ON THE SCALE OF ATMOSPHERIC MOTIONS

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In a recent publication entitled *The Dynamics of Long Waves in a Baroclinic Westerly Current*[1] (1947) the writer pointed out that, in the study of atmospheric wave motion, the problem of integration is greatly complicated by the simultaneous existance of a discrete set of wave motions all of which satisfy the conditions of the problem, namely that the motion be simple-harmonic and of a specified wave-length. Whereas only the long inertially-propagated waves are important for the study of large-scale weather phenomena, one is forced by the generality of the equations of motion to contend with each of the theoretically possible wave types. This extreme generality whereby the equations of motion apply to the entire spectrum of possible motions — to sound waves as well as to cyclone waves — constitutes a serious defect of the equations from the meteorological point of view. It means that the investigator must take into account modifications to the large-scale motions of the atmosphere which are of little meteorological importance and which only serve to make the integration of the equations a virtual impossibility.

One does not encounter difficulties of this kind in other branches of applied hydrodynamics, where the special characteristics of the motions dealt with are used as a means for simplifying the basic equations. For example, the fundamental equations of aerodynamics have been considerably simplified by the introduction of the incompressibility, homogeneity, and boundary layer approximations.

The successful procedure of such related sciences suggests that a corresponding set of simplifying principles, characterizing the meteorologically significant motions, can be utilized to "filter out the noise" from the meteorological equations. In the search for such a set of principles, one is guided by the experience of synoptic meteorologists who have found that the weather producing motions of the free atmosphere can be characterized as quasi-hydrostatic, quasi-adiabatic, quasi-horizontal, and quasi-geostrophic. But here one encounters difficulties; although the first three approximations can be introduced without difficulty, no acceptable method has been proposed for using the geostrophic approximation in dynamic analysis. Instead, one can point to instances in which this approximation breaks down, such as in the application to the calculation of pressure changes. Nevertheless it was found in DLW that the use of the geostrophic approximation in *certain terms* of the equations of motion has just the effect of filtering out the meteorologically insignificant wave solutions.

The method of simplification which was employed in the special case of wave motion has been extended in the present paper to apply to the most general large-scale motions. It will be shown how the geostrophic approximation, together with the other three approximations mentioned above, can be incorporated into the general equations of motion to obtain a dynamically consistent set of simplified equations applicable to all large-scale motions.

But if the present theory is to be free of inconclusive empirical elements, a means of estimating the accuracy of the approximations used must first be given; in particular, one must demonstrate the validity of the geostrophic

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[1] Hereafter referred to by the abbreviation DLW.

approximation in a manner which will be acceptable to those meteorologists who have questioned its applicability. In the first section of this work an attempt is therefore made to formulate an adequate theory of meteorological approximations.

Since the meteorologically significant motions are distinguished from all other types of atmospheric motion only by a great difference in scale, it is clear that any attempt to justify the peculiarly meteorological approximations must take into account the scale of the motion. The present theory is therefore based on a kind of dimensional analysis similar to that used in the boundary layer theory of aerodynamics.[2] In the latter theory, the motion is characterized by a length parameter and a velocity parameter in terms of which the orders of magnitude of the individual terms in the equations of motion are evaluated. In the present theory, the parameters are chosen to characterize the horizontal and vertical scales of the motion, the speed of propagation of the streamline pattern, the horizontal particle speed, and the internal static stability. It is then shown that the geostrophic deviation is negligible for those disturbances whose characteristic frequency is small compared to the frequency of an horizontal inertial oscillation, i.e., for the primary large-scale perturbations of the atmosphere.

A further consequence of the theory is that the terms comprising the horizontal divergence in rectangular coordinates compensate in such a way that they are individually one order of magnitude larger than the horizontal divergence itself. It is this circumstance that makes it impossible to evaluate the horizontal divergence by means of the geostrophic wind; for the error incurred thereby would have the same order of magnitude as the horizontal divergence itself. But, if the horizontal divergence is excluded, it may be shown that the geostrophic wind can be used to approximate the horizontal velocity field in all other terms in the equations of motion. Hence if the equations of motion are so transformed as to eliminate the horizontal divergence both implicitly and explicitly, the geostrophic approximation can be applied to derive a dynamically consistant simplification of the equations of motion.

[2] See, for example, Goldstein (1938).

The elimination of the horizontal divergence and the application of the geostrophic, hydrostatic, adiabatic and quasi-horizontal approximations yield a set of equations which express the following physical principle: The large-scale motion of the atmosphere is governed by the laws of conservation of potential temperature (or wet-bulb potential temperature) and absolute potential vorticity, and by the conditions that the motion be in hydrostatic and geostrophic equilibrium. Thus the conservation equations of potential temperature and absolute potential vorticity, together with the hydrostatic and geostrophic equations, form a closed, mutually consistant, dynamical system which applies only to the meteorologically significant motions and is therefore free of the defect of too great generality. By way of illustration it is shown that the simplified system *does* filter out the meteorologically insignificant wave components from the wave equations for barotropic and baroclinic motion.

These results also have an important application to the problem of numerical integration. The difficulty that has attended this problem so far is the practical impossibility of evaluating the initial distributions of horizontal acceleration and horizontal divergence with sufficient accuracy. But if the simplified equations are used, the integration presupposes only a knowledge of the initial pressure field, a field which is given directly by available radiosonde data.

The theory of approximations offered here contains a justification of the rule that the individual time derivative of density is due almost entirely to the vertical motion. This rule furnishes the basic reason for the failure of the tendency equation to serve as a means for calculating pressure changes; it implies that the local pressure change at the ground is a small difference between two large quantities — the total positive and total negative horizontal mass divergence — whose values cannot be evaluated with sufficient accuracy from observations. However, if the tendency equation is regarded, not as an instrument for calculating pressure tendencies, but as a statement of the approximate balance between the total horizontal mass convergence and divergence, it may be converted into a useful tool for calculating the speed of systems by the simple expedient of evaluating the horizontal divergence in terms of the individual time derivative of the vertical vorticity component. If this method is applied to small amplitude wave motions, it leads to the known results of Rossby (1939), Haurwitz (1940), and Holmboe (1945) for a barotropic atmosphere, and to those of the writer (1947) for a baroclinic atmosphere. But that the method is quite general is shown by its application to the calculation of the velocity of propagation of the large amplitude cyclone wave containing closed streamlines at low levels.

1. A THEORY OF METEOROLOGICAL APPROXIMATIONS.

We shall let x be the west to east distance measured along a fixed latitude from a fixed meridian to the meridian through the variable point, y the south to north distance measured along the fixed meridian from the fixed lätitude to the latitude through the measured point, and z the vertical distance measured upwards. Then in order to avoid unnecessary geometrical complications in the analysis, we shall suppose that the equations of motion in this curvilinear system take the same form as in a rectangular system whose x and y axes are tangent respectively to the fixed latitude and meridian at their point of intersection, and whose z axis is directed vertically upwards from this point. This approximation ignores the influence of curvature on the motion but not the variation with latitude of the coriolis parameter. Whereas the neglect of curvature produces only a certain distortion in the kinematics of the flow, which is of minor importance except for extremely large-scale motions, the variability of the coriolis parameter is essential for the explanation of the local dynamics of the motion. It can be shown that to ignore the variation of the coriolis parameter in a barotropic atmosphere is virtually equivalent to ignoring the effect of the earth's rotation altogether.

The Eulerian equations may now be written

$$\frac{du}{dt} - fv + jw = -\frac{1}{\varrho}\frac{\partial p}{\partial x}, \tag{1}$$

$$\frac{dv}{dt} + fu = -\frac{1}{\varrho}\frac{\partial p}{\partial y}, \tag{2}$$

$$\frac{dw}{dt} - ju + g = -\frac{1}{\varrho}\frac{\partial p}{\partial z}, \tag{3}$$

and the equation of continuity

$$\frac{d}{dt}(\ln \varrho) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \tag{4}$$

Here u, v and w are the x, y, and z velocity components respectively; p is the pressure; ϱ the density; f the z-component of the earth's vorticity, $2\,\Omega \sin\varphi$; j the corresponding y-component, $2\Omega\cos\varphi$; Ω the angular speed of the earth's rotation about its axis; and φ the latitude. The operator d/dt is the time derivative following the motion of a particle, i. e.,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

or

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla,$$

where ∇ is the gradient operator $\boldsymbol{i}\,\partial/\partial x + \boldsymbol{j}\partial/\partial y + \boldsymbol{k}\,\partial/\partial z$, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ being unit vectors along the x, y, and z axes respectively.

As it is not our purpose to enter into the theory of thermodynamic approximations, although these may be treated by an analogous method of dimensional analysis, we shall assume that the motion is adiabatic. Hence

$$\frac{d\theta}{dt} = 0\,, \tag{5}$$

where θ is the potential temperature, defined by

$$\theta = \text{constant} \times p^{1/\varepsilon}\varrho^{-1}, \tag{6}$$

ε being the ratio c_p/c_v of the specific heat of dry air at constant pressure to that at constant volume.

Finally we add the equation of state of the atmosphere, which is assumed to be a perfect gas,

$$p = \varrho RT. \tag{7}$$

Here T is the absolute temperature and R the specific gas constant, equal to $c_p - c_v$.

The scale properties of a given motion are determined as follows: The spatial dimensions are characterized by S, the mean horizontal distance between points at which the velocity components take extreme values, and by H, the corresponding mean vertical distance. The time dimensions are characterized by V, the mean magnitude of the horizontal velocity component, by W, the mean magnitude of the vertical velocity component, and by C, the mean speed of propagation of the horizontal streamline pattern. Roughly speaking, S is the mean distance between trough and wedge in the streamline pattern, and H is the height of the tropopause if defined for u and v, and usually somewhat smaller, though of the same order of magnitude if defined for w.

For large-scale motions — represented by the major waves and vortices on the upper level weather maps — S is of the order of 10^6 m, H is of the order of 10^4 m, C is of the order 10 m sec^{-1}, and V is of the order 10 m sec^{-1} throughout the greatest part of the atmosphere. The order of W may not be assigned independently for it is dependent on the remaining characteristic parameters.

Finally, to characterize the static stability of the atmosphere it is convenient to choose the non-dimensional parameter

$$K = \frac{H}{\theta}\frac{\partial\theta}{\partial z} = \frac{H}{T}(\gamma_d - \gamma), \tag{8}$$

where γ_d is the dry-adiabatic lapse rate of temperature g/c_p, and γ is the actual lapse rate in the atmosphere. The values of K as a function of γ for a mean temperature of 260° C are given in table 1. They are seen to have the order of magnitude 10^{-1} for the normally observed lapse rates in the free atmosphere.

Table 1.

Values of the static stability parameter.

γ C km^{-1}	0	2	4	6	8	10
K	0.38	0.31	0.23	0.15	0.08	0.00

The following list summarizes the various orders of the characteristic parameters:

$$\left.\begin{aligned} S &\sim 10^6 \text{ m} \\ H &\sim 10^4 \text{ m} \\ C &\sim 10 \text{ msec}^{-1} \\ V &\sim 10 \text{ msec}^{-1} \\ K &\sim 10^{-1} \\ g &\sim 10 \text{ msec}^{-2} \\ f, j &\sim 10^{-4} \text{ sec}^{-1} \end{aligned}\right\} \tag{9}$$

The orders of f and j between the latitudes 15° and 75° and the order of g have been added for later reference. The symbol "$\sim$" denotes equality in orders of magnitude.

We are now in a position to evaluate the orders of magnitude of all quantities appearing in the equations of motion. This is done by replacing differentials by finite increments and expressing the incremental ratios in terms of S, H, C, V, and K. Thus to determine the order of $\partial u/\partial s$, s being a horizontal distance coordinate, we replace $\partial u/\partial s$ by $\Delta u/\Delta s$ and choose Δs equal to S. Then, by definition, Δu has the same order of magnitude as u itself, and we have,

$$\frac{\partial u}{\partial s} \sim \frac{\Delta u}{\Delta s} \sim \frac{V}{S}. \tag{10}$$

In the same way

$$\frac{\partial v}{\partial s} \sim \frac{V}{S} \tag{11}$$

and

$$\frac{\partial w}{\partial s} \sim \frac{W}{S}. \tag{12}$$

Finally, by taking increments in the z direction,

$$\frac{\partial u}{\partial z} \sim \frac{V}{H}, \quad \frac{\partial v}{\partial z} \sim \frac{V}{H}, \quad \frac{\partial w}{\partial z} \sim \frac{W}{H}. \tag{13}$$

The space derivatives of p and ϱ are estimated in a similar manner. But here, in order to avoid having to introduce separate characteristic values for p and ϱ, we evaluate their logarithmic derivatives instead. Since the fields of pressure and density have the same horizontal scale as the velocity field, and since their horizontal space variations are not greater in order of magnitude than their mean values, $\bar{p}$ and $\bar{\varrho}$, we may write

$$\left.\begin{aligned} \frac{\partial}{\partial s}(\ln p) &= \frac{1}{p}\frac{\partial p}{\partial s} \sim \frac{1}{\bar{\varrho}}\frac{\Delta p}{\Delta s} \lesssim \frac{1}{S}, \\ \frac{\partial}{\partial s}(\ln \varrho) &= \frac{1}{\varrho}\frac{\partial \varrho}{\partial s} \sim \frac{1}{\bar{\varrho}}\frac{\Delta \varrho}{\Delta s} \lesssim \frac{1}{S}. \end{aligned}\right\} \tag{14}$$

On the other hand, since the vertical increments in p and ϱ through the distance H have the same order as p and ϱ themselves, we have

$$\frac{\partial}{\partial z}(\ln p) \sim \frac{\partial}{\partial z}(\ln \varrho) \sim \frac{1}{H}. \tag{15}$$

To estimate the orders of magnitude of the time derivatives we may make use of the fact that the streamline, isobaric, and isopycnic patterns translate horizontally with the speed C. Then the system moves the distance ds in the time ds/C, and the time variation $\partial/\partial t$ is given by

$$\frac{\partial}{\partial t} \sim \frac{\partial}{\partial s/C} = C\frac{\partial}{\partial s}. \quad (16)$$

The case where the local time variation is produced by a change in amplitude — in addition to a translational motion — is provided for by supposing C to be composed of the speed of propogation plus an additional term which allows for the change in amplitude. The order of magnitude of C (10 m sec^{-1}) is kept the same as before, because the specification of this order for C is precisely what distinguishes the meteorologically significant motions from the several varieties of theoretically possible motions having the same values of S, H, and V that may exist. Thus C has the order 10^2 m sec^{-1} in external gravity waves and the order 10^3 m sec^{-1} in tidal waves, and by assigning the order 10 m sec^{-1} we exclude such motions. To put the matter in another way: the motion of the atmosphere is not determined by the initial space distribution of the kinematic variables; it is also necessary to assign initial time derivatives. Hence when the order of V is determined, we may regard the relation

$$C \sim V \quad (17)$$

as the one which distinguishes the meteorologically significant motions from all other types of atmospheric motion.

The order of W can now be estimated as follows. Writing the equation of continuity (4) in the form

$$\frac{\partial}{\partial t}(\ln \varrho) + u\frac{\partial}{\partial x}(\ln \varrho) + v\frac{\partial}{\partial y}(\ln \varrho) + \\ + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -w\frac{\partial}{\partial z}(\ln \varrho) - \frac{\partial w}{\partial z}, \quad (18)$$

and evaluating its terms by means of (10—16), we obtain, in the same order,

$$\frac{C}{S} + \frac{V}{S} + \frac{V}{S} + \frac{V}{S} + \frac{V}{S} \lesssim \frac{W}{H} + \frac{W}{H}.$$

The inequality sign must be added as a possibility because $\partial u/\partial x$ and $\partial v/\partial y$ may tend to compensate. It then follows as a consequence of (17) that

$$\frac{W}{H} \lesssim \frac{V}{S}, \quad (19)$$

so that the equation of continuity establishes an upper limit for the magnitude of W.

The above relation, together with (10—13) and (17), now permits the evaluation of the operator d/dt as applied to u, v, and w. We find that

$$\frac{d}{dt} \sim \frac{C}{S} \sim \frac{V}{S}, \quad (20)$$

and if this order of magnitude is inserted into the hydrodynamical equations (1—3), we obtain

$$\frac{C}{S}V + fV + jW \sim \frac{1}{\varrho}\frac{\partial p}{\partial x}, \quad (21)$$

$$\frac{C}{S}V + fV \sim \frac{1}{\varrho}\frac{\partial p}{dy}, \quad (22)$$

$$\frac{C}{S}W + jV + g \sim \frac{1}{\varrho}\frac{\partial p}{\partial z}. \quad (23)$$

It then follows from (9) and (19) that the term involving w in the x-component of the coriolis force is at least one order smaller than the term involving v and can therefore be ignored. Hence by (21) and (22) we obtain

$$\frac{\text{horizontal acceleration}}{\text{horizontal coriolis force}} \sim \frac{C/S}{f} = \frac{N}{N_i}. \quad (24)$$

The quantity N is the *characteristic frequency*, CS^{-1}, of the motion, and N_i is equal to f, the frequency of an horizontal inertial oscillation. In terms of these quantities equation (24) states that *the geostrophic deviation decreases with the ratio of the characteristic frequency to the frequency of a horizontal inertial oscillation.*

This criterion may be used to prove that the principle atmospheric perturbations are quasi-geostrophic, for by substituting the values of C, S, and f from (9) into (24), we find

$$\frac{\text{horizontal acceleration}}{\text{horizontal coriolis force}} \sim \frac{10/10^6}{10^{-4}} \sim \frac{1}{10}, \quad (25)$$

which shows that the horizontal acceleration is one order of magnitude less than that of the horizontal coriolis force. (Here, as well as in other sections, an approximation is said to be valid if the error is less than the term to be approximated by at least one order of magnitude.) We may therefore regard the geostrophic approximation to be substantiated for the primary large-scale perturbations of the atmosphere — as manifested, for example, in the isobaric patterns on the upper level pressure maps.[3]

It is not strictly proper to regard tbe large-scale motions as independent of the small-scale motions, for the governing equations are non-linear and the motions are not superposable. But if, as in the atmosphere, the bulk of the energy is associated with the large-scale systems, the small-scale motions may be regarded as turbulent fluctuations giving rise to small Reynold's stresses and heat transports which may be ignored in the first approximation.

The following additional relationships, obtained from (23), (19), and (9),

$$\frac{\text{vertical coriolis force}}{\text{acceleration of gravity}} \sim \frac{jV}{g} \sim \frac{10^{-4}\times 10}{10} \sim 10^{-4},$$

$$\frac{\text{vertical acceleration}}{\text{acceleration of gravity}} \sim \frac{CW}{gS} \lesssim \frac{C^2H}{gS^2} \sim \frac{10^2\times 10^4}{10\times 10^{12}} \sim 10^{-7},$$

when taken together, serve to justify the hydrostatic approximation.

The establishment of the geostrophic approximation for large-scale motions makes it possible to derive a more precise value for W than is furnished by (19). Thus the expansion of (5) gives

$$w = -\frac{d_h\theta}{dt}\Big/\frac{\partial\theta}{\partial z} = \frac{d_h}{dt}(\ln\theta)\Big/\frac{\partial}{\partial z}(\ln\theta), \tag{26}$$

where

$$\frac{d_h}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_h\cdot\nabla_h, \tag{27}$$

and $\boldsymbol{v}_h$ and ∇_h denote the horizontal components of $\boldsymbol{v}$ and ∇ respectively. Now from the geostrophic wind equation

$$\boldsymbol{v}_h \approx \frac{\boldsymbol{k}}{\varrho f}\times\nabla_h p \tag{28}$$

and the hydrostatic equation

$$\frac{\partial p}{\partial z} \approx -g\varrho, \tag{29}$$

we obtain

$$\boldsymbol{k}\times\nabla_h(\ln p) \approx -\frac{f}{g}\boldsymbol{v}_h\frac{\partial}{\partial z}(\ln p),$$

and therefore, with the aid of (15),

$$\frac{\partial}{\partial s}(\ln p) \sim \frac{fV}{gH}. \tag{30}$$

Furthermore, differentiation of (28) with respect to z and substitution of (29) gives

$$\boldsymbol{k}\times\nabla_h(\ln\varrho) \approx -\frac{f}{g}\boldsymbol{v}_h\frac{\partial}{\partial z}(\ln\varrho) - \frac{f}{g}\frac{\partial\boldsymbol{v}_h}{\partial z};$$

whence, by (13) and (15),

$$\frac{\partial}{\partial s}(\ln\varrho) \sim \frac{fV}{gH}. \tag{31}$$

By differentiation of (6) with respect to s and substitution of (30) and (31) we then obtain

$$\frac{\partial}{\partial s}(\ln\theta) = \frac{1}{\varepsilon}\frac{\partial}{\partial s}(\ln p) - \frac{\partial}{\partial s}(\ln\varrho) \sim \frac{fV}{gH} \tag{32}$$

since $\varepsilon = 1.4$; and from (16)

$$\frac{\partial}{\partial t}(\ln\theta) \sim \frac{fCV}{gH}.$$

Hence, by (17) and (27),

$$\frac{d_h}{dt}(\ln\theta) \sim \frac{fCV}{gH}. \tag{33}$$

Finally since by (8) $\partial(\ln\theta)/\partial z \sim K/H$, (26) gives

$$W \sim \frac{fCV}{gK} \sim \frac{10^{-4}\times 10\times 10}{10\times 10^{-1}} \sim 10^{-2}\ \text{m sec}^{-1}. \tag{34}$$

If we now substitute this order of magnitude for W into the equation

$$\frac{d\boldsymbol{v}_h}{dt} = \frac{d_h\boldsymbol{v}_h}{dt} + w\frac{\partial\boldsymbol{v}_h}{\partial z}$$

we obtain, with the aid of (10—13) and (16)

$$\frac{du}{dt} \sim \frac{dv}{dt} \sim \frac{CV}{S} + \frac{fCV^2}{gHK} \sim 10^{-4}(1+10^{-2}/K).$$

Since K has the order 10^{-1} we may conclude that *the term in $d\boldsymbol{v}_h/dt$ involving w is one order of magnitude smaller than the others*, i. e.,

$$\frac{d\boldsymbol{v}_h}{dt} \approx \frac{d_h\boldsymbol{v}_h}{dt}. \tag{35}$$

Thus the acceleration of the horizontal wind may be computed as if the motion were purely horizontal.

[3] A good illustration of the fact that significant geostrophic deviations are associated only with high-frequency, and therefore small-scale perturbations, is provided in an article by Houghton and Austin (1946). Figures 1, 2, and 3 of this article show that the value of S corresponding to the large-scale motion — determined by the 10,000 ft pressure field — is 4 to 5 times greater than the value of S corresponding to the observed field of the geostrophic deviation.

We also observe that the accuracy of the approximation (35) increases with the static stability. This condition may be attributed to the inhibitory effect of the stability on the vertical motion; for if the parameters S, H, C, and V are held constant and the stability parameter K is decreased, the isentropic surfaces, and therefore the particle motions, become more and more horizontal.

Application of (15), (16), (30), and (34) now gives

$$\frac{d}{dt}(\ln p) = \frac{d_h}{dt}(\ln p) + w\frac{\partial}{\partial z}(\ln p)$$
$$\sim \frac{fCV}{gH} + \frac{fCV}{gHK}$$
$$\sim 10^{-7} + 10^{-7}/K;$$

and it follows from (9) that *the individual change in pressure is due almost entirely to the vertical motion*, i.e.,

$$\frac{d}{dt}(\ln p) \approx w\frac{\partial}{\partial z}(\ln p), \text{ or } \frac{dp}{dt} \approx w\frac{\partial p}{\partial z}. \quad (36)$$

But here in contrast with (35) the accuracy of the approximation diminishes with increasing stability.

By exactly the same reasoning we obtain

$$\frac{d\varrho}{dt} \approx w\frac{\partial \varrho}{\partial z}, \quad (37)$$

which shows that *the individual time rate of change of density is also due almost entirely to the vertical motion.* This last relationship is often used for the computation of vertical velocities.[4]

Substituting (34), (37), (13), and (15) into (18), we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \sim \frac{fCV}{gHK} \sim 10^{-6}\,\text{sec}^{-1}, \quad (38)$$

and contrasting this result with the relations

$$\frac{\partial u}{\partial x} \sim \frac{\partial v}{\partial y} \sim \frac{V}{S} \sim 10^{-5}\,\text{sec}^{-1} \quad (39)$$

obtained from (10) and (11), we see that the terms $\partial u/\partial x$ and $\partial v/\partial y$ comprising the horizontal divergence tend to compensate. This fact helps to explain why different methods of computation lead to wide discrepancies in the values obtained for the horizontal divergence. In the first place the magnitude of the error in the observed winds is only one order less than that of the winds themselves, so that the error in computing the large-scale divergence directly from wind observations will have the same order of magnitude as the divergence itself. But what is probably of even greater importance is that the small-scale motions superimposed on the large-scale systems may have greater divergences than the large-scale motions themselves. This is because the small-scale motions are low level phenomena which quickly damp out with height so that H is small in (38), and because the lapse-rate in the small-scale systems may approach the adiabatic so that K may also be small.

Beers (1946) remarked that the values of the divergence computed by Fleagle (1946) are of the order 10^{-5} sec^{-1}, whereas those computed by Namias and Clapp (1946) are of the order 10^{-6} sec^{-1}. To explain this discrepancy one need only point out that Fleagle's results were based on actual wind observations from which, presumably, the small-scale perturbations with large divergences were not smoothed out, whereas Namias and Clapp's computations were based on 5-day mean charts from which the small-scale perturbations are surely eliminated. Moreover, in the latter case, the divergence was computed by means of the vorticity equation (42), an equation which can be justified only for large-scale motions. One is apparently forced to the conclusion that it is impossible to determine the values of the horizontal divergence pertaining to the large-scale motions from instantaneous wind observations.

The approximations (29) and (35) together with the inequality $V >> W$ at last enable the Eulerian equations to be written

$$\frac{d_h \boldsymbol{v}_h}{dt} + f\boldsymbol{k} \times \boldsymbol{v}_h = -\frac{1}{\varrho}\nabla_h p, \quad (40)$$

$$g = -\frac{1}{\varrho}\frac{\partial p}{\partial z}, \quad (41)$$

the form most often used in meteorology.

It must of course be clearly understood that the validity of the foregoing theory of approximations depends upon the correctness of the orders of magnitude that have been assigned to the characteristic parameters S, H, C, V, and K. For the most part, these values are observed to be correct; there are, however, regions near the tropopause in which V has the order 10^2 m sec^{-1}. In such regions, the approximations are of a

[4] See, for example, Panofsky (1946).

doubtful character and must be used, if at all, with a great deal of caution. In any case, it is advisable to test the approximations by using the observed values of S, H, C, V, and K. It will then often be found that though V is large S is also large, and the approximations remain valid.

2. INCORPORATION OF THE GEOSTROPHIC APPROXIMATION INTO THE EQUATIONS OF MOTION.

In spite of the fact that the geostrophic approximation is successfully applied in synoptic practice, it has never been incorporated into the equations of motion in an acceptable manner. The difficulty can in part be attributed to the fact that the outright use of the geostrophic approximation destroys the possibility of accounting for changes in the motion. To neglect the acceleration terms entirely in the Eulerian equations is to throw out the baby with the bath; if the motion is geostrophic and hydrostatic the acceleration is zero, and the motion will not change.

There have been a number of attempts — notably by Hesselberg (1915), Brunt and Douglas (1928), and Phillips (1939) — to overcome this difficulty by a method which consists essentially in the formal inversion of the operator $d_h/dt + f\boldsymbol{k}\times$ in the horizontal Eulerian equations

$$\left(\frac{d_h}{dt} + f\boldsymbol{k}\times\right)\boldsymbol{v}_h = -\frac{1}{\varrho}\nabla_h p$$

One obtains an expression for $\boldsymbol{v}_h$ in the form of an infinite series of iterated individual derivatives of the horizontal pressure force. But in addition to the fact that this series has not been shown to converge, it is doubtful whether the terms retained in practice provide an acceptable approximation to the wind. In the case of the Brunt-Douglas isallobaric approximation, Haurwitz (1946) has shown that they do not; and though it seems possible that the inclusion of more terms would yield a better approximation, there is evidently a need for a more justifiable approach to the whole problem.

The failure of the geostrophic approximation as a means of calculating changes in the motion is traceable to the fact that it fails to provide a valid approximation to the horizontal divergence. We have seen from (38) and (39) that the two terms $\partial u/\partial x$ and $\partial v/\partial y$ comprising the horizontal divergence are individually one order greater in magnitude than their sum. If these terms were approximated, as by the geostrophic wind, with an error less by only one order of magnitude than the terms themselves, the error would have as great an order of magnitude as the horizontal divergence itself.[5] On the other hand, an application of the scale theory of approximations shows that the horizontal wind $\boldsymbol{v}_h$ may be replaced by the geostrophic wind

$$\boldsymbol{v}_g = \frac{\boldsymbol{k}}{\varrho f}\times\nabla_h p$$

in all other terms occuring in the equations of motion. These circumstances suggest that the way to incorporate the geostrophic approximation is to eliminate $\nabla_h\cdot\boldsymbol{v}_h$ from the equations of motion and then introduce the geostrophic approximation $\boldsymbol{v}_h = \boldsymbol{v}_g$. Now the horizontal divergence occurs implicitly in the horizontal Eulerian equations as well as explicitly in the equation of continuity. This may be shown by deriving the equation for the vertical component of vorticity ζ. Thus by taking the horizontal curl of (40) we obtain

$$\frac{d_h}{dt}(\zeta + f) + (\zeta + f)\nabla_h\cdot\boldsymbol{v}_h = -\boldsymbol{k}\cdot\nabla_h\left(\frac{1}{\varrho}\right)\times\nabla_h p, \tag{42}$$

which now contains $\nabla_h\cdot\boldsymbol{v}_h$ in explicit form. Since the geostrophic approximation is valid for all terms except $\nabla_h\cdot\boldsymbol{v}_h$, the elimination of $\nabla_h\cdot\boldsymbol{v}_h$ from (4) and (42) will yield an equation in which the geostrophic approximation may be consistently introduced; and this equation, together with the adiabatic equation (5), the hydrostatic equation (29), and the geostrophic equations (28), will constitute a dynamically consistent set of equations which may be used to study the large-scale motion of the atmosphere. However, instead of proceding with the direct elimination

[5] The fact that owing to the coordinate approximation used the horizontal divergence here used is in error by an amount $(v/\mathfrak{R})\tan\varphi$, with $\mathfrak{R}$ the radius of the earth, does not effect the reasoning; for this term has also the order of the horizontal divergence (10^{-6} sec^{-1}).

it will be more convenient to derive an equivalent system of equations by a method which has the additional advantage of furnishing a greater insight into their physical significance. For this purpose we derive a theorem whose approximate form was first given by Rossby (1940).

We consider two isentropic surfaces infinitesimally close together and characterized by the values θ and $\theta + \delta\theta$ of the potential temperature. A cylinder with sides perpendicular to these surfaces and with infinitesimal cross-section cuts them in the congruent closed curves δc enclosing the infinitesimal areas δA. If we denote the vertical distance between the surfaces by δn, then $\delta A\,\delta n$ is the volume and $\varrho\delta A\,\delta n$ the mass enclosed by the part of the cylinder contained within the isentropic surfaces. The law of conservation of mass requires that $\varrho\delta A\,\delta n$ be constant during the motion. Now the circulation theorem of V. Bjerknes states that the time rate of change of absolute circulation around δc is equal to the number of pressure-volume solenoids enclosed by the curve, which in this case is zero because the curve is always contained in an isentropic surface. But the circulation around an infinitesimal curve is equal to the area enclosed by the curve times the absolute vorticity component perpendicular to the plane of the curve — in the present case to $q_\theta\delta A$, where q_θ is the absolute vorticity component perpendicular to the isentropic surface. Hence $q_\theta\delta A$ is constant, and since we have established that $\varrho\delta A\,\delta n$ is constant, we may conclude that $q_\theta/\varrho\delta n$ is also constant. Since, moreover, $\delta\theta = (\delta\theta/\delta n)\;\delta n$ is constant for isentropic motion, we finally obtain that $(\delta\theta/\delta n)\,(q_\theta/\varrho)$ is a conservative quantity, or, in differential form, that

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial n}\,\frac{q_\theta}{\varrho}\right) = 0. \tag{43}$$

It is noteworthy that the only assumptions made in deriving the above equation of conservation are that the motion is isentropic and frictionless.

Since (43) does not contain the horizontal divergence, it may be used together with (5), (28) and (29) in place of the system consisting of the equation obtained by elimination of $\triangle_h\cdot\boldsymbol{v}_h$ between (4) and (42) together with (5), (28), and (29)

Before proceding further we shall derive a simplification of (43). Writing

$$\left(\frac{\partial\theta}{\partial n}\right)^2 = \left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2$$

and utilizing (8), (9), and (32), we have

$$\left.\begin{aligned}\left(\frac{1}{\theta}\frac{\partial\theta}{\partial x}\right)^2 \sim \left(\frac{1}{\theta}\frac{\partial\theta}{\partial y}\right)^2 &\sim \frac{f^2V^2}{g^2H^2} \sim 10^{-16}\,m^{-2};\\ \left(\frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)^2 &\sim \frac{K^2}{H^2} \;\;\sim 10^{-10}\,\mathrm{m}^{-2};\end{aligned}\right\} \tag{44}$$

which show that $\partial\theta/\partial n$ may be replaced by $\partial\theta/\partial z$. We may therefore write

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial z}\,\frac{q_\theta}{\varrho}\right) = 0 \tag{45}$$

in place of (43); or, in view of the hydrostatic relationship (29),

$$\frac{(q_\theta)_1}{(\delta p)_1} = \frac{(q_\theta)_0}{(\delta p)_0}, \tag{46}$$

where δp is the difference in pressure, measured along a vertical, between the two isentropic surfaces with potential temperatures θ and $\theta + \delta\theta$ respectively, and the quantities $(\delta p)_0$, $(q_\theta)_0$, and (δp), (q_θ), refer to the values of δp and q_θ at two different positions of a moving particle. If we choose a standard value for $(\delta p)_0$, then $(q_\theta)_0$ is constant for the motion, and shall call it the absolute potential vorticity to conform to the terminology introduced by Rossby (1940).

Equations (5) and (45), together with the stipulation that ϱ, and $\boldsymbol{v}_h$ are to be evaluated in terms of p by means of (6), (28) and (29), are the mathematical expressions of the following physical principle: *the motion of large-scale atmospheric disturbances is governed by the laws of conservation of potential temperature*[6] *and absolute potential vorticity, and by the conditions that the horizontal velocity be quasi-geostrophic and the pressure quasi-hydrostatic.*

For purposes of numerical integration it is necessary to eliminate not only the horizontal divergence from the equations of motion — as has been done — but also the vertical velocity component, for the later quantity likewise can not be evaluated with the necessary accuracy

[6] It is evident that the potential temperature in equations (5) and (45) may be replaced by any other conservative quantity. In practical applications it may be preferable to use the wet-bulb potential temperature, for this quantity is conserved in both dry and saturated adiabatic processes. To do this would of course necessitate a knowledge of the water vapor distribution in the atmosphere.

from the available data. To eliminate w a further simplification of (45) is required. If $\boldsymbol{v}_a$ denotes the absolute velocity, we may write

$$q_\theta = (\nabla \times \boldsymbol{v}_a \cdot \nabla\theta)/(\partial\theta/\partial n)$$
$$= q_x \frac{\partial\theta}{\partial x}\Big/\frac{\partial\theta}{\partial n} + q_y \frac{\partial\theta}{\partial y}\Big/\frac{\partial\theta}{\partial n} + q_z \frac{\partial\theta}{\partial z}\Big/\frac{\partial\theta}{\partial n}$$

where q_x, q_y, and q_z denote respectively the x, y, and z components of absolute vorticity. By (9—13) and (34) we have

$$q_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \sim 10^{-3}\,\mathrm{sec}^{-1};$$

$$q_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} + 2\,\Omega\cos\varphi \sim 10^{-3}\,\mathrm{sec}^{-1};$$

$$q_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + 2\,\Omega\sin\varphi \sim 10^{-4}\,\mathrm{sec}^{-1};$$

and taking these magnitudes together with those of (44) we see that q_θ may be replaced by q_z with a negligible error. Hence (45) may finally be written

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial z}\frac{\zeta + f}{\varrho}\right) = 0. \qquad (47)$$

Now if the individual derivatives in (5) and (47) are expanded, we may eliminate w to obtain

$$\frac{\partial\theta}{\partial z}\frac{d_h}{dt}\left(\frac{\partial\theta}{\partial z}\frac{\zeta + f}{\varrho}\right) - \frac{d_h\theta}{dt}\frac{\partial}{\partial z}\left(\frac{\partial\theta}{\partial z}\frac{\zeta + f}{\varrho}\right) = 0, \qquad (48)$$

which, when ϱ and $\boldsymbol{v}_h$ are expressed in terms of p by means of (28) and (29), is alone sufficient to determine the motion.

Since (48) involves only p as dependent variable, and is moreover of the first order in t, its integration requires only a knowledge of the initial pressure distribution. This circumstance makes (48) particularly well-adapted for numerical forecasting. Indeed, some such equation or system of equations is necessary; it is quite illusory to suppose that the primitive equations (1—5) can be used for numerical forecasting for the reason that neither the horizontal acceleration nor the horizontal divergence pertaining to the large-scale motion can be evaluated from the observed data.

The evaluation of ζ in equation (48) is facilitated by the following approximation. Taking the horizontal curl of (28) we obtain

$$\zeta = \boldsymbol{k}\cdot\nabla_h \times \left(\frac{\boldsymbol{k}}{\varrho f} \times \nabla_h p\right) = \frac{\nabla_h^2 p}{\varrho f}$$
$$- \frac{\nabla_h p}{\varrho f}\cdot\frac{\nabla_h \varrho}{\varrho} - \frac{1}{f}\frac{df}{dy}\frac{1}{\varrho f}\frac{\partial p}{\partial y}.$$

Now from (9—11), (28), and (31), we have

$$\frac{\nabla_h p}{\varrho f}\cdot\frac{\nabla_h \varrho}{\varrho} \sim V\frac{fV}{gH} \sim 10^{-7}\,\mathrm{sec}^{-1},$$

$$-\frac{1}{f}\frac{df}{dy}\left(\frac{1}{\varrho f}\frac{\partial p}{\partial y}\right) \approx \frac{u}{2\Omega\sin\varphi}\frac{2\,\Omega\cos\varphi}{\Re} \sim \frac{V}{\Re} \sim 10^{-6}\,\mathrm{sec}^{-1},$$

$$\zeta \sim 10^{-5}\,\mathrm{sec}^{-1}$$

where $\Re$ is the radius of the earth. It is then seen that the last two terms in the expression for ζ can be ignored in comparison with the first, so that we have

$$\zeta \approx \frac{\nabla_h^2 p}{\varrho f} = \frac{1}{\varrho f}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right). \qquad (49)$$

This expression for ζ has an advantage for numerical calculation over those in which ζ is expressed in terms of the wind field, for the operations of calculating wind components are eliminated. Thus if h denotes a small space increment, we may write simply

$$\zeta = \frac{4}{h^2 \varrho f}(\bar{p} - p), \qquad (50)$$

where $\bar{p}$ is the mean of the pressure values at the points $(x + h, y)$, $(x - h, y)$, $(x, y + h)$, and $(x, y - h)$.

A single equation analogous to (48) for the case of barotropic motion cannot be derived by specialization of (48) because of indeterminacies that arise. It is necessary to return to the basic equations. Since ϱ is a function of p in a barotropic atmosphere, and since $\boldsymbol{v}_h$ is nearly independent of height, it follows that (40) is almost exact and that (42) may be written

$$\frac{d_h}{dt}(\zeta + f) + (\zeta + f)\,\nabla_h\cdot\boldsymbol{v}_h = 0\,. \qquad (51)$$

If further the equation of continuity (4) is written

$$\frac{d_h\varrho}{dt} = -\varrho\,\nabla_h\cdot\boldsymbol{v}_h - \frac{\partial}{\partial z}(\varrho w) \qquad (52)$$

and integrated from the ground to the top of the atmosphere, we obtain, when the ground is assumed level,

$$\frac{d_h p_0}{dt} = -p_0 \nabla_h \cdot \boldsymbol{v}_h. \qquad (53)$$

Here use is made of the facts that $\boldsymbol{v}_h$ is independent of height and ϱw vanishes at the ground and at the top of the atmosphere. The quantity p_0 denotes the pressure at the ground. Elimination of $\nabla_h \cdot \boldsymbol{v}_h$ between (51) and (53) then gives

$$\frac{d_h}{dt}\left(\frac{\zeta + f}{p_0}\right) = 0. \qquad (54)$$

This equation suffices to determine the motion providing we add the condition of geostrophic balance,

$$\boldsymbol{v}_h = \boldsymbol{k} \times \frac{\nabla_h p_0}{\varrho_0 f} = \boldsymbol{k} \times \frac{\nabla_h \pi}{f}, \qquad (55)$$

where ϱ_0 is the surface density and π the barotropic pressure function $\int^{p_0} \frac{\delta p_0}{\varrho_0}$. It follows from (55) that $\boldsymbol{v}_h \cdot \nabla_h \pi = 0$, so that we have

$$\frac{d_h}{dt}(\zeta + f) = \frac{\zeta + f}{RT_0} \frac{\partial \pi}{\partial t}, \qquad (56)$$

where T_0 is the ground temperature.

The linearized form of (48) for small perturbations in a baroclinic zonal current $\bar{u}(z)$ with constant lapse rate is found, with the aid of (49), to be

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\frac{\nu_b^2}{f^2}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) + \frac{\partial^2 p'}{\partial z^2} - a\frac{\partial p'}{\partial z}\right] + \left(\frac{\nu_b^2}{f^2}\frac{df}{dy} - \frac{d^2\bar{u}}{dz^2} + \frac{1}{H}\frac{d\bar{u}}{dz}\right)\frac{\partial p'}{\partial x} = 0, \qquad (57a)$$

where primes denote perturbations, bars denote undisturbed values, and

$$\nu_b^2 = \frac{g}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z} = \frac{g}{\bar{T}}(\gamma_d - \bar{\gamma}),$$

$$H = \frac{R\bar{T}}{g},$$

$$a = \frac{2}{\bar{\varrho}}\frac{\partial \bar{\varrho}}{\partial z} + \frac{1}{H}.$$

The quantity ν_b is the frequence of a buoyancy oscillation and H is the height of a homogeneous atmosphere with surface temperature T.

In case the motion is independent of the y-coordinate, a somewhat simpler form of (57a) can be obtained by changing the dependent variable from p to v. By differentiation of (57a) with respect to x and use of the geostrophic relationship $\partial p'/\partial x = \bar{\varrho} f v'$, we obtain

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\frac{\nu_b^2}{f^2}\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial z^2} - \frac{1}{H}\frac{\partial v'}{\partial z}\right] + \left(\frac{\nu_b^2}{f^2}\frac{df}{dy} - \frac{d^2\bar{u}}{dz^2} + \frac{1}{H}\frac{d\bar{u}}{dz}\right)\frac{\partial v'}{\partial x} = 0, \qquad (57b)$$

where it is assumed that f and df/dy are to be replaced by suitable mean values.

In a similar manner the linearized form of (56) for a constant zonal current is found to be

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \pi'}{\partial x^2} + \frac{\partial^2 \pi'}{\partial y^2}\right) + \frac{df}{dy}\frac{\partial \pi'}{\partial x} - \frac{f^2}{gH_0}\frac{\partial \pi'}{\partial t} = 0, \qquad (58a)$$

or, when the motion is independent of y, in terms of v',

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2 v'}{\partial x^2} + \frac{df}{dy}\frac{\partial v'}{\partial x} - \frac{f^2}{gH_0}\frac{\partial v'}{\partial t} = 0. \qquad (58b)$$

In the next section it will be demonstrated by applying (57b) and (58b), the linearized forms of (48) and (56), that the simplified equations (48) and (56) act as filters to eliminate the meteorologically insignificant wave components from the equations for baroclinic and barotropic wave motion.

3. APPLICATION OF THE SIMPLIFIED EQUATIONS TO WAVE MOTION.

Let us first consider the case of wave motion in a constant barotropic zonal current. This problem has previously been treated by Rossby (1939) and by Holmboe (1945). The frequency equation is found to be

$$\bar{u} - c - u_c = \frac{f^2}{\mu^2}\frac{c}{gH_0 - (\bar{u} - c)^2}, \qquad (59)$$

when the motion is assumed to be quasi-hydrostatic and the velocity independent of the y-coordinate. Here $\bar{u}$ is the mean zonal wind, c the wave-velocity, $\bar{T}_0$ the mean surface temperature, and μ and u_c are defined by

$$\left.\begin{aligned} \mu &= \frac{2\pi}{L}, \\ u_c &= \frac{1}{\mu^2}\frac{df}{dy}, \end{aligned}\right\} \qquad (60)$$

where L is the wave length, and it is understood that f and df/dy denote mean values.

In the derivation of (59) it is specified only that the waves be simple-harmonic vibrations traveling in the x-direction. The solution of (59) must therefore yield the velocity of the quasi-hydrostatic gravitational waves as well as the velocity of the meteorologically important inertially-propagated long waves first studied by Bjerknes (1937) and Rossby (1939).[7] And indeed it can be shown that two of the roots of (59) are closely approximated by those of

$$(\bar{u} - c)^2 = gH_0, \tag{61}$$

the equation for the velocity of gravitational waves in a barotropic current on a non-rotating earth, and the third by that of

$$\bar{u} - c - u_c = \frac{f^2}{\mu^2 g H_0} c, \tag{62}$$

an equation derived from (59) by use of the inequality

$$1 >> \frac{(\bar{u} - c)^2}{gH_0} \tag{63}$$

stating that the relative velocity of the long inertially propagated wave is small compared to the gravitational wave velocity. Since the latter is equal to $(R\bar{T}_0)^{\frac{1}{2}}$ and therefore has the same order as $(\varepsilon R\bar{T}_0)^{\frac{1}{2}}$, the velocity of sound, the approximation is surely justified.

Now it may be inferred from the fact that f does not appear in (61) that the rotation of the earth exerts virtually no influence on the propagation of external gravitational waves and consequently that the gravitational wave motion is non-geostrophic. Then, since (58 b) is designed to govern only the large-scale quasi-geostrophic motions, we should expect it to filter out the non-geostrophic gravitational wave components from (59) and reduce directly to (62). That it does so can be seen by introducing the wave expression

$$v' = V e^{i\mu(x-ct)}$$

into (58 b); we obtain

$$-i\mu^3 (\bar{u} - c) V + i\mu \frac{df}{dy} V + i\mu \frac{f^2}{gH_0} cV = 0,$$

or

$$\bar{u} - c - u_c = \frac{f^2}{\mu^2 g H_0} c,$$

which is identical to (62).

A more general verification of the effectiveness of the simplified equations for excluding meteorological "noise" is obtained by applying (48) to wave motion in a baroclinic zonal current. It was shown in DLW (pg. 147, eq. 58) that the motion of small amplitude waves of infinite lateral extent is governed by the equation

$$(\bar{u} - c) \frac{d^2 V}{dz^2} - \frac{(\bar{u} - c)}{H} \frac{dV}{dz} - \frac{\mu^2}{f^2} \nu_b^2 (\bar{u} - c - u_c) V + \frac{1}{H} \frac{d\bar{u}}{dz} V = 0, \tag{64}$$

where V is the amplitude of the v'-wave, i.e.,

$$v' = V(z)\, e^{i\mu(x-ct)}. \tag{65}$$

Equation (64) was obtained by a systematic use of the inequality

$$1 >> \left(\frac{2\pi c}{L} / f\right)^2 = \left(\frac{\nu}{N_i}\right)^2,$$

where N_i is the frequency of an horizontal inertial oscillation, and ν the frequency of the long wave. This inequality has the effect of eliminating gravitational waves and indeed all waves whose frequency is of a higher order of magnitude than that of an horizontal inertial oscillation. But these are precisely the waves that are excluded by the geostrophic approximation (cf. equation (24)). We should therefore expect (64) to follow directly from (48). That it does so is seen by substituting (65) into (57 b), the linearized form of (48); it is found that the resulting equation reduces exactly to (64).

4. THE PROGNOSTIC USE OF THE TENDENCY EQUATION.

If the equation of continuity in the form (52) is integrated with respect to z from 0 to ∞, we obtain the so-called tendency equation

$$g \int_0^\infty \frac{\partial \varrho}{\partial t} \delta z = \frac{\partial p_0}{\partial t} = -g \int_0^\infty \nabla_h \cdot \varrho \mathbf{v}_h \, \delta z, \tag{66}$$

where, as before, p_0 is the pressure at the ground, which is assumed to be level. Equation (37), combined with the equation of continuity, implies that $\partial\varrho/\partial t$ is lower in order of magnitude than $\nabla_h \cdot \varrho \mathbf{v}_h$, whereas (66) states that the z-integrals of these quantities are equal. It must follow, therefore, that the magnitude of the integral on the right of (66) is at least one order smaller than the magnitude of either its positive or negative

[7] Sound waves are excluded by the hydrostatic assumption.

parts, i.e., the pressure tendency is a small difference between the two large quantities — the total horizontal mass convergence and the total horizontal mass divergence. Since neither of these quantities can be evaluated with the necessary accuracy from wind observations (see last part of sec. 1), we must abandon the idea of using (66) in its primitive form as a prognostic tool and seek a more suitable reformulation.

Since $\partial p_0/\partial t$ will always be small compared to the individual terms on the right of (66), we shall begin by setting it equal to zero, we obtain

$$\int_0^\infty \nabla_h \cdot \varrho \boldsymbol{v}_h \, \delta z = 0. \tag{67}$$

This equation is of course dynamically inconsistent since it states that the pressure change is zero, but it can be used as an approximation as long as it is not intended as a means for calculating pressure changes. It states that the total horizontal mass divergence in a vertical column extending from the ground to the top of the atmosphere is approximately balanced by the horizontal mass convergence. Equation (67) can be transformed into a more useful form in the following manner. By applying the scale theory of approximations we find that the right-hand term in (42) has the order of magnitude 10^{-11} sec^{-2}, whereas the left-hand terms have the order 10^{-10} sec^{-2}. Also if the integrand in (67) is expanded to give

$$\nabla_h \cdot \varrho \boldsymbol{v}_h = \varrho \nabla_h \cdot \boldsymbol{v}_h + \boldsymbol{v}_h \cdot \nabla_h \varrho ,$$

it is found that the density advection term is one order of magnitude less than the divergence term. Hence we may write (42) in the form

$$\frac{d_h}{dt} \ln (\zeta + f) = - \nabla_h \cdot \boldsymbol{v}_h ,$$

and (67) in the form

$$\int_0^\infty \varrho \nabla_h \cdot \boldsymbol{v}_h \, \delta z = 0.$$

Then, by combining the two equations, we obtain

$$\int_0^\infty \varrho \frac{d_h}{dt} \ln (\zeta + f) \, \delta z = - \frac{1}{g} \int_{p_0}^0 \frac{d_h}{dt} \ln (\zeta + f) \, \delta p = 0 \tag{68}$$

which states that the mean value of the individual logarithmic derivative of the absolute vertical vorticity component, averaged with respect to pressure from the bottom to the top of the atmosphere, is zero.

The last equation derives its usefulness from the fact that it may be used to determine the speed of propagation of a system of streamlines from an empirical knowledge of the geometrical shape of the streamline pattern alone. Thus by supposing, to a first approximation, that the system is moving with a constant speed c in the fixed direction specified by the unit vector $\boldsymbol{n}$, we have

$$\frac{\partial \zeta}{\partial t} = - c \boldsymbol{n} \cdot \nabla_h \zeta ,$$

and if this relation is substituted into (68), we obtain

$$\int_{p_0}^0 \left[\frac{c \boldsymbol{n} \cdot \nabla_h \zeta}{\zeta + f} - \frac{\boldsymbol{v}_h \cdot \nabla_h (\zeta + f)}{\zeta + f} \right] \delta p = 0 ,$$

or

$$c = \frac{\displaystyle\int_{p_0}^0 \frac{\boldsymbol{v}_h \cdot \nabla_h (\zeta + f)}{\zeta + f} \, \delta p ,}{\displaystyle\int_{p_0}^0 \frac{\boldsymbol{n} \cdot \nabla_h \zeta}{\zeta + f} \, \delta p} . \tag{69}$$

The velocity c may be evaluated by numerical integration from observed streamline patterns or by integration of a suitable idealized model of the observed patterns.

The utility of (69) may be illustrated by using it to determine the velocity of both infinitesimal and finite amplitude long waves. In the case of long waves of infinite lateral extent in a barotropic zonal current, the horizontal velocity field is independent of height, and substitution of $v = V \exp i \mu (x - ct)$ into (69) gives, by the method of small perturbations,

$$\bar{u} - c - u_c = 0 , \tag{70}$$

which is the same as (62) except for the absence of the small right-hand term. This term is the contribution of $\partial p/\partial t$ in the tendency equation, and it can be verified that it is smaller in order of magnitude than the left-hand terms.

Again, if the amplitude factor V in (65) is assumed to be a function of z, substitution into (69) yields the following formula for the velocity of baroclinic waves of infinite lateral extent moving in the x-direction and imbedded in a variable zonal current $\bar{u}(z)$:

$$c = \frac{\int_{\bar{p}_0}^{0} \bar{u} V \delta \bar{p}}{\int_{\bar{p}_0}^{0} V \delta \bar{p}} - u_c. \qquad (71)$$

This equation is not the complete solution to the equations of motion because V is undetermined (an expression for V is derived in DLW under the assumption that u is a linear function of z). However the equation is well-adapted for numerical integration from the observed distributions of V and u (see also DLW, pp. 146—147). If the phase of the wave changes with height, V and c are complex numbers and the wave is damped or unstable.

The velocity of waves of finite lateral extent can be found in the same way. Assuming

$$\left.\begin{aligned} u &= \bar{u}(z) - 2\pi \frac{A(z)}{D} e^{\frac{2\pi i}{L}(x-ct)} \cos \frac{2\pi}{D} y, \\ v &= \qquad\quad 2\pi \frac{B(z)}{L} e^{\frac{2\pi i}{L}(x-ct)} \sin \frac{2\pi}{D} y, \end{aligned}\right\} \qquad (72)$$

and substituting in (69) and (67), we obtain

$$c = \frac{\int_{\bar{p}_0}^{0} \left(\frac{A}{D^2} + \frac{B}{L^2}\right) \bar{u} \delta \bar{p}}{\int_{\bar{p}_0}^{0} \left(\frac{A}{D^2} + \frac{B}{L^2}\right) \delta \bar{p}} - \frac{\int_{\bar{p}_0}^{0} \frac{B}{L^2} \delta \bar{p}}{\int_{\bar{p}_0}^{0} \left(\frac{A}{D^2} + \frac{B}{L^2}\right) \delta \bar{p}} u_c, \qquad (73)$$

and

$$\int_{\bar{p}_0}^{0} (A - B) \delta \bar{p} = 0. \qquad (74)$$

The streamlines corresponding to (72) are represented in fig. 1 by dashed lines.

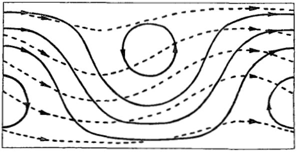

Fig. 1.

If the atmosphere is barotropic, A and B are independent of z, and (73) together with (74) reduces to

$$c = \bar{u} - \frac{D^2}{L^2 + D^2} u_c,$$

which is identical to the original equation derived by Haurwitz (1940).

If the restriction that A and B be small is removed, u and v in (72) may be taken as the velocity components of a wave of finite amplitude. We make use of the fact that $f >> \zeta$, which follows from (9) and (39). It is then found that (72) always satisfies (67) and (68) providing (a) c is given by (73), (b) $L = D$, and (c) A and B satisfy (74). By a proper choice of A and B, subject to restriction (c), the streamlines can be made to approximate the typical structure of a young cyclone wave with closed streamlines at low levels and open streamlines aloft. An example of the type of pattern that can be obtained is shown in fig. 1. The dashed curves represent the high level streamlines, and the solid curves the low level streamlines. The change in phase of the streamline pattern with height is obtained by assigning complex values to A, B, and s.

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