may further in the approximation introduced in the preceding section (disregarding the curvature of the earth) be written:

$$\frac{d\boldsymbol{I}}{dt} = -q\left[2\Omega_{z_0}\boldsymbol{i}\iint_{\Sigma}\frac{\partial\psi}{\partial x}\,\delta x\,\delta y + \beta\boldsymbol{i}\iint_{\Sigma} y\frac{\partial\psi}{\partial x}\,\delta x\,\delta y + 2\,\Omega_{z_0}\boldsymbol{j}\iint_{\Sigma}\frac{\partial\psi}{\partial y}\,\delta x\delta y + \beta\boldsymbol{j}\iint_{\Sigma} y\frac{\partial\psi}{\partial y}\,\delta x\,\delta y\right] - \boldsymbol{i}\oint_L p\delta y + \boldsymbol{j}\oint_L p\delta x,$$

or

$$(5{,}1)\quad \frac{d\boldsymbol{I}}{dt} = \boldsymbol{i}\left\{\oint_L [q\,(2\,\Omega_{z_0} + \beta y)\,\psi - p]\,\delta y\right\} + \boldsymbol{j}\left\{\oint_L [q\,(2\,\Omega_{z_0} + \beta y)\,\psi - p]\,\delta x + q\iint_{\Sigma}\psi\,\delta x\delta y\right\}.$$

If the curve L is a stream-line, we can assume $\psi = 0$ at this stream-line, so that in this case the expression reduces to

$$(5{,}2)\quad \frac{d\boldsymbol{I}}{dt} = -\boldsymbol{i}\oint_L p\delta y - \boldsymbol{j}\left\{\oint_L p\delta x - q\iint_{\Sigma}\psi\,\delta x\,\delta y\right\}.$$

The last term here is the expression for the forces on an isolated vortex given by Rossby [6] and Davies [7]. They assume p to be a constant on L, so that the resultant of the pressure forces drops out, and we obtain simply

$$(5{,}3)\qquad \frac{d\boldsymbol{I}}{dt} = \boldsymbol{j}q\iint_{\Sigma}\psi\,\delta x\,\delta y.$$

As shown by the above mentioned authors the impulse will for a cyclone get an increase directed towards the north, and for an anticyclone an increase directed towards the south. The acting forces is now only the Coriolis forces, and since the Coriolis force for the same velocity is greater the farther north the particle is situated, the result is quite obvious.

As long as we consider L to be a stream-line, we have no transport of momentum out of the considered region, so that

$$(5{,}4)\quad \frac{d\boldsymbol{I}}{dt} = \frac{\partial\boldsymbol{I}}{\partial t} = -\boldsymbol{i}\oint_L p\delta y - \boldsymbol{j}\left\{\oint_L p\delta x - q\iint_{\Sigma}\psi\,\delta x\,\delta y\right\},$$

where $\frac{\partial\boldsymbol{I}}{\partial t}$ is the locally determined variation per unit time of the impulse. If the motion shall be stationary, this variation must vanish, i. e. we must have

$$(5{,}5)\qquad \oint_L p\delta y = 0,\quad \oint_L p\delta x = q\iint_{\Sigma}\psi\,\delta x\,\delta y.$$

From this equation it is seen that the "Rossby force" gives a measure for the departures from geostrophic wind which are necessary to keep the motion stationary. We see that in a cyclone the pressure must on a stream-line be greatest in the north, smallest in the south, while the opposite is true in an anticyclone.

The results deduced above are in accordance with those deduced in section 2 for the geostrophic departures in stationary cyclones and anticyclones.

6. Instability Waves.

From equation (1,9) we obtain the linearized equation

$$(6,1)\quad \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2\psi+\left(\beta-\frac{d^2U}{dy^2}\right)\frac{\partial\psi}{\partial x}=0,$$

neglecting terms of the second and higher orders, and utilizing the simplification introduced in section 4. U is the velocity in the undisturbed zonal flow, ψ the stream function of the perturbations.

In the following we shall confine ourselves to the case that U is a constant, say U_0. Equation (6,1) then assumes the form

$$(6,2)\quad \left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)\nabla^2\psi+\beta\frac{\partial\psi}{\partial x}=0.$$

We shall now assume that we have some source of instability outside the considered layer, and investigate the kinematics of the instability waves then produced in the layer. Two cases shall be discussed: 1. That the perturbation component v_x is zero for $y=0$, and 2. that the perturbation component v_y is zero for $y=0$.

It is easily verified that

$$(6,3)\quad \psi=Ae^{\sigma t}[\cosh\lambda y\cos\varkappa y\sin\mu(x+ct) - \sinh\lambda y\sin\varkappa y\cos\mu(x+ct)]$$

is a solution of equation (6,2), and that it fulfills the condition

$$v_x=\frac{\partial\psi}{\partial y}=0 \text{ for } y=0.$$

We find

$$(6,4)\quad \sigma=\frac{2\mu\lambda\varkappa}{(\mu^2+\varkappa^2-\lambda^2)^2+4\lambda^2\varkappa^2}\beta,$$
$$c+U_0=\frac{\mu^2+\varkappa^2-\lambda^2}{(\mu^2+\varkappa^2-\lambda^2)^2+4\lambda^2\varkappa^2}\beta.$$

The phase α of the wave is given by

$$(6,5)\quad \tan\alpha=-\tanh\lambda y\ \tan\varkappa y,$$

from which we obtain

$$(6,6)\quad \frac{1}{\cos^2\alpha}\frac{d\alpha}{dy}= -\frac{\lambda\varkappa y}{\cosh^2\lambda y\cos^2\varkappa y}\left[\frac{\sinh 2\lambda y}{2\lambda y}+\frac{\sin 2\varkappa y}{2\varkappa y}\right].$$

The term within the brackets is positive definite, so that we may write, since we without loss of generality may assume $\varkappa>0$,

$$(6,7)\quad \frac{d\alpha}{dy}=-k^2\lambda y.$$

From the first of equations (6,4) we see that $\lambda>0$ gives $\sigma>0$, i. e. a wave with increasing amplitude.

Hence, for a growing perturbation we get

$$(6,8)\quad \frac{d\alpha}{dy}<0 \text{ for } y>0,\ \frac{d\alpha}{dy}>0 \text{ for } y<0,$$

while for a wave with decreasing amplitude, $\lambda<0$, we obtain

$$(6,9)\quad \frac{d\alpha}{dy}>0 \text{ for } y>0,\ \frac{d\alpha}{dy}<0 \text{ for } y<0.$$

The stream function

$$(6,10)\quad \psi=Ae^{\sigma t}[\cosh\lambda y\sin\varkappa y\sin\mu(x+ct) + \sinh\lambda y\cos\varkappa y\cos\mu(x+ct)]$$

also fulfills equation (6,2). For this streamfunction we get

$$v_y=-\frac{\partial\psi}{\partial x}=0 \text{ for } y=0.$$

The equations (6,4) is also valid in this case. The phase is given by

$$(6,11)\quad \tan\alpha=\tanh\lambda y\ \cot\varkappa y,$$

from which

$$(6,12)\quad \frac{1}{\cos^2\alpha}\frac{d\alpha}{dy}= -\frac{\lambda\varkappa y}{\cosh^2\lambda y\sin^2\varkappa y}\left[\frac{\sinh 2\lambda y}{2\lambda y}-\frac{\sin 2\varkappa y}{2\varkappa y}\right],$$

is obtained. Again the expression within the brackets is positive definite, so we get the same results for the sign of $\frac{d\alpha}{dy}$ as in the first case. In both cases we will for growing perturbations for positive y have stream-lines as drawn in Fig. 18 A, and for negative y stream-lines as drawn in Fig. 18 B.

It is reasonable to assume that these laws for the phase, deduced here for small disturbances, will hold good also for finite disturbances. Then assuming a kinematics of one of the kinds

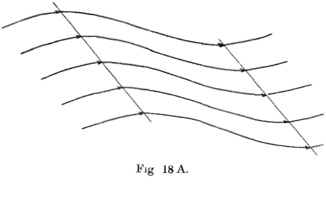

Fig 18 A.

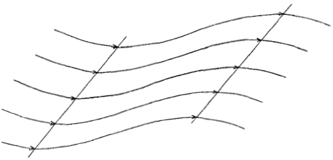

Fig 18 B

discussed above, we would for growing (deepening) cyclones, for instance, get an asymmetry in the east-west direction as shown in Fig. 19 A for $y > 0$ and in Fig. 19 B for $y < 0$. For a weakening (filling) cyclone we should get the opposite assymmetry.

The asymmetry obtained is of course dependent on the somewhat arbitrarily chcsen

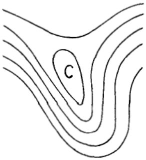

Fig. 19 A.

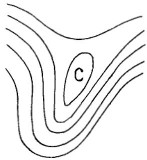

Fig. 19 B.

kinematic constraints for $y = 0$. I think, however, that the method used above seems to indicate a relationship between deepening and filling of cyclones and the asymmetry in west-east direction of the stream-line pattern.

7. Waves in a Rectangular Basin.

We have in section 2 drawn attention to the fact that the induced vorticity can produce no standing oscillations. In this section we shall investigate what kind of periodic motion we will have in a rectangular basin. We put U_0 equal to zero in equation (6,2), obtaining

$$\frac{\partial}{\partial t} \nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} = 0. \tag{7,1}$$

This equation is fulfilled for

$$\psi = A \sin \varkappa y \sin \gamma x \sin \mu (x + ct), \tag{7,2}$$

with

$$\gamma = \sqrt{\mu^2 - \varkappa^2} \text{ and } c = \frac{\beta}{2\mu^2}. \tag{7.3}$$

If our rectangle has sides given by

$$y = 0,\ y = H,\ x = 0,\ x = L,$$

we must have

$$\varkappa H = n\pi,\ a L = \sqrt{\mu^2 - \varkappa^2}\, L = m\pi,$$

where n and m are integers. From these equations we obtain for the wavelength

$$l = \frac{2HL}{\sqrt{n^2 L^2 + m^2 H^2}}. \tag{7,4}$$

This equation determines the waves which can exist in the basin. They correspond to and are a generalization of the ordinary standing waves met with in the theory of gravitational and inertia waves. The waves are propagated towards the west with a velocity of propagation equal to half of the velocity of propagation of the Rossby waves.

The possible wave-lengths which can occur are easily found. Put for instance $n = 1$ and $m = 1$. Then

$$l = \frac{2HL}{D}, \tag{7,5}$$

where D is the diagonal of the rectangular basin. l is found by a simple construction, see Fig. 20 A. This gives the greatest wave-length that can occur. The velocity of propagation of the wave is given by

$$c = \frac{\beta}{2\pi^2} \frac{H^2 L^2}{D^2}. \tag{7,6}$$

For $n = 1$ and $m = 2$, the construction of the wave-length is shovn in Fig. 20 B, and for $n = 2$ and $m = 1$, the construction is shown in Fig. 20 C. It is easily seen how the other possible l-values may be constructed.

The ordinary standing oscillations may emerge from superposition of two ordinary propagating waves. So also for the generalized "standing oscillations" considered here. It is easily verified that we have

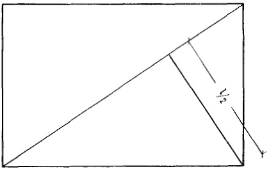

Fig. 20 A.

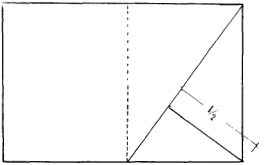

Fig. 20 B.

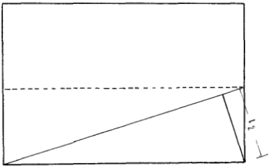

Fig. 20 C.

(7,7) $A \sin \varkappa y \sin \gamma x \sin \mu (x + ct)$
$= \frac{1}{2} A \sin \varkappa y [\cos \mu_1 (x + c_1 t) - \cos \mu_2 (x + c_2 t)]$
where

$$\mu_1 = \mu - \gamma, \quad c_1 = \frac{\beta}{(\mu - \gamma)^2 + \varkappa^2} = \frac{\beta}{\mu_1^2 + \varkappa^2},$$

(7,8) $$\mu_2 = \mu + \gamma, \quad c_2 = \frac{\beta}{(\mu + \gamma)^2 + \varkappa^2} = \frac{\beta}{\mu_2^2 + \varkappa^2}$$

and $\gamma^2 = \mu^2 - \varkappa^2$,

relations which in section 4 are developed for ordinary propagating waves of the type discovered by Haurwitz.

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