THE QUASI-STATIC EQUATIONS OF MOTION WITH PRESSURE AS INDEPENDENT VARIABLE

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Introduction and Summary.

In the treatment of many meteorological problems, it proves convenient to introduce pressure as an independent variable instead of height. This method has been used by many writers in order to simplify the calculations. Thus in "Dynamic Meteorology and Hydrography" [2], pressure is used as a vertical coordinate in the theory of vertical motions, and also in a discussion of the prognostic value of the tendency equation. In "Physikalische Hydrodynamik" [3], the method is used in the theory of quasi-static wave motions in autobarotropic layers, and also in the theory of turbulent friction.

In the present paper, it is shown that pressure may be used consistently as an independent variable instead of height, in any quasi-static treatment of atmospheric motion, and that this method has certain advantages. Thus the equations become simpler in many ways, and are better suited for use in connection with the aerological charts.

In chapter I, the quasi-static equations of atmospheric motion are given in ordinary Cartesian coordinates. For simplicity, the following assumptions are made: (i) The earth is flat and g is a constant. (ii) Turbulent stresses and dissipation of energy are negligible. (iii) The air is completely dry.

It is pointed out that in the quasi-static theory, it is possible to eliminate density and vertical velocity, the instantaneous state and motion of the atmosphere being completely known from the instantaneous fields of pressure and horizontal velocity. The possibility of prognostic utilization of the quasi-static equations is briefly discussed, but this method is found to be unserviceable, at least as far as no further simplifying assumptions are made.

In chapter II, pressure is introduced as an independent variable instead of height. Formulae are derived to transform the derivatives of a function, when x, y, z and t are independent variables, into derivatives when the independent variables are x, y, p and t. The geometry and kinematics of the fields of the dependent variables in the coordinates x, y, p are discussed, and it is shown that the differential analysis in this coordinate system is directly applicable to the isobaric surface charts used in synoptic aerology. On the same assumptions as were made in chapter I, the hydrodynamic equations are transformed into the system of independent variables x, y, p and t. In this form, the equations prove simpler than in the usual form; and some of the equations become formally identical to the equations for a homogeneous and incompressible fluid. Thus the density drops out in the horizontal equation of motion, and the equation of continuity expresses that the three-dimensional velocity field is solenoidal. Some simple applications of the equations are shown.

Chapter III deals with the effect of the earth's curvature. The quasi-static equations for an atmosphere in a curvilinear potential field are derived with geopotential as a vertical coordinate. In these equations, pressure is introduced as a vertical coordinate instead of geopotential; and the use of pressure as a vertical coordinate proves convenient also when the earth's curvature is taken into consideration.

The effect of the curvature is to introduce certain additional terms in the equations, thus making the mathematical analysis more complicated. As regards the tendency equation, the effect of the curvature is shown to be relatively small. When pressure variations are considered from a fundamental point of view, the assumption of a flat earth with constant g is thus reasonable.

In Chapter IV, some simple gravity waves are treated, partly with pressure, and partly with the logarithm of pressure as a vertical coordinate; and a criterion is given for the legitimacy of the quasi-static approximation.

Chapters V and VI deal with dynamic-meteorological problems. In both chapters, the logarithm of pressure is used consistently as a vertical coordinate.

Quasi-static perturbations of a westerly current are considered in chapter V. It is shown that Solberg—Høiland's stability criteria can be derived from the quasi-static theory without taking into consideration the curvature of the current. Some simple solutions are found in the case of a motion which does not vary in the direction of the current.

Wave perturbations are considered in the case of autobarotropy. The formulae for the wave-velocity by Rossby and Holmboe are derived under more general conditions than those originally assumed.

In Chapter VI a wind formula is derived, on the assumption that the wind is approximately geostrophic. This quasi-geostrophic approximation is applied to the theory of wave perturbations in a baroclinic, westerly current; this leads to a differential perturbation equation for the potential. The form of this equation shows that the stability conditions for perturbations which do not vary in the direction of the current will be a factor of decisive importance also in the case of wave perturbations.

A possible method of numerical weather forecasting, based upon the quasi-geostrophic approximation, is suggested.

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Note.

After having written this paper, I have become aware of a recent paper by Sutcliffe [22], where pressure is used as a vertical coordinate in the discussion of cyclonic development. Sutcliffe writes the equation of continuity, as well as the equations of motion and the vorticity equation, with pressure as an independent variable, and refers to a paper of Sutcliffe and Godart (Isobaric Analysis, Met. Off., London, S. D. T. M., No. 50, 1942), which has not yet been available here.

CHAPTER I. THE QUASI-STATIC EQUATIONS IN CARTESIAN COORDINATES.

1. The hydrodynamic equations.

Assuming the earth to be flat, we choose a Cartesian system of coordinates with the x-axis pointing eastward, the y-axis northward and z-axis vertically upward. The unit vectors along these three axes are denoted by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively.

A vector quantity is given by its components along these three axes. However, it proves convenient to deal with horizontal components and vertical components of vector quantities separately. Therefore, vector symbols (heavy types) are used to denote horizontal vectors only, with the exception of the unit vector $\boldsymbol{k}$, which is vertical. Thus the horizontal wind velocity will be denoted by $\boldsymbol{v} = \boldsymbol{i}v_x + \boldsymbol{j}v_y$; and the three-dimensional wind velocity may be written $\boldsymbol{v} + \boldsymbol{k}w$, where w is the vertical velocity. In the same way, $\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y}$ is the horizontal del-operator, and the three-dimensional del-operator is $\nabla + \boldsymbol{k}\frac{\partial}{\partial z}$.

Individual differentiation with respect to time is denoted by the symbol $\frac{D}{dt}$. In Eulerian representation with x, y, z and t as independent variables we have:

$$\frac{D}{dt}=\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla+w\frac{\partial}{\partial z}. \tag{1.1}$$

The horizontal component of the equation of motion may be written:

$$\frac{D\boldsymbol{v}}{dt}=-s\nabla p-2\Omega_z\,\boldsymbol{k}\times\boldsymbol{v}-2\Omega_y w\boldsymbol{i}. \tag{1.2}$$

Here p denotes pressure, s specific volume, and Ω_y and Ω_z are the components of the angular velocity of the earth.

The vertical component of the equation of motion is replaced by the hydrostatic equation:

$$s\frac{\partial p}{\partial z}+g=0,\ \text{ or }\ q=-\frac{1}{g}\frac{\partial p}{\partial z}, \tag{1.3}$$

where $q=\frac{1}{s}$ means density.

The quasi-static method is based upon the assumption that this equation yields sufficient accuracy, and also on the additional assumption that differentiation of the hydrostatic equation with respect to the independent variables is allowed.

The equation of continuity may be written in one of the two equivalent forms:

$$\frac{\partial q}{\partial t}+\nabla\cdot(q\boldsymbol{v})+\frac{\partial(qw)}{\partial z}=0, \tag{1,4}$$

$$\frac{1}{q}\frac{Dq}{dt}+\nabla\cdot\boldsymbol{v}+\frac{\partial w}{\partial z}=0. \tag{1.5}$$

Differentiating eq. (1.3) with respect to time, and eliminating $\frac{\partial q}{\partial t}$ by means of (1.4), we get

$$\frac{\partial}{\partial z}\left(-\frac{\partial p}{\partial t}+gqw\right)=-g\,\nabla\cdot(q\boldsymbol{v}),$$

when g is considered as a constant. Integrating between the limits z and ∞, and applying the boundary condition at the "upper limit" ($z=\infty$) of the atmosphere:

$$\frac{\partial p}{\partial t}=0,\ \ qw=0\ \text{ at }\ z=\infty, \tag{1.6}$$

we obtain the well-known "tendency equation":

$$\left(\frac{\partial p}{\partial t}-g\,q\,w\right)_z=-g\int_z^{\infty}\nabla\cdot(q\boldsymbol{v})\,d\zeta. \tag{1.7}$$

Here the variable of integration is denoted by ζ, to prevent confusion.

2. The thermodynamic equations.

In the following we will disregard the effect of water vapor on the density and the specific heat of the air, thus considering the air as completely dry. Then we can take the two variables p and q to define the thermodynamic state of the air.

The temperature T is defined as a function of p and q by the equation of state for an ideal gas:

$$T=\frac{p}{Rq},\quad \frac{dT}{T}=\frac{dp}{p}-\frac{dq}{q}, \tag{2.1}$$

where R is the gas constant referred to unit mass of dry air.

The potential temperature ϑ is defined by:

$$\vartheta=\frac{\pi_0^{\frac{\varkappa-1}{\varkappa}}\,p^{\frac{1}{\varkappa}}}{Rq},\quad \frac{d\vartheta}{\vartheta}=\frac{dp}{\varkappa p}-\frac{dq}{q}, \tag{2.2}$$

where $\pi_0=1000$ mb. and $\varkappa$ is the ratio of specific heat at constant pressure (c_p) to specific heat at constant volume (c_v).

Designating by H the heat received by a unit mass of air per unit time, we may write the first law of thermodynamics:

$$\frac{1}{\varkappa p}\frac{Dp}{dt}-\frac{1}{q}\frac{Dq}{dt}=\frac{H}{c_pT}, \tag{2.3}$$

or, by introducing the potential temperature:

$$\frac{1}{\vartheta}\frac{D\vartheta}{dt}=\frac{H}{c_pT}. \tag{2.4}$$

The first law of thermodynamics may be combined, in different ways, with the equations in section 1. Thus, by expanding $\frac{D\vartheta}{dt}$ in eq. (2.4) according to (1.1), and putting

$$\frac{1}{\vartheta}\frac{\partial\vartheta}{\partial t}=\frac{1}{\varkappa p}\frac{\partial p}{\partial t}-\frac{1}{q}\frac{\partial q}{\partial t}=\frac{1}{\varkappa p}\frac{\partial p}{\partial t}+\frac{1}{gq}\frac{\partial}{\partial z}\frac{\partial p}{\partial t}$$

in virtue of (2.2) and (1.3), we find:

$$(2.5)\quad \frac{\partial}{\partial z}\frac{\partial p}{\partial t}+\frac{gq}{\varkappa p}\frac{\partial p}{\partial t} = \frac{gq}{c_p T}H - gq\,\boldsymbol{v}\cdot\frac{\nabla\vartheta}{\vartheta} - g\,q\,w\frac{1}{\vartheta}\frac{\partial\vartheta}{\partial z}.$$

In this equation, the variation of pressure tendency with height is expressed by the heat conveyed to the air, horizontal advection and vertical motion.

We may also combine the first law of thermodynamics in the form (2.3) with the equation of continuity (1.5) by eliminating $\frac{Dq}{dt}$ between them. There results then:

$$(2.6)\quad \frac{Dp}{dt}=\frac{\partial p}{\partial t}+\boldsymbol{v}\cdot\nabla p+w\frac{\partial p}{\partial z} = (\varkappa-1)\,qH-\varkappa p\nabla\cdot\boldsymbol{v}-\varkappa p\frac{\partial w}{\partial z}.$$

Here ∇p, according to eq. (2.2), can be expressed in terms of $\nabla\vartheta$ and ∇q:

$$\nabla p=\varkappa p\left(\frac{\nabla\vartheta}{\vartheta}+\frac{\nabla q}{q}\right).$$

Inserting this, we obtain the following formula for the pressure tendency:

$$(2.7)\quad \frac{\partial p}{\partial t}=-w\frac{\partial p}{\partial z}+(\varkappa-1)\,qH -\varkappa p\boldsymbol{v}\cdot\frac{\nabla\vartheta}{\vartheta}-\varkappa RT\nabla\cdot(q\boldsymbol{v})-\varkappa p\frac{\partial w}{\partial z}.$$

3. The vertical velocity.

The hydrostatic equation is deduced from the vertical equation of motion by neglecting the vertical acceleration $\left(\frac{Dw}{dt}\right)$ and the vertical component of the Coriolis force $(2\Omega_y v_x)$. The hydrostatic equation is thus an approximate one, and its application does not mean that the vertical acceleration and the vertical Coriolis force shall really vanish, which would obviously lead to absurd results. Thus, it is quite reasonable to deal with vertical velocities and accelerations within the quasistatic theory.

L. F. Richardson [18] has shown that in the quasi-static theory, the vertical velocity is given by the instantaneous distribution of the horizontal velocity, the variables of state and the heat received by the air.

The matter may be elucidated by the following reasoning: Consider an air column in hydrostatic equilibrium, bounded by rigid vertical walls. Snppose that this air column is perturbed in different ways: at some levels, we pump air into or out of the column; further we remove some air from the column and replace it by air of different temperature; and finally we let parts of the column be heated or cooled. As a result of these processes, the static conditions of the column are disturbed. To restore the equilibrium, the parcels of air in the column must undergo certain vertical displacements, which can obviously be calculated if we know quantitatively the perturbations initially given to the column.

In an atmospheric air column, such perturbations are going on continuously: air is conveyed to, or removed from every part of the column, at the rate $\nabla\cdot(q\boldsymbol{v})$ per unit time; parts of the column are by horizontal advection replaced by air of different temperature, and heat is conveyed to or from the air. Thus it may be understood that the maintenance of static conditions within the column requires the air parcels to have vertical displacements, varying with time in accordance with the disturbing effects mentioned above. In other words, the vertical velocity in the column must be given as a function of these disturbing effects.

This is expressed by the equation of Richardson, which is obtained by eliminating $\frac{\partial p}{\partial t}$ between the eqs. (1.7) and (2.7)[1]):

$$(3.1)\quad \frac{\partial w}{\partial z}=\frac{H}{c_pT}-\boldsymbol{v}\cdot\frac{\nabla\vartheta}{\vartheta}-\frac{1}{q}\nabla\cdot(q\boldsymbol{v}) +\frac{g}{\varkappa p}\int_z^\infty\nabla\cdot(q\boldsymbol{v})\,d\zeta.$$

At the surface of the earth, which we assume to be plane, we have the boundary condition:

$$(3.2)\qquad w=0 \text{ when } z=0.$$

[1]) In this form, the equation is not exactly identical with the equation given by Richardson, because he took into consideration the curvature of the earth, and also provided for the water vapor in the air.

Hence we obtain by integration of (3.1) between the limits $z = 0$ and $z = h$:

$$(3.3)\quad w_{z=h} = \int_0^h \frac{H}{c_p T}\,dz - \int_0^h \boldsymbol{v}\cdot\frac{\nabla\vartheta}{\vartheta}\,dz - \int_0^h \frac{1}{q}\nabla\cdot(q\boldsymbol{v})\,dz + \frac{g}{\varkappa}\int_0^h \left(\int_z^\infty \nabla\cdot(q\boldsymbol{v})\,d\zeta\right)\frac{dz}{p}.$$

This expression yields a calculation of w when the instantaneous distribution of $\boldsymbol{v}$ and the variables of state is known.

The first and the second term on the right express the effect of vertical expansion due to heat conveyed to the air below the level $z = h$, and due to advective supply of potentially warmer air below $z = h$, respectively. The third term shows an effect of horizontal mass-divergence below $z = h$, giving descending motion for positive mass-divergence. The last term is an effect of compressibility, expressing expansion in the vertical direction due to horizontal mass-divergence at higher levels; this term contains mass-divergence below $z = h$ as well as mass-divergence above $z = h$.

It is seen that the two last terms to some extent are counter-acting, since they are affected in opposite ways by a mass-divergence below $z = h$. Therefore, it is not readily seen whether the contribution given to the vertical velocity by a positive mass-divergence below $z = h$ will be positive or negative. A better account of these things is obtained by a modification of the two last terms of eq. (3.3). Integration by parts in the third term gives:

$$-\int_0^h \frac{1}{q}\nabla\cdot(q\boldsymbol{v})\,dz = -\frac{1}{q_0}\int_0^h \nabla\cdot(q\boldsymbol{v})\,dz + \int_0^h \frac{1}{q^2}\frac{\partial q}{\partial z}\left(\int_z^h \nabla\cdot(q\boldsymbol{v})\,d\zeta\right)dz.$$

The last term may be written:

$$\frac{g}{\varkappa}\int_0^h \left(\int_z^\infty \nabla\cdot(q\boldsymbol{v})\,d\zeta\right)\frac{dz}{p} = \frac{g}{\varkappa}\int_0^h \left(\int_z^h \nabla\cdot(q\boldsymbol{v})\,d\zeta\right)\frac{dz}{p} + \frac{g}{\varkappa}\left(\int_0^h \frac{dz}{p}\right)\left(\int_h^\infty \nabla\cdot(q\boldsymbol{v})\,d\zeta\right),$$

showing separately the effect of horizontal mass-divergence below the level $z = h$, and of horizontal mass-divergence above that same level. Inserting these expressions into eq. (3.3), and noting that

$$\frac{gq}{\varkappa p} + \frac{1}{q}\frac{\partial q}{\partial z} = -\frac{1}{\vartheta}\frac{\partial\vartheta}{\partial z}$$

according to (1,3) and (2.2), we find:

$$(3.4)\quad w_{z=h} = \int_0^h \frac{H}{c_p T}\,dz - \int_0^h \boldsymbol{v}\cdot\frac{\nabla\vartheta}{\vartheta}\,dz - \frac{1}{q_0}\int_0^h \nabla\cdot(q\boldsymbol{v})\,dz - \int_0^h \frac{1}{q\vartheta}\frac{\partial\vartheta}{\partial z}\left(\int_z^h \nabla\cdot(q\boldsymbol{v})\,d\zeta\right)dz + \frac{g}{\varkappa}\left(\int_0^h \frac{dz}{p}\right)\left(\int_h^\infty \nabla\cdot(q\boldsymbol{v})\,d\zeta\right).$$

Here the effect of horizontal mass-divergence below $z = h$ is shown by the third and the fourth terms on the right-hand side. It is seen that positive mass-divergence below $z = h$ always gives a negative contribution to the vertical velocity at $z = h$, if the stratification below $z = h$ is stable or indifferent $\left(\frac{\partial\vartheta}{\partial z} \geqq 0\right)$. The effect of horizontal mass-divergence above $z = h$ is represented separately by the last term.

4. Prognostic use of the quasi-static system of equations.

Eqs. (1.2), (1.3), (1.7) and (3.3) form a complete system of equations, when H is known as a function of the independent and the dependent variables. These equations determine the future motion, when the initial state and motion is known (initial condition). The boundary conditions at $z = 0$ and $z = \infty$ are already taken into consideration, being implied in eqs. (3.3) and (1.7).

The main simplification attained by the quasi-static treatment consists in the fact that two of these equations, viz. eqs. (1.3) and (3.3), are free from time derivatives. For this reason, these equations may be characterized as "diagnostic", as distinct from the "prognostic" equations (1.2) and (1.7), which involve time derivatives. By means of the two "diagnostic" equations the quantities q and w are computable from the instantaneous distribution of $\boldsymbol{v}$ and p[1]). Thus, in the quasi-static theory, the hydrodynamic and thermodynamic state of the atmosphere is completely defined if we know the fields of pressure and horizontal velocity, whereas density and vertical velocity appear as merely auxiliary quantities (which might have been eliminated).

The problem of weather prognosis may therefore be reduced to the determination of the changes in the fields of pressure and horizontal velocity, when these fields are known at a certain initial instant (initial condition). This is attained by integration of the prognostic equations (1.2) and (1.7). This is what Richardson called a "marching problem", and it should be possible to carry out the integration by using a step-wise procedure, starting at the initial time. Such a prognostic utilization of the quasi-static system of equations entails the use of the tendency equation as a prognostic equation, starting with the observed initial velocities. It is well known, however, that the tendency equation is unsuitable for this purpose, because it is impossible to measure the wind with sufficient accuracy. This was shown by Margules [16]. The equation of motion (1.2) is probably also unserviceable as a prognostic equation, because the local acceleration must then be computed from the deviations from the geostrophic wind, which cannot be observed with sufficient accuracy.

We may conclude, therefore, that weather prognosis by numerical integration of the quasi-static system of equations, starting from a certain initial state which is determined by observation, is impracticable, at least as long as no further simplifying assumptions are applied than those made in this chapter. It will be shown later (section 33) that by utilizing the assumption of quasi-geostrophic approximation, one obtains a system of equations which may be better fitted for prognostic use.

[1]) T and ϑ should be interpreted as functions of p and q according to eqs. (2.1) and (2.2).

CHAPTER II. THE QUASI-STATIC EQUATIONS WITH PRESSURE AS THE VERTICAL COORDINATE.

5. Introductory remarks.

In the preceding chapter, the equations of motion were written down with x, y, z and t as independent variables. The corresponding method of three-dimensional representation of the state of the atmosphere is the drawing of isopleths of pressure, temperature, etcetera, at several fixed levels (analysis of constant levels). By this representation, the different horizontal and vertical ascendents, which occur in the equations, can be evaluated from the charts in a simple manner.

Another method of representation is the drawing of level curves, isotherms, etcetera, of several isobaric surfaces. This latter method, including the technique of building up the analysis layer by layer, starting from the ground chart, is worked out by V. Bjerknes and J. W. Sandstrøm [2]. As pointed out by these investigators and later by Petterssen [17], the latter method has many advantages as compared with the analysis of constant levels. Therefore, the isobaric surface analysis is the method used in practice in most countries.

The isobaric surface analysis corresponds to the interpretation of pressure as vertical coordinate instead of height. This also applies to the aerological ascents, which give measurements of temperature and humidity as functions of pressure. Therefore, when the hydrodynamic equations are to be used in connection with the aerological data, it appears convenient to have the equations transformed into the system of independent variables x, y, p and t. In this chapter, the transformation will be carried out, and the kinematics and dynamics in the coordinate system x, y, p will be discussed.

6. The equations of transformation.

For brevity, the independent variables x, y, z, t will be referred to in the following as "system z" whereas x, y, p, t will be called "system p".

The new coordinate p is a function of space and time,

$$p = p\,(x, y, z, t). \tag{6.1}$$

When x, y and t are constant, this equation expresses a one-to-one relationship between p and z. Solved with respect to z, it becomes:

$$z = z\,(x, y, p, t). \tag{6.2}$$

Thus, in the system p, z becomes a dependent variable. It seems more convenient, however, to introduce as a dependent variable instead of z the geopotential

$$\Phi = \Phi\,(x, y, p, t), \tag{6.3}$$

defined by

$$d\Phi = g dz. \tag{6.4}$$

Eq. (6.2) is the equation of transformation. By substituting this function for z, we can get any function of x, y, z and t transformed into a function of x, y, p and t. Hence, if α is any scalar function, we can write:

$$\alpha = A\,(x, y, z, t) = A\,(x, y, z\,(x, y, p, t), t) = B\,(x, y, p, t). \tag{6.5}$$

Eq. (6.1) is used if the reverse transformation is desired.

Partial differentiation of eq. (6.5) with constant x, y and t gives:

$$\frac{\partial \alpha}{\partial z} = \frac{\partial \alpha}{\partial p}\frac{\partial p}{\partial z} = -g q \frac{\partial \alpha}{\partial p}, \tag{6.6}$$

on account of the hydrostatic equation (1.3).

In partial differentiation of eq. (6.5) with respect to x, y and t, it must be remembered that the derivatives of the functions A and B are not equal. Thus $\frac{\partial A}{\partial x}$ or $\frac{\partial A}{\partial y}$ expresses the change of α in horizontal direction (with constant z) whereas $\frac{\partial B}{\partial x}$ or $\frac{\partial B}{\partial y}$ expresses the change of α along an isobaric surface (p constant). In like manner, $\frac{\partial A}{\partial t}$ gives the change of α with time at a fixed level (z constant), whereas $\frac{\partial B}{\partial t}$ gives the change of α at a fixed isobaric surface (p constant). To prevent confusion, we shall in the following denote by a subscript the quantity which is held constant by the operation in question; thus we write:

$$\frac{\partial A}{\partial x} = \left(\frac{\partial \alpha}{\partial x}\right)_z, \quad \frac{\partial A}{\partial y} = \left(\frac{\partial \alpha}{\partial y}\right)_z, \quad \frac{\partial A}{\partial t} = \left(\frac{\partial \alpha}{\partial t}\right)_z,$$

$$\frac{\partial B}{\partial x} = \left(\frac{\partial \alpha}{\partial x}\right)_p, \quad \frac{\partial B}{\partial y} = \left(\frac{\partial \alpha}{\partial y}\right)_p, \quad \frac{\partial B}{\partial t} = \left(\frac{\partial \alpha}{\partial t}\right)_p.$$

In this notation, all the derivatives with respect to x, y and t, which occur in chapter I, must be written with the subscript z.

Partial differentiation of eq. (6.5) with respect to x gives, according to a well known principle of differential analysis:

$$\left(\frac{\partial \alpha}{\partial x}\right)_p = \left(\frac{\partial \alpha}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial \alpha}{\partial z}.$$

Writing here, according to eq. (6.4):

$$\left(\frac{\partial z}{\partial x}\right)_p = \frac{1}{g}\left(\frac{\partial \Phi}{\partial x}\right)_p,$$

and applying eq. (6.6), we obtain:

$$\left(\frac{\partial \alpha}{\partial x}\right)_z = \left(\frac{\partial \alpha}{\partial x}\right)_p + q\left(\frac{\partial \Phi}{\partial x}\right)_p \frac{\partial \alpha}{\partial p}. \tag{6.7}$$

In the same way we find:

$$\left(\frac{\partial \alpha}{\partial y}\right)_z = \left(\frac{\partial \alpha}{\partial y}\right)_p + q\left(\frac{\partial \Phi}{\partial y}\right)_p \frac{\partial \alpha}{\partial p}, \tag{6.8}$$

$$\left(\frac{\partial \alpha}{\partial t}\right)_z = \left(\frac{\partial \alpha}{\partial t}\right)_p + q\left(\frac{\partial \Phi}{\partial t}\right)_p \frac{\partial \alpha}{\partial p}. \tag{6.9}$$

The horizontal del-operator used in chapter I must now be written, in accordance with the notation introduced above:

$$\nabla_z = \boldsymbol{i}\left(\frac{\partial}{\partial x}\right)_z + \boldsymbol{j}\left(\frac{\partial}{\partial y}\right)_z. \tag{6.10}$$

Similarly, we may in the system p define a horizontal vector operator, expressing the variation along an isobaric surface:

$$\nabla_p = \boldsymbol{i}\left(\frac{\partial}{\partial x}\right)_p + \boldsymbol{j}\left(\frac{\partial}{\partial y}\right)_p. \tag{6.11}$$

From (6.7) and (6.8) we find:

$$\nabla_z \alpha = \nabla_p \alpha + q \nabla_p \Phi \frac{\partial \alpha}{\partial p}. \tag{6.12}$$

The quantity $\nabla_p \alpha$ is a *horizontal* vector, denoting the increase of α along an *isobaric surface* per *horizontal* length unit, and pointing in the direction where this increase is greatest. $\nabla_p \alpha$ can be evaluated from the isopleths of α on an isobaric surface chart, in the same manner as the horizontal ascendent $\nabla_z \alpha$ is found from the isopleths of α in an horizontal level chart.

It is desirable to have a name for the vector $\nabla_p a$, and for lack of a better one, it will in this paper be called *the horizontal p-ascendent of* a. In the text book by Holmboe, Forsythe and Gustin [11] the vector $\nabla_p a$, which is introduced in order to simplify the thermal wind equation, is called "the horizontal isobaric ascendent". I have not adopted that name here, bceause it seems to be a little confusing.

By applying the operator ∇_p to a horizontal vector field $\boldsymbol{A} = A_x \boldsymbol{i} + A_y \boldsymbol{j}$, we may define a horizontal p-divergence and a vertical p-vorticity:

$$\nabla_p \cdot \boldsymbol{A} = \left(\frac{\partial A_x}{\partial x}\right)_p + \left(\frac{\partial A_y}{\partial y}\right)_p, \tag{6.13}$$

$$\boldsymbol{k} \cdot \nabla_p \times \boldsymbol{A} = \left(\frac{\partial A_y}{\partial x}\right)_p - \left(\frac{\partial A_x}{\partial y}\right)_p. \tag{6.14}$$

These quantities can be evaluated from a representation of the field of $\boldsymbol{A}$ on isobaric surface charts, in the same manner as the horizontal divergence and the vertical vorticity are determined from a representation of the field of $\boldsymbol{A}$ on horizontal level charts. From (6.7) and (6.8) we find:

$$\nabla_z \cdot \boldsymbol{A} = \nabla_p \cdot \boldsymbol{A} + q \nabla_p \Phi \cdot \frac{\partial \boldsymbol{A}}{\partial p}, \tag{6.15}$$

$$\boldsymbol{k} \cdot \nabla_z \times \boldsymbol{A} = \boldsymbol{k} \cdot \nabla_p \times \boldsymbol{A} + q \boldsymbol{k} \cdot \nabla_p \Phi \times \frac{\partial \boldsymbol{A}}{\partial p}. \tag{6.16}$$

Eqs. (6.12), (6.15) and (6.16) may be interpreted as applications of an operator equation:

$$\nabla_z = \nabla_p + q \nabla_p \Phi \frac{\partial}{\partial p}. \tag{6.17}$$

7. Geometry and motion of scalar fields.

Consider a scalar function $a = a(x, y, z, t)$. The equation of an equiscalar surface at a certain instant is obtained by putting a constant and t constant; then z becomes a function of x and y. Let us denote the ascendent of this function by $\nabla_a z$. This is a horizontal vector pointing in the direction where z has the most rapid increase, i. e. perpendicular to the curves $a =$ constant in the horizontal plane. The numerical value of the vector is seen to represent the slope of the tangent plane of the equiscalar surface. Thus the orientation of this tangent plane is represented in a very perspicuous way by the vector $\nabla_a z$. We may call this vector the "slope vector" of the equiscalar surface. It is easily seen that:

$$\nabla_a z = - \frac{\nabla_z a}{\dfrac{\partial a}{\partial z}}. \tag{7.1}$$

In the system p, an analogous consideration gives the result that the orientation of the tangent plane of the equiscalar surface is determined by the horizontal vector

$$\nabla_a p = - \frac{\nabla_p a}{\dfrac{\partial a}{\partial p}}. \tag{7.2}$$

This vector is directed perpendicular to the curves $a =$ constant in the isobaric surface. Its numerical value represents the "slope" of the equiscalar surface relative to the isobaric surface, expressed in millibars per meter. For lack of a better name, this slope of a surface relative to an isobaric surface, expressed in mb/m, will in the following be called "p-slope", and the vector $\nabla_a p$ will be called the "p-slope vector".

Applying eqs. (6.6) and (6.12) we find:

$$\nabla_a z = - \frac{1}{gq} \nabla_a p + \frac{1}{g} \nabla_p \Phi = - \frac{1}{gq} \nabla_a p + \nabla_p z, \tag{7.3}$$

showing the relation between the "slope vector" and the "p-slope-vector". Here $\nabla_p z$ is the slope vector of the isobaric surface. It will be seen that if the surface $a =$ constant is much steeper than the isobaric surface, then $\nabla_a z$ and $\nabla_a p$ point in nearly opposite directions; and this is strictly true if the isobaric surface is horizontal.

Now let us consider the vertical motion of the equiscalar surface $a =$ constant. In the system z, the vertical velocity of the surface is given by:

$$\left(\frac{\partial z}{\partial t}\right)_a = - \frac{\left(\dfrac{\partial a}{\partial t}\right)_z}{\dfrac{\partial a}{\partial z}}. \tag{7.4}$$

On the other hand, in the system p, the quantity

$$\left(\frac{\partial p}{\partial t}\right)_a = - \frac{\left(\dfrac{\partial a}{\partial t}\right)_p}{\dfrac{\partial a}{\partial p}} \tag{7.5}$$

gives the vertical velocity of the surface relative to the isobaric surface, expressed in millibars per second. This quantity will be called "the vertical p-velocity" of the equiscalar surface. From eqs. (6.6) and (6.9) we find:

$$(7.6)\qquad \left(\frac{\partial z}{\partial t}\right)_\alpha = -\frac{1}{gq}\left(\frac{\partial p}{\partial t}\right)_\alpha + \left(\frac{\partial z}{\partial t}\right)_p,$$

where the last term represents the vertical velocity of the isobaric surface.

Finally let us consider the horizontal motion of the equiscalar surface $\alpha =$ constant. In the system z, the horizontal velocity ($\boldsymbol{c}_z$) of a curve $\alpha =$ constant in the horizontal plane is given by

$$(7.7)\qquad \left(\frac{\partial \alpha}{\partial t}\right)_z + \boldsymbol{c}_z \cdot \nabla_z \alpha = 0.$$

Only the component of $\boldsymbol{c}_z$ perpendicular to the curve α=constant is involved in this equation; the component parallel to the curve $\alpha =$ constant is seen to be immaterial. By using eqs. (7.1) and (7.4), this equation may also be written:

$$(7.8)\qquad \left(\frac{\partial z}{\partial t}\right)_\alpha + \boldsymbol{c}_z \cdot \nabla_\alpha z = 0.$$

In the system p, we have an analogous formula for the horizontal velocity ($\boldsymbol{c}_p$) of the curve $\alpha =$ constant in the isobaric surface:

$$(7.9)\qquad \left(\frac{\partial \alpha}{\partial t}\right)_p + \boldsymbol{c}_p \cdot \nabla_p \alpha = 0.$$

By using eqs. (7.2) and (7.5), this may also be written:

$$(7.10)\qquad \left(\frac{\partial p}{\partial t}\right)_\alpha + \boldsymbol{c}_p \cdot \nabla_\alpha p = 0.$$

8. The field of geopotential.

Putting $\alpha = \Phi$ in eq. (6.6), we obtain

$$(8.1)\qquad \frac{\partial \Phi}{\partial p} = -s,$$

which is the form assumed by the hydrostatic equation in the system p.

From eqs. (6.9) and (6.12) we get, by putting $\alpha = p$:

$$(8.2)\qquad \left(\frac{\partial p}{\partial t}\right)_z = q\left(\frac{\partial \Phi}{\partial t}\right)_p,$$

$$(8.3)\qquad \nabla_z p = q\,\nabla_p \Phi,$$

showing that the pressure tendency and the horizontal pressure ascendent are expressible in the system p as derivatives of Φ. Notice that according to eqs. (7.2) and (7.5), $q\nabla_p\Phi$ and $q\left(\frac{\partial \Phi}{\partial t}\right)_p$ may be interpreted as the "p-slope vector" and the "vertical p-velocity" of the equipotential surfaces, respectively.

9. The fields of the thermodynamic quantities.

We have assumed that the temperature and the potential temperature are functions of p and q only. Hence, if equidensity curves are drawn in an isobaric surface, T and ϑ will be constant along these curves. In other words: in an isobaric surface, the isopleths of the various functions of state will coincide. This makes the representation of the state of the air by means of isobaric surface charts very simple, as pointed out by Petterssen [17].

The derivatives with respect to x, y and t of the various quantities of state are also related in a very simple way in the system p. If $f(p, s)$ is any function of state, then we have:

$$(9.1)\qquad \nabla_p f = \frac{\partial f}{\partial s}\nabla_p s,$$

$$(9.2)\qquad \left(\frac{\partial f}{\partial t}\right)_p = \frac{\partial f}{\partial s}\left(\frac{\partial s}{\partial t}\right)_p.$$

Here, $\nabla_p s$ and $\left(\frac{\partial s}{\partial t}\right)_p$ may also be expressed by the derivatives of Φ, using the hydrostatic equation (8.1). Thus, from eqs. (2.1) and (2.2),

$$(9.3)\qquad \frac{1}{s}\nabla_p s = \frac{1}{T}\nabla_p T = \frac{1}{\vartheta}\nabla_p\vartheta = -q\frac{\partial}{\partial p}\nabla_p\Phi,$$

$$(9.4)\qquad \frac{1}{s}\left(\frac{\partial s}{\partial t}\right)_p = \frac{1}{T}\left(\frac{\partial T}{\partial t}\right)_p = \frac{1}{\vartheta}\left(\frac{\partial \vartheta}{\partial t}\right)_p = -q\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right)_p.$$

The relation between the derivatives of the same quantities with respect to p is

$$(9.5)\qquad \frac{1}{s}\frac{\partial s}{\partial p} = \frac{1}{T}\frac{\partial T}{\partial p} - \frac{1}{p} = \frac{1}{\vartheta}\frac{\partial \vartheta}{\partial p} - \frac{1}{\varkappa p} = -q\frac{\partial^2 \Phi}{\partial p^2}.$$

By means of these equations, we can find the "p-slope-vectors" and the vertical p-velocities of the equidense, isothermal and isentropic surfaces, expressed by the derivatives of Φ.

Notice that in the case of barotropy, $\nabla_p s = 0$, and all the terms of (9.3) vanish. In a baroclinic atmosphere, $\nabla_p s \neq 0$, $\nabla_p T \neq 0$ and $\nabla_p \vartheta \neq 0$.

10. Velocity components. Individual differentiation.

In the system p, the horizontal motion of the air will be represented by the horizontal velocity $\boldsymbol{v}$, defined in the same way as in the system z. This is because the same horizontal coordinates (x, y) are used in both systems.

The vertical coordinate, however, is not the same in the two systems, and therefore, the vertical motion will not be represented in the same way in both systems. In the system z, we have used the vertical velocity $w = \frac{Dz}{dt}$. In the system p we use the individual pressure change:

$$\omega = \frac{Dp}{dt}. \tag{10.1}$$

This quantity represents the vertical velocity relative to the isobaric surfaces, expressed in millibars per second, and will be called the vertical p-velocity of the air. Positive ω means descending, and negative ω ascending motion relatively to the isobaric surfaces.

The vertical motions in the atmosphere are important especially because they cause individual changes of state (e. g. condensation or subsidence). Such changes of state are directly related to ω, but not to w; this shows that the system p gives a simpler connection between the thermodynamics and the kinematics of the motion.

The Eulerian expansion of the individual derivative becomes, by analogy to eq. (1.1):

$$\frac{D}{dt} = \left(\frac{\partial}{\partial t}\right)_p + \boldsymbol{v}\cdot\nabla_p + \omega\frac{\partial}{\partial p}. \tag{10.2}$$

Applying this formula to Φ and providing for eq. (8.1), we find:

$$gw = \frac{D\Phi}{dt} = \left(\frac{\partial \Phi}{\partial t}\right)_p + \boldsymbol{v}\cdot\nabla_p\Phi - s\omega, \tag{10.3}$$

showing the relation between w and ω. The two first terms on the right-hand side correspond to vertical velocities which seldom exceed 1 cm per second. Thus, if no great accuracy is needed, we may write:

$$w = -\frac{\omega}{gq}. \tag{10.4}$$

11. The equation of motion. Vorticity equation.

We are now able to transform the hydrodynamic equations into the system p, and we start with the horizontal equation of motion (1.2). By means of eqs. (8.3), (10.2) and (10.4), this equation assumes the form:

$$\frac{D\boldsymbol{v}}{dt} = \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_p + \boldsymbol{v}\cdot\nabla_p\boldsymbol{v} + \omega\frac{\partial \boldsymbol{v}}{\partial p}$$
$$= -\nabla_p\Phi - 2\Omega_z\boldsymbol{k}\times\boldsymbol{v} + \frac{2\Omega_y}{gq}\omega\boldsymbol{i}. \tag{11.1}$$

The last term of eq. (1.2), representing the Coriolis effect due to the vertical motion, is very small. Hence, it seems to be of negligible consequence that we in this term have used the approximate eq. (10.4), instead of the exact eq. (10.3).

For most of the applications, the Coriolis force due to the vertical motion can be neglected:

$$\frac{D\boldsymbol{v}}{dt} = -\nabla_p\Phi - 2\Omega_z\boldsymbol{k}\times\boldsymbol{v}. \tag{11.2}$$

Here, the density has dropped out, and the equation corresponds formally to the equation of motion for a homogeneous and incompressible fluid with pressure Φ and density 1.

From (11.2) we derive the equation for "vertical p-vorticity" $\zeta_p = \boldsymbol{k}\cdot\nabla_p\times\boldsymbol{v}$ by applying the operation $\boldsymbol{k}\cdot\nabla_p\times$. Since $\boldsymbol{k}\cdot\nabla_p\times\frac{D\boldsymbol{v}}{dt} = \frac{D}{dt}(\boldsymbol{k}\cdot\nabla_p\times\boldsymbol{v}) + (\boldsymbol{k}\cdot\nabla_p\times\boldsymbol{v})(\nabla_p\cdot\boldsymbol{v}) + \boldsymbol{k}\cdot\nabla_p\omega\times\frac{\partial \boldsymbol{v}}{\partial p}$, we find:

$$\frac{D}{dt}(2\Omega_z + \zeta_p)$$
$$= -(2\Omega_z + \zeta_p)\nabla_p\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\nabla_p\omega\times\frac{\partial \boldsymbol{v}}{\partial p}. \tag{11.3}$$

This equation is quite analogous to the equation for vertical vorticity in the system z, except that the solenoid term is absent. This is not surprising, since an isobaric surface can obviously not be intersected by solenoids.

12. Geostrophic wind.

Putting in eq. (11.2) $\frac{D\boldsymbol{v}}{dt} = 0$, we obtain the formula for the geostrophic wind:

$$\nabla_p\Phi = -2\,\Omega_z\,\boldsymbol{k}\times\boldsymbol{v}_g, \tag{12.1}$$

$$\text{or}\quad \boldsymbol{v}_g = \frac{1}{2\Omega_z}\boldsymbol{k}\times\nabla_p\Phi,$$

which is simpler than the corresponding formula in the system z. Writing eq. (12.1) for two different points on a vertical line, and subtracting, we get the well known relation between thermal wind and thickness of an isobaric layer.

Differentiating eq. (12.1) with respect to p an using (9.3), we find:

$$(12.2)\quad \frac{\partial \boldsymbol{v}_g}{\partial p}=\frac{1}{2\Omega_z}\boldsymbol{k}\times\nabla_p\frac{\partial\Phi}{\partial p}=-\frac{1}{2\Omega_z}\boldsymbol{k}\times\nabla_p s$$

$$=-\frac{1}{2\Omega_z}\frac{s}{T}\boldsymbol{k}\times\nabla_p T=-\frac{1}{2\Omega_z}\frac{s}{\vartheta}\boldsymbol{k}\times\nabla_p\vartheta,$$

or
$$\frac{\nabla_p s}{s}=\frac{\nabla_p T}{T}=\frac{\nabla_p\vartheta}{\vartheta}=2\,\Omega_z q\,\boldsymbol{k}\times\frac{\partial \boldsymbol{v}_g}{\partial p}.$$

Thus the variation of the geostrophic wind with height is determined by the horizontal p-ascendent of any quantity of state. And, conversely, if we content ourselves with the geostrophic wind as an approximation to the true wind, the horizontal p-ascendent of any quantity of state can be evaluated from the wind observations and the ascent curve. With the same approximation, we are also able to determine the "p-slope vectors" of the equidensity, isothermal and isentropic surfaces, when the ascent curve and wind observations are available. For instance, we have for the p-slope vector of the isentropic surfaces, according to eq. (7.2)

$$(12.3)\quad \nabla_\vartheta p=-\frac{\nabla_p\vartheta}{\dfrac{\partial\vartheta}{\partial p}}=-\frac{2\,\Omega_z q}{\dfrac{1}{\vartheta}\dfrac{\partial\vartheta}{\partial p}}\boldsymbol{k}\times\frac{\partial \boldsymbol{v}_g}{\partial p}.$$

This method may be useful in drawing of cross-section diagrams.

Taking the isobaric divergence of the geostrophic wind, we get

$$(12.4)\quad \nabla_p\cdot\boldsymbol{v}_g=-\boldsymbol{v}_g\cdot\frac{\nabla_p\,\Omega_z}{\Omega_z}.$$

Thus $\nabla_p\cdot\boldsymbol{v}_g$ is due to the variation of the Coriolis parameter with latitude, and vanishes if Ω_z is considered as a constant. In the system z, this quality is attributed to $\nabla_z\cdot(q\boldsymbol{v}_g)$; and in fact we have

$$(12.5)\quad \nabla_z\cdot(q\boldsymbol{v}_g)=q\nabla_p\cdot\boldsymbol{v}_g\,,$$

which is easily verified by means of the equations of transformation.

13. Surface of discontinuity. Orientation and movement of a frontal surface.

A frontal surface may, approximately, be considered as a surface of discontinuity (of the order zero) in the air. At a surface of discontinuity, the dependent variables must satisfy the dynamic and kinematic boundary conditions (see "Physikalische Hydrodynamik" [3]). These conditions will now be written down in the system p. Let $\boldsymbol{v}_1$, ω_1, Φ_1, s_1 and $\boldsymbol{v}_2$, ω_2, Φ_2, s_2 denote the dependent variables on the two sides of a surface of discontinuity. The dynamical boundary condition expresses that the pressure field, in the system z, is continuous at the surface of discontinuity. In the system p, this is expressed by:

$$(13.1)\quad F=\Phi_1(x,y,p,t)-\Phi_2(x,y,p,t)=0$$

at the surface. This equation may be interpreted as the equation of the surface of discontinuity.

From (13.1) we may deduce formulae for the orientation and movement of a frontal surface. Thus we find the p-slope vector of a frontal surface, by putting $a=F$ in eq. (7.2)

$$(13.2)\quad \nabla_F p=-\frac{\nabla_p\Phi_1-\nabla_p\Phi_2}{\dfrac{\partial\Phi_1}{\partial p}-\dfrac{\partial\Phi_2}{\partial p}}.$$

By means of the hydrostatic equation (8.1) and the geostrophic wind formula (12.1), this equation may be rewritten as:

$$(13.3)\quad \nabla_F p=-\frac{2\Omega_z\,\boldsymbol{k}\times(\boldsymbol{v}_{g_1}-\boldsymbol{v}_{g_2})}{s_1-s_2}.$$

According to (7.5), the vertical p velocity of the frontal surface is expressed by:

$$(13.4)\quad \left(\frac{\partial p}{\partial t}\right)_F=\frac{\left(\dfrac{\partial\Phi_1}{\partial t}\right)_p-\left(\dfrac{\partial\Phi_2}{\partial t}\right)_p}{s_1-s_2}.$$

The horizontal velocity of the front in an isobaric surface is seen from (7.9) to fulfil the condition

$$(13.5)\quad \left(\frac{\partial F}{\partial t}\right)_p+\boldsymbol{c}\cdot\nabla_p F=0\,,$$

or, dividing by $\dfrac{\partial F}{\partial p}$,

$$(13.6)\quad \left(\frac{\partial p}{\partial t}\right)_F+\boldsymbol{c}\cdot\nabla_F\,p=0.$$

In these two equations, only the component of $\boldsymbol{c}$ perpendicular to the front in an isobaric surface is involved.

The kinematical boundary condition expresses that the surface of discontinuity (frontal surface) all the time consists of the same air particles. In mathematical form, this may be written:

$$\frac{DF}{dt} = 0 \text{ when } F = 0,$$

applying to particles on either side of the surface.

Expanding the operator $\frac{D}{dt}$ according to (10.2), we get:

$$(13.7)\left\{\begin{array}{l} \left(\frac{\partial F}{\partial t}\right)_p + \boldsymbol{v}_1 \cdot \nabla_p F + \omega_1 \frac{\partial F}{\partial p} = 0 \\ \left(\frac{\partial F}{\partial t}\right)_p + \boldsymbol{v}_2 \cdot \nabla_p F + \omega_2 \frac{\partial F}{\partial p} = 0 \end{array}\right\} \text{when } F = 0.$$

Dividing by $\frac{\partial F}{\partial p}$, we obtain

$$(13.8) \quad \left\{\begin{array}{l} \omega_1 = \left(\frac{\partial p}{\partial t}\right)_F + \boldsymbol{v}_1 \cdot \nabla_F p \\ \omega_2 = \left(\frac{\partial p}{\partial t}\right)_F + \boldsymbol{v}_2 \cdot \nabla_F p \end{array}\right\} \text{ when } F = 0.$$

In this latter form, the kinematical boundary condition could have been derived directly by geometrical considerations in the x, y, p-space.

Using (13.6), eqs. (13.8) may also be written:

$$(13.9) \quad \left.\begin{array}{l} \omega_1 = (\boldsymbol{v}_1 - \boldsymbol{c}) \cdot \nabla_F p \\ \omega_2 = (\boldsymbol{v}_2 - \boldsymbol{c}) \cdot \nabla_F p \end{array}\right\} \text{when } F = 0.$$

Eqs. (13.7), (13.8) and (13.9) are equivalent forms of the kinematical boundary conditions for a frontal surface. Subtracting the two eqs. (13.8), and using (13.3), we obtain:

$$(13.10) \quad \omega_1 - \omega_2 = (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \nabla_F p$$
$$= -\frac{2\,\Omega_z}{s_1 - s_2} (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{k} \times (\boldsymbol{v}_{g_1} - \boldsymbol{v}_{g_2})$$
$$\text{when } F = 0.$$

From this it will be seen that if the wind is geostrophic on both sides of the frontal surface ($\boldsymbol{v}_1 = \boldsymbol{v}_{g_1}$, $\boldsymbol{v}_2 = \boldsymbol{v}_{g_2}$), then $\omega_1 - \omega_2$ must be zero, i. e. the individual pressure changes are equal in the two air masses near the front. Hence, if $\omega_1 - \omega_2$ is different from zero, there must be deviations from the geostrophic wind. Denoting by $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$ the deviations from the geostrophic winds, eq. (13.10) may be written:

$$(13.11) \quad \omega_1 - \omega_2 = (\boldsymbol{v}'_1 - \boldsymbol{v}'_2) \cdot \nabla_F p.$$

Thus $\omega_1 - \omega_2$ is related to the component of $(\boldsymbol{v}'_1 - \boldsymbol{v}'_2)$ perpendicular to the front. If the trajectories of the air are anticyclonically curved in the warm mass and cyclonically curved in the cold mass, this will mean that ω, for a warm front, is smaller in the warm mass than in the cold mass, corresponding to upsliding motion of the warm air relative to the cold air. For a cold front, the converse is seen to be true, corresponding to subsidence in the warm air. Isallobaric winds directed towards the front in both air masses are seen to imply that ω, for all types of fronts, is smaller in the warm air than in the cold air, corresponding to upsliding motion of warm air relative to cold air.

14. Boundary conditions at the earth's surface and the upper limit of the atmosphere.

The surface of the earth, while a rigid surface in the system z, is moving in the system p. If the earth's surface is taken as the zero level of geopotential, then

$$(14.1) \quad \Phi(x, y, p, t) = 0, \text{ or } p = p_0(x, y, t)$$

is the equation of this surface in the system p. The kinematical boundary condition at this surface is:

$$(14.2) \quad \frac{D\Phi}{dt} = \left(\frac{\partial \Phi}{\partial t}\right)_p + \boldsymbol{v} \cdot \nabla_p \Phi - s\omega = 0$$
$$\text{when } p = p_0.$$

The form of this boundary condition is seen to be more complicated in the system p than in the system z, because the earth's surface is moving in the system p. This difficulty may perhaps in some cases counter-balance the advantages attained in using the system p

Consider now the upper limit of the atmosphere. The equation

$$(14.3) \quad p = 0$$

may be interpreted as the equation of this upper limit, which is thus seen to be a well-defined, "rigid" surface in the system p. The boundary condition at this surface becomes:

$$(14.4) \quad \frac{Dp}{dt} = \omega = 0 \text{ when } p = 0.$$

In the system p, we may formally interpret p as a vertical, geometrical coordinate, and hence, ω as the vertical velocity. Furthermore, Φ, on many accounts, plays the role of pressure (see for instance eq. (11.2)). Hence, we may interpret eq. (14.2) as the boundary condition at a free surface, and (14.4) as the boundary condition at a rigid, horizontal bounding plane. Thus we have the following paradox: In the system p, the plane surface of the earth corresponds to a free surface, whereas the upper limit of the atmosphere corresponds to a rigid bounding plane.

15. The equation of continuity. Tendency equation.

With the notation of this chapter eq. (1.4) becomes:

$$\left(\frac{\partial q}{\partial t}\right)_z + \nabla_z \cdot (q\boldsymbol{v}) + \frac{\partial (qw)}{\partial z} = 0.$$

This equation will now be transformed to the system p. From (6.15) and (8.1), we find:

$$\text{(15.1)}\qquad \nabla_z \cdot (q\boldsymbol{v}) = q\nabla_p \cdot \boldsymbol{v} + q\frac{\partial}{\partial p}(q\boldsymbol{v}\cdot\nabla_p\Phi).$$

Applying further eqs. (6.6), (6.9) and (10.3), and assuming g to be a constant, we arrive at

$$\text{(15.2)}\qquad \nabla_p \cdot \boldsymbol{v} + \frac{\partial \omega}{\partial p} = 0.$$

This is the remarkably simple form assumed by the equation of continuity in the system p. The simplicity is due to the fact that the coordinate p, with constant g, represents the mass per unit area of the air above the corresponding isobaric surface.

Eq. (15.2) can be derived more directly by geometrical considerations in the x, y, p-space. Consider a fixed infinitesimal parallelepiped in the x, y, p-space with side lengths dx, dy and dp. The mass of air contained in it is $\frac{1}{g}\,dxdydp =$ constant. The net outflow of mass from the parallelepiped, which can be shown by simple geometrical considerations to be $\frac{1}{g}\left(\nabla_p \cdot \boldsymbol{v} + \frac{\partial\omega}{\partial p}\right) dxdydp$, must then be zero.

Interpreting p as a vertical, geometrical coordinate, and hence, ω as a vertical velocity, eq. (15.2) is seen to express that in the x, y, p-space, the air moves as an incompressible fluid. Owing to this quality, the mathematical treatment of the motion may be considerably simplified by using the system p.

It may be convenient to satisfy eq. (15.2) by introducing a vector potential, and it can be shown that this vector potential may be chosen horizontal, without laying any restrictions upon the velocity field. Denoting this horizontal vector potential by $\boldsymbol{A}$, we have:

$$\text{(15.3)}\qquad \boldsymbol{v} = \boldsymbol{k}\times\frac{\partial \boldsymbol{A}}{\partial p},\quad \omega = \boldsymbol{k}\cdot\nabla_p\times\boldsymbol{A}.$$

Integrating the equation of continuity between the limits 0 and p, we obtain, on account of the boundary condition (14.4):

$$\text{(15.4)}\qquad \omega = -\int_0^p \nabla_p \cdot \boldsymbol{v}\,d\pi,$$

where the variable of integration is denoted by π to prevent confusion. This equation shows that the field of ω is determined by the field of $\boldsymbol{v}$, and is the parallel to the complicated expression for the vertical velocity (3.3) in the system z.

Combining (15.4) with (14.2), and denoting the values at the ground by the subscript 0, we find

$$\text{(15.5)}\qquad \left(\frac{\partial \Phi}{\partial t}\right)_{p,0} = -\boldsymbol{v}_0\cdot(\nabla_p\Phi)_0 - s_0\int_0^{p_0}\nabla_p\cdot\boldsymbol{v}\,dp.$$

This is the equation in the system p for the tendency at the ground, corresponding to eq. (1.7) in the system z. It should be noted that in (15.5), the tendency is expressed as an integral effect of $\nabla_p\cdot\boldsymbol{v}$, whereas in eq. (1.7) the integral effect of $\nabla_z\cdot(q\boldsymbol{v})$ occurs.

Since $\int_0^{p_0}\boldsymbol{v}\,dp$ is a function of x, y and t only, we have:

$$s_0\nabla\cdot\int_0^{p_0}\boldsymbol{v}\,dp = s_0\int_0^{p_0}\nabla_p\cdot\boldsymbol{v}\,dp + \boldsymbol{v}_0\cdot s_0\nabla_z p_0$$
$$= s_0\int_0^{p_0}\nabla_p\cdot\boldsymbol{v}\,dp + \boldsymbol{v}_0\cdot(\nabla_p\Phi)_0,$$

in virtue of (8.3). Hence, eq. (15.5) may be rewritten as

$$\text{(15.6)}\qquad \left(\frac{\partial \Phi}{\partial t}\right)_{p,0} = -s_0\nabla\cdot\int_0^{p_0}\boldsymbol{v}\,dp,$$

in accordance with a formula given by Petterssen [17].

16. The first law of thermodynamics.

When $\frac{D\vartheta}{dt}$ is expanded according to (10.2), eq. (2.4) becomes

$$(16.1) \quad \frac{1}{\vartheta}\left(\frac{\partial\vartheta}{\partial t}\right)_p + \boldsymbol{v}\cdot\frac{\nabla_p\vartheta}{\vartheta} + \omega\frac{1}{\vartheta}\frac{\partial\vartheta}{\partial p} = \frac{H}{c_pT},$$

which is the first law of thermodynamics, written in the system p. By using (9.3), (9.4) and (9.5), the derivatives of ϑ may be replaced by the derivatives of other quantities of state. Using (9.4), we can also deduce a formula for the variation of tendency with height:

$$(16.2) \quad \frac{\partial}{\partial p}\left(\frac{\partial\Phi}{\partial t}\right)_p = -\frac{sH}{c_pT} + \frac{s}{\vartheta}\left(\boldsymbol{v}\cdot\nabla_p\vartheta + \omega\frac{\partial\vartheta}{\partial p}\right)$$
$$= -\frac{\varkappa-1}{\varkappa}\frac{H}{p} + \frac{R}{\pi_0^{\frac{\varkappa-1}{\varkappa}}}\frac{1}{p^{\frac{1}{\varkappa}}}\left(\boldsymbol{v}\cdot\nabla_p\vartheta + \omega\frac{\partial\vartheta}{\partial p}\right).$$

This equation is seen to be simpler than the corresponding equation (2.5) in the system z. Integrating (16.1) between the arbitrary pressure values p_1 and p_2 $(p_1 < p_2)$, we get:

$$(16.3) \quad \frac{\partial(\Phi_1-\Phi_2)}{\partial t} = \frac{\varkappa-1}{\varkappa}\int_{p_1}^{p_2} H\frac{dp}{p}$$
$$- \frac{R}{\pi_0^{\frac{\varkappa-1}{\varkappa}}}\int_{p_1}^{p_2}\left(\boldsymbol{v}\cdot\nabla_p\vartheta + \omega\frac{\partial\vartheta}{\partial p}\right)\frac{dp}{p^{\frac{1}{\varkappa}}},$$

expressing the variation with time of the dynamic thickness of an isobaric layer. It is seen that the thickness may increase for three reasons: (i) heat conveyed to the air, (ii) "isobaric" advection of warmer air, and (iii) individual pressure increase (with stable stratification), i e. subsidence.

Putting in eq. (16.3) $\Phi_2 = \Phi_0$ and $p_2 = p_0$ (where subscript 0 denotes values at the earth's surface), and applying eq. (15.5), we find an expression for the tendency at the arbitrary level p_1:

$$(16.4) \quad \left(\frac{\partial\Phi}{\partial t}\right)_{p=p_1} = -\boldsymbol{v}_0\cdot(\nabla_p\Phi)_0 - s_0\int_0^{p_0}\nabla_p\cdot\boldsymbol{v}\,dp$$
$$+ \frac{\varkappa-1}{\varkappa}\int_{p_1}^{p_0} H\frac{dp}{p} - \frac{R}{\pi_0^{\frac{\varkappa-1}{\varkappa}}}\int_{p_1}^{p_0}\left(\boldsymbol{v}\cdot\nabla_p\vartheta + \omega\frac{\partial\vartheta}{\partial p}\right)\frac{dp}{p^{\frac{1}{\varkappa}}}.$$

Eq. (15.5) is a special case of this equation, obtained by putting $p_1 = p_0$.

17. The complete system of equations.

In section 4, we considered the complete quasi-static system of equations, which control the state and motion in the future when the initial state and motion are known and H is known as a function of the independent and the dependent variables. The equations were then written in the system z; now we will consider the corresponding equations in the system p.

From a fundamental point of view, it is obviously of no consequence whether we operate in the system z or in the system p; the system of equations will in both cases express the same physical meaning, and nothing can be deduced from the equations in the system p, which could not also have been found in the system z. But even though the equations have the same meaning in both cases, they will not have the same form; and the system p seems to be preferable in most cases, on account of its greater simplicity.

In the system p, the quasi-static system of equations consists of eqs. (8.1), (11.2), (15.4) and (16.4). The boundary conditions at the earth's surface and at the upper limit of the atmosphere are included in eqs. (15.4) and (16.4).

Two of the equations are of the "diagnostic" type, viz. the hydrostatic equation (8.1) and the equation of continuity (15.4). The hydrostatic equation is used in synoptic aerology to find the instantaneous field of Φ when the instantaneous fields of T and p_0 (pressure at the ground) are known from the observations:

$$\Phi = \Phi_0 + \int_p^{p_0} s\,dp = \Phi_0 + R\int_p^{p_0} T\frac{dp}{p}.$$

Conversely, the instantaneous field of s (and hence, the field of T or ϑ or any other function of s and p) can be determined from the hydrostatic equation, when the field of Φ is known at the same instant. In similar manner, the instantaneous field of ω is determined, from eq. (15.4), by the instantaneous distribution of $\boldsymbol{v}$. Therefore, the instantaneous state and motion of the atmosphere is completely determined, if the fields of Φ and $\boldsymbol{v}$ are given at the same instant (or, alternatively, if the fields of p_0, T and $\boldsymbol{v}$ are given).

The variation with time of the fields of Φ and $\boldsymbol{v}$ is controlled by the "prognostic" equations (11.2) and (16.4). These equations are not better fitted for prognostic use than the corresponding prognostic equations in the system z. What was mentioned about this in section 4 applies here too.

18. The quasi-static equations with $-\ln p$ as the vertical coordinate.

If the linearized equations, which apply to atmospheric wave motions, are written in the system p, then the coefficients of these equations appear to vary with p. In order to attain constant or quasi-constant coefficients, it proves convenient to use

$$P = -\ln p \tag{18.1}$$

as a vertical coordinate[1]). With a view to later applications, the equations will now be written with P as an independent variable.

From (18.1) we find

$$\frac{\partial}{\partial p} = -\frac{1}{p}\frac{\partial}{\partial P} = -e^{P}\frac{\partial}{\partial P}. \tag{18.2}$$

The partial derivatives with respect to x, y or t are the same, whether p or P is used as the vertical coordinate. Denoting by ψ the individual variation of P,

$$\frac{DP}{dt} = \psi = -\frac{\omega}{p}, \tag{18.3}$$

the formula of individual differentiation (10.2) becomes

$$\frac{D}{dt} = \left(\frac{\partial}{\partial t}\right)_p + \boldsymbol{v}\cdot\nabla_p + \psi\frac{\partial}{\partial P}. \tag{18.4}$$

The hydrostatic equation (8.1) becomes

$$\frac{\partial \Phi}{\partial P} = RT. \tag{18.5}$$

Hence, in an isothermal atmosphere, P measures the true height above a certain isobaric surface.

[1]) To get correct dimensions, the formula should have been written $P = -\ln\frac{p}{p_0}$, where p_0 is a constant. However, this constant may just as well be put equal to the pressure unit, since its value is immaterial.

The equation of motion, written in the form (11.2) is unaffected by the change of variable. The vorticity equation (11.3), however, assumes the form

$$\frac{D}{dt}(2\Omega_z + \zeta_p) = -(2\Omega_z + \zeta_p)\,\nabla_p\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\nabla_p\psi\times\frac{\partial \boldsymbol{v}}{\partial P}. \tag{18.6}$$

The formula for geostrophic wind (12.1) still holds, but the variation of geostrophic wind with height is now, in virtue of (18.5), given by:

$$\frac{\partial \boldsymbol{v}_g}{\partial P} = \frac{1}{2\Omega_z}\boldsymbol{k}\times\nabla_p\frac{\partial\Phi}{\partial P} = \frac{R}{2\Omega_z}\boldsymbol{k}\times\nabla_p T = \frac{RT}{2\Omega_z\vartheta}\boldsymbol{k}\times\nabla_p\vartheta. \tag{18.7}$$

The equation of continuity (15.2) assumes the form:

$$\nabla_p\cdot\boldsymbol{v} + \frac{\partial\psi}{\partial P} - \psi = 0\,, \tag{18.8}$$

and finally, the first law of thermodynamics, in the form (16.2), becomes:

$$\frac{\partial}{\partial P}\left(\frac{\partial\Phi}{\partial t}\right)_p = \frac{R}{c_p}H - RT\left(\boldsymbol{v}\cdot\nabla_p \ln\vartheta + \psi\frac{\partial}{\partial P}\ln\vartheta\right). \tag{18.9}$$

We shall return to these equations later.

CHAPTER III. THE EFFECT OF THE CURVATURE OF THE EARTH.

19. The quasi-static equations with Φ as the vertical coordinate.

In the preceding chapter, we have emphasized the simple form assumed by the hydrodynamic equations when pressure is used as an independent variable. The geopotential surfaces were then, however, assumed to be parallel planes, and the force of gravity was considered as a constant. The question now arises whether it will still be convenient to use pressure as an independent variable when these simplifying assumptions are given up. This question is considered in the present chapter.

We start by writing the quasi-static equations for a curved earth, with geopotential as vertical coordinate. The other coordinates are curvilinear coordinates in the geopotential surfaces (e. g. latitude and longitude). These horizontal coordinates must be defined in such a way that they are constant along a vertical line (perpendicular to the geopotential surfaces). In the following, the choice of horizontal coordinates will not be specified.

As before, we introduce a unit vector $\boldsymbol{k}$, directed vertically upwards. On account of the curvature and divergence of the vertical lines, $\boldsymbol{k}$ points in different directions at different places, and, therefore, cannot be treated as a constant.

Let $\boldsymbol{v}$ denote the horizontal velocity. The vertical velocity is $\frac{1}{g}\frac{D\Phi}{dt}$. Hence, the three-dimensional velocity may be written as

$$\boldsymbol{v}+\frac{1}{g}\frac{D\Phi}{dt}\boldsymbol{k}. \tag{19.1}$$

In similar manner, we write the three-dimensional del-operator in the form:

$$\nabla_{\Phi}+\boldsymbol{k}g\frac{\partial}{\partial\Phi}. \tag{19.2}$$

Here, ∇_{Φ} is the horizontal del-operator, which may be defined by:

$$d\boldsymbol{r}\cdot\nabla_{\Phi}=d_{\Phi}, \tag{19.3}$$

where d_{Φ} denotes the change of a quantity in horizontal direction corresponding to the horizontal line element $d\boldsymbol{r}$. The suffix Φ is used to denote that Φ is constant by the operation. The formula of individual differentiation becomes:

$$\frac{D}{dt}=\left(\frac{\partial}{\partial t}\right)_{\Phi}+\left(\boldsymbol{v}+\frac{1}{g}\frac{D\Phi}{dt}\boldsymbol{k}\right)\cdot\left(\nabla_{\Phi}+\boldsymbol{k}\,g\frac{\partial}{\partial\Phi}\right) = \left(\frac{\partial}{\partial t}\right)_{\Phi}+\boldsymbol{v}\cdot\nabla_{\Phi}+\frac{D\Phi}{dt}\frac{\partial}{\partial\Phi}. \tag{19.4}$$

We are now able to write down the hydrodynamic equations with Φ, t and two horizontal coordinates (not specified) as independent variables. We start with the hydrostatic equation, which becomes:

$$\frac{\partial p}{\partial\Phi}=-q. \tag{19.5}$$

To find the form of the horizontal component of the equation of motion, consider the three-dimensional acceleration:

$$\frac{D}{dt}\left(\boldsymbol{v}+\frac{1}{g}\frac{D\Phi}{dt}\boldsymbol{k}\right) = \frac{D\boldsymbol{v}}{dt}+\frac{1}{g}\frac{D\Phi}{dt}\frac{D\boldsymbol{k}}{dt}+\boldsymbol{k}\frac{D}{dt}\left(\frac{1}{g}\frac{D\Phi}{dt}\right).$$

Here the first term on the right involves both a horizontal and a small vertical component; the second term is strictly horizontal, and the third term is strictly vertical. Neglecting the second term, which is very small, the horizontal acceleration may be written as the horizontal component of $\frac{D\boldsymbol{v}}{dt}$, which will be denoted by $\left(\frac{D\boldsymbol{v}}{dt}\right)_{\text{hor}}$. Neglecting also the horizontal Coriolis-acceleration due to the vertical motion, the equation of motion in horizontal direction becomes:

$$\left(\frac{D\boldsymbol{v}}{dt}\right)_{\text{hor}}=\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_{\Phi}+(\boldsymbol{v}\cdot\nabla_{\Phi}\boldsymbol{v})_{\text{hor}}+\frac{D\Phi}{dt}\left(\frac{\partial\boldsymbol{v}}{\partial\Phi}\right)_{\text{hor}} = -s\nabla_{\Phi}p-2\,\Omega\sin\varphi\,\boldsymbol{k}\times\boldsymbol{v}, \tag{19.6}$$

where φ is the angle between $\boldsymbol{k}$ and the equatorial plane.

The developed form of the term $(\boldsymbol{v}\cdot\nabla_{\Phi}\boldsymbol{v})_{\text{hor}}$ in spherical coordinates, is given in section 21.

The first law of thermodynamics, written in the form (2.3) or (2.4), is unaffected by the earth's curvature.

The equation of continuity, however, requires a more thorough examination. The three-dimensional velocity-divergence is obtained by forming the scalar product of the three-dimensional del-operator (19.2) and the three-dimensional velocity (19.1):

$$\left(\nabla_{\Phi}+\boldsymbol{k}g\frac{\partial}{\partial\Phi}\right)\cdot\left(\boldsymbol{v}+\frac{1}{g}\frac{D\Phi}{dt}\boldsymbol{k}\right)=\nabla_{\Phi}\cdot\boldsymbol{v}+\frac{\partial}{\partial\Phi}\frac{D\Phi}{dt} +\frac{1}{g}\frac{D\Phi}{dt}\nabla_{\Phi}\cdot\boldsymbol{k}+g\boldsymbol{k}\cdot\frac{\partial\boldsymbol{v}}{\partial\Phi}+g\frac{\partial}{\partial\Phi}\left(\frac{1}{g}\right)\frac{D\Phi}{dt}.$$

Here $\nabla_{\Phi}\cdot\boldsymbol{v}$ is the horizontal divergence, which may be defined as the relative increase per unit time of a horizontal substantial area. The developed form of $\nabla_{\Phi}\cdot\boldsymbol{v}$ in spherical coordinates is given in section 21. Notice further, that although the vectors ∇_{Φ} and $\boldsymbol{k}$ are formally orthogonal, the expression $\nabla_{\Phi}\cdot\boldsymbol{k}$ does not vanish, because the derivatives of $\boldsymbol{k}$ are horizontal. Since $\boldsymbol{k}\cdot\boldsymbol{v}=0$,

$$\boldsymbol{k}\cdot\frac{\partial\boldsymbol{v}}{\partial\Phi}+\boldsymbol{v}\cdot\frac{\partial\boldsymbol{k}}{\partial\Phi}=0,$$

so that the three-dimensional velocity-divergence may be rewritten as

$$(19.7)\quad \nabla_\Phi \cdot \boldsymbol{v} + \frac{\partial}{\partial \Phi}\frac{D\Phi}{dt} + \frac{1}{g}\frac{D\Phi}{dt}\left(\nabla_\Phi \cdot \boldsymbol{k} - \frac{\partial g}{\partial \Phi}\right) - g\,\boldsymbol{v}\cdot\frac{\partial \boldsymbol{k}}{\partial \Phi}.$$

It will now be shown that the derivatives of $\boldsymbol{k}$, which occur in this formula, can be expressed as derivatives of g.

Consider the force of gravity, $-g\boldsymbol{k}$, which involves the centrifugal force owing to the earth's rotation. The three-dimensional divergence of $-g\boldsymbol{k}$ is due to the centrifugal force alone, since the force of pure gravity may be considered as solenoidal on account of the small density of the air. Thus we have:

$$\left(\nabla_\Phi + \boldsymbol{k} g \frac{\partial}{\partial \Phi}\right)\cdot(-g\boldsymbol{k}) = 2\,\Omega^2,$$

or

$$(19.8)\qquad \nabla_\Phi \cdot \boldsymbol{k} = -\frac{\partial g}{\partial \Phi} - \frac{2\,\Omega^2}{g}.$$

The three-dimensional vorticity of $-\boldsymbol{k}g$ must vanish:

$$\left(\nabla_\Phi + \boldsymbol{k} g \frac{\partial}{\partial \Phi}\right)\times(-g\boldsymbol{k}) = 0,$$

or[1])

$$(19.9)\qquad \frac{\partial \boldsymbol{k}}{\partial \Phi} = \frac{1}{g^2}\nabla_\Phi g.$$

By means of (19.8) and (19.9), the three-dimensional velocity-divergence (19.7) becomes:

$$(19.10)\quad \nabla_\Phi \cdot \boldsymbol{v} + \frac{\partial}{\partial \Phi}\frac{D\Phi}{dt} + \varepsilon\frac{D\Phi}{dt} - \frac{1}{g}\boldsymbol{v}\cdot\nabla_\Phi g,$$

where

$$(19{,}11)\qquad \varepsilon = -\frac{2}{g}\frac{\partial g}{\partial \Phi} - \frac{2\,\Omega^2}{g^2}.$$

Consequently, the equation of continuity assumes the form:

$$(19.12)\quad \frac{1}{q}\frac{Dq}{dt} + \nabla_\Phi \cdot \boldsymbol{v} + \frac{\partial}{\partial \Phi}\frac{D\Phi}{dt} = -\varepsilon\frac{D\Phi}{dt} + \boldsymbol{v}\cdot\frac{\nabla_\Phi g}{g}.$$

[1]) We got first:

$$\nabla_\Phi g \times \boldsymbol{k} + g\nabla_\Phi \times \boldsymbol{k} + g^2 \boldsymbol{k} \times \frac{\partial \boldsymbol{k}}{\partial \Phi} = 0.$$

Hence, cross-multiplicating by $\boldsymbol{k}$:

$$\nabla_\Phi g + g\,(\nabla_\Phi \boldsymbol{k})\cdot\boldsymbol{k} - g\,\boldsymbol{k}\cdot\nabla_\Phi \boldsymbol{k} - g^2\frac{\partial \boldsymbol{k}}{\partial \Phi} = 0.$$

Here $(\nabla_\Phi \boldsymbol{k})\cdot\boldsymbol{k} = 0$, because any derivative of $\boldsymbol{k}$ is horizontal. Further, $\boldsymbol{k}\cdot\nabla_\Phi \boldsymbol{k} = 0$, and we arrive at (19.9).

The first term on the right is partly due to the divergence of the vertical lines, and partly to the use of Φ as measure of height. The second term on the right is due to the convergence of the geopotential surfaces towards the poles.

The corresponding form of the tendency equation is:

$$(19.13)\quad \frac{\partial p}{\partial t} - q\frac{D\Phi}{dt} = -\int_\Phi^\infty \nabla_\Phi\cdot(q\boldsymbol{v})\,d\Phi - \int_\Phi^\infty \varepsilon\frac{D\Phi}{dt}q\,d\Phi + \int_\Phi^\infty \boldsymbol{v}\cdot\frac{\nabla_\Phi g}{g}q\,d\Phi.$$

Here the second and third term on the right represent the effect of the curvature. Since ε is positive, the second term gives pressure decrease at the ground for ascending motion, and pressure increase for descending motion. The third term is positive for motion towards the poles, and negative for motion towards the equator. To estimate the magnitude of the tendency at the ground represented by the two last terms of eq. (19.13), we insert some reasonable numerical values. Putting

$$g = 9.8\ \text{m/sec}^2,$$

$$\frac{\partial g}{\partial \Phi} = -0.315\cdot10^{-6}\,\text{m}^{-1},\quad \Omega = 0.729\cdot10^{-4}\,\text{sec}^{-1},$$

we obtain:

$$\varepsilon = 0.642\cdot10^{-7}\,\text{m}^{-2}\,\text{sec}^2.$$

Since ε is nearly constant, we may write the second term on the right of eq. (19.13):

$$-\varepsilon\int_0^{p_0}\frac{d\Phi}{dt}dp = -\varepsilon\,\overline{\frac{D\Phi}{dt}}\,p_0.$$

Putting $\overline{\frac{D\Phi}{dt}} = 1\ \text{m}^2/\text{sec}^2$, corresponding to a mean vertical velocity of 0.1 m/sec, and $p_0 = 1000$ mb, we obtain:

$$\int_\Phi^\infty \varepsilon\frac{D\Phi}{dt}q\,d\Phi \sim 0.6\ \text{millibars/3 hours}.$$

The vector $\frac{1}{g}\nabla_\Phi g$ is directed northwards (on the northern hemisphere) with a maximum value of $0.83\cdot10^{-9}\text{m}^{-1}$ at 45° latitude. Adopting this value, we get for the last term of (19.13):

$$0.83\cdot10^{-9}\text{m}^{-1}\int_0^{p_0} v_N\,dp = 0.83\cdot10^{-9}\text{m}^{-1}\,\bar{v}_N\,p_0,$$

where v_N is the northwards velocity. Putting $v_N = 20$ m/sec, we find:

$$\int_{\Phi_0}^{\infty} \boldsymbol{v} \cdot \frac{\nabla_\Phi g}{g} \, q \, d\Phi \sim 0.2 \text{ millibars/3 hours.}$$

These figures must be considered as extreme values, since the values of $\frac{D\Phi}{dt}$ and $\bar{v}_N$ have been chosen very large. The two last terms of eq. (19.13) represent therefore, as might have been expected, very small effects, which may be neglected, since the tendency equation cannot be used anyhow for accurate calculations of the tendency.

It will therefore be sufficiently accurate to write the tendency equation on a curved earth in the form:

$$(19.14) \quad \frac{\partial p}{\partial t} - q \frac{D\Phi}{dt} = -\int_{\Phi}^{\infty} \nabla_\Phi \cdot (q\boldsymbol{v}) \, d\Phi ,$$

in agreement with the usual form (1.7). In the same approximation, the equation of continuity assumes the form:

$$(19.15) \quad \frac{1}{q}\frac{Dq}{dt} + \nabla_\Phi \cdot \boldsymbol{v} + \frac{\partial}{\partial \Phi}\frac{D\Phi}{dt} = 0 ,$$

which corresponds to (1.5). Using these approximate equations, we can also derive an expression for $\frac{D\Phi}{dt}$, which corresponds to eq. (3.3) for the vertical velocity.

20. Pressure as the vertical coordinate on a curved earth.

Now let us introduce p as independent variable instead of Φ, in the case of curved geopotential surfaces. The dependent variables will then be interpreted as functions of p, t and two horizontal coordinates. Partial differentiation with constant p will be denoted, as before, with a suffix p.

Differentiation with respect to the horizontal coordinates may be expressed, without specifying these coordinates, by means of the horizontal del-operator ∇_p, defined by:

$$(20.1) \quad d\boldsymbol{r} \cdot \nabla_p = d_p$$

where d_p is the change of a quantity in an isobaric surface, corresponding to the horizontal line element $d\boldsymbol{r}$. The formula (10.2) for individual differentiation is then valid also in the case of a curved earth. Also the formulae in section 7 for the "p-slope-vector", the "vertical p-velocity" and the horizontal motion of equiscalar surfaces are still valid in the case of a curved earth. The hydrostatic equation assumes the same form (8.1) as before, and what was said in section 9 about the geometry of the fields of the thermodynamic quantities is still true.

The equations of transformation are obtained in the same way as in section 6, with the result:

$$(20.2) \quad \frac{\partial}{\partial \Phi} = - q \frac{\partial}{\partial p}$$

$$(20.3) \quad \nabla_\Phi = \nabla_p + q \nabla_p \Phi \frac{\partial}{\partial p}$$

$$(20.4) \quad \left(\frac{\partial}{\partial t}\right)_\Phi = \left(\frac{\partial}{\partial t}\right)_p + q \left(\frac{\partial \Phi}{\partial t}\right)_p \frac{\partial}{\partial p},$$

in agreement with eqs. (6.6), (6.12) and (6.9).

Transformation of the equation of motion in the form (19.6) gives:

$$(20.5) \quad \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_p + (\boldsymbol{v} \cdot \nabla_p \boldsymbol{v})_{\text{hor}} + \omega \left(\frac{\partial \boldsymbol{v}}{\partial p}\right)_{\text{hor}} = - \nabla_p \Phi - 2\,\Omega \sin\varphi \, \boldsymbol{k} \times \boldsymbol{v}.$$

The equation above corresponds to eq. (11.2) in the case of a flat earth; and it follows that the expression (12.1) for geostrophic wind and (12.2) for thermal wind are still valid in the case of a curved earth.

Transformation of the first law of thermodynamics can be carried out in the same way as was done in the case of a flat earth, and the equations of section 16 are valid in the case of a curved earth as well.

Transformation of the equation of continuity in the form (19.12) gives

$$(20.6) \quad \nabla_p \cdot \boldsymbol{v} + \frac{\partial \omega}{\partial p} = - \varepsilon \left[\left(\frac{\partial \Phi}{\partial t}\right)_p + \boldsymbol{v} \cdot \nabla_p \Phi - s\,\omega\right] + \boldsymbol{v} \cdot \frac{\nabla_\Phi g}{g}.$$

Here $\nabla_\Phi g$ is not transformed into $\nabla_p g$, since g is most conveniently interpreted as a function of Φ.

Comparison with eq. (15.2) shows that the terms on the right are due to the curvature of the earth. As was shown in section 19, these terms are small and may be disregarded, if the accuracy needed is not very great. Doing so, we arrive at the same equation (15.2) as was found in the case of a flat earth with constant g. We may therefore conclude that pressure may be used as the vertical coordinate also when the earth's curvature is taken into consideration.

For the sake of completeness, some remarks on the effect of the right-hand terms of (20.6) will be added. Neglecting $\left(\frac{\partial \Phi}{\partial t}\right)_p$ and $\boldsymbol{v}\cdot\nabla_p\Phi$ compared to $s\omega$, eq. (20.6) becomes:

$$(20.7) \qquad \nabla_p\cdot\boldsymbol{v}+\frac{\partial\omega}{\partial p}=\varepsilon s\omega+\boldsymbol{v}\cdot\frac{\nabla_\Phi g}{g}.$$

When ε is considered as a constant, this equation can be integrated to:

$$(20.8) \qquad \omega=-\int_0^p\left(\nabla_p\cdot\boldsymbol{v}-\boldsymbol{v}\cdot\frac{\nabla_\Phi g}{g}\right)e^{\varepsilon\int_\pi^p s\,d\pi}\,d\pi,$$

where the pressure, as a variable of integration, is denoted by π. This equation yields a computation of ω, and replaces eq. (15.4) which was found in the case of a flat earth.

Equation (20.8) can be combined with the boundary condition at the ground, to give an equation for the tendency $\left(\frac{\partial\Phi}{\partial t}\right)_p$ at the ground, corresponding to eq. (15.5) in the case of a flat earth.

The weight function $e^{\varepsilon\int_\pi^p s\,d\pi}$ is equal to unity when $\pi=p$ and increases slowly as π decreases. Thus, for instance, in the case of an isothermal atmosphere we find

$$e^{\varepsilon\int_\pi^p s\,d\pi}=\left(\frac{p}{\pi}\right)^{\varepsilon RT}.$$

With $R=287$, $T=240$, $\varepsilon=0.642.10^{-7}$, and hence $\varepsilon RT=0.00443$, we find a very slow increase of the function, as shown in the following table:

π/p	1	0.5	0.3	0.2	0.1	0.01	0.001	0
$(p/\pi)^{0.00443}$	1	1.003	1.005	1.007	1.010	1.020	1.030	∞

With another temperature distribution, nearly the same figures would have been found. The effect of this weight function is to make the divergences in the upper atmosphere slightly more "effective" in producing pressure changes than the divergences in the lower atmosphere.

21. Horizontal differential operations in spherical geopotential surfaces.

As the meaning of the symbols $\nabla_\Phi\cdot\boldsymbol{v}$, $(\boldsymbol{v}\cdot\nabla_\Phi\boldsymbol{v})_{\text{hor}}$, $\nabla_p\cdot\boldsymbol{v}$ and $(\boldsymbol{v}\cdot\nabla_p\boldsymbol{v})_{\text{hor}}$ used in this chapter may perhaps not be quite clear, the developed form of these expressions will now be given.

As to the differential operations in horizontal direction, it is sufficiently accurate to assume the geopotential surfaces to be spheres. As coordinates on these it is appropriate to choose longitude, λ (positive in easterly direction), and latitude, φ (positive northwards). Further we introduce two unit vectors: $\boldsymbol{i}$ pointing eastwards, and $\boldsymbol{j}$ pointing northwards. These are functions of λ and φ, and by geometrical considerations we find:

$$(21.1)\begin{cases}\left(\frac{\partial\boldsymbol{i}}{\partial\lambda}\right)_\Phi=\boldsymbol{j}\sin\varphi-\boldsymbol{k}\cos\varphi, & \left(\frac{\partial\boldsymbol{j}}{\partial\lambda}\right)_\Phi=-\boldsymbol{i}\sin\varphi\\ \left(\frac{\partial\boldsymbol{i}}{\partial\varphi}\right)_\Phi=0 & ,\left(\frac{\partial\boldsymbol{j}}{\partial\varphi}\right)_\Phi=-\boldsymbol{k}.\end{cases}$$

In eq. (19.3) we have

$$(21.2) \qquad d\boldsymbol{r}=\boldsymbol{i}\;a\;\cos\varphi\,d\lambda+\boldsymbol{j}\;a\;d\varphi,$$

where a is the radius of the earth; and hence

$$(21.3) \qquad \nabla_\Phi=\frac{\boldsymbol{i}}{a\cos\varphi}\left(\frac{\partial}{\partial\lambda}\right)_\Phi+\frac{\boldsymbol{j}}{a}\left(\frac{\partial}{\partial\varphi}\right)_\Phi.$$

The horizontal velocity may be written:

$$(21.4) \qquad \boldsymbol{v}=\boldsymbol{i}\,v_E+\boldsymbol{j}\,v_N.$$

By means of these formulae we find for the horizontal divergence:

$$(21.5)\quad \begin{aligned}\nabla_\Phi\cdot\boldsymbol{v}&=\left[\frac{\boldsymbol{i}}{a\cos\varphi}\left(\frac{\partial}{\partial\lambda}\right)_\Phi+\frac{\boldsymbol{j}}{a}\left(\frac{\partial}{\partial\varphi}\right)_\Phi\right]\cdot(\boldsymbol{i}v_E+\boldsymbol{j}v_N)\\ &=\frac{1}{a\cos\varphi}\left(\frac{\partial v_E}{\partial\lambda}\right)_\Phi+\frac{1}{a}\left(\frac{\partial v_N}{\partial\varphi}\right)_\Phi-\frac{v_N}{a\operatorname{cotg}\varphi},\end{aligned}$$

and similarly for the horizontal convective acceleration:

(21 6) $(\boldsymbol{v}\cdot\nabla_{\Phi}\boldsymbol{v})_{\text{hor}} =$

$$= \boldsymbol{i}\left[\frac{v_E}{a\cos\varphi}\left(\frac{\partial v_E}{\partial\lambda}\right)_{\Phi} + \frac{v_N}{a}\left(\frac{\partial v_E}{\partial\varphi}\right)_{\Phi} - \frac{v_E v_N}{a\operatorname{cotg}\varphi}\right]$$

$$+\boldsymbol{j}\left[\frac{v_E}{a\cos\varphi}\left(\frac{\partial v_N}{\partial\lambda}\right)_{\Phi} + \frac{v_N}{a}\left(\frac{\partial v_N}{\partial\varphi}\right)_{\Phi} + \frac{v_E^2}{a\operatorname{cotg}\varphi}\right].$$

In the above equations, it is permitted to replace the suffix Φ by the suffix p, and so we obtain similar formulae for the quantities $\nabla_p\cdot\boldsymbol{v}$ and $(\boldsymbol{v}\cdot\nabla_p\boldsymbol{v})_{\text{hor}}$.

We may conclude that there are three different effects of the earth's curvature:

(i) The Coriolis parameter varies with latitude. This effect may be taken into account, without introducing a curvilinear coordinate system, by interpreting Ω_z as a function of y.

(ii) The formulae of horizontal differential operations, e. g. horizontal acceleration and horizontal divergence, are more complicated than the corresponding formulae in plane coordinates.

(iii) Certain additional terms appear in the equation of continuity and in the tendency equation. These additional terms are, however, small and may be disregarded in most cases.

CHAPTER IV. ON QUASI-STATIC GRAVITATIONAL WAVES ON A NON-ROTATING EARTH.

22. The legitimacy of the quasi-static approximation.

Consider a fluid at rest in a constant gravity field. Denoting by $\bar{p}$ and $\bar{q}$ the pressure and density in this state, we have:

$$\frac{\partial\bar{p}}{\partial z} = -g\,\bar{q},\quad \frac{\partial\bar{q}}{\partial z} = -\Gamma g\,\bar{q}, \tag{22.1}$$

where Γ is the coefficient of barotropy.

Consider now small oscillations of this fluid in the xz-plane. Let ξ and ζ denote the horizontal and vertical displacements respectively, and let p' and q' designate the local perturbations of pressure and density. The individual perturbations of pressure and density may then be written:

$$p_i' = p' - g\bar{q}\zeta,\quad q_i' = q' - \Gamma g\bar{q}\zeta.$$

Assuming piezotropic changes of state, $q_i' = \gamma p_i'$, or

$$q' = (\Gamma - \gamma)\,g\,\bar{q}\,\zeta + \gamma p', \tag{22.2}$$

where γ is the coefficient of piezotropy. By using (22.1) and (22.2), the equations of motion for small perturbations become, when the vertical acceleration is taken into consideration:

$$\begin{cases}\bar{q}\dfrac{\partial^2\xi}{\partial t^2} = -\dfrac{\partial p'}{\partial x}\\[2mm] \bar{q}\dfrac{\partial^2\zeta}{\partial t^2} = -\dfrac{\partial p'}{\partial z} - \gamma g p' - (\Gamma-\gamma)g^2\bar{q}\zeta.\end{cases} \tag{22.3}$$

From these equations, Høiland [12] has derived a frequency formula for standing oscillations along rigid streamlines, by assuming a sinusoidal time dependency and taking the line integral along a closed curve. The local pressure perturbation, however, is not completely eliminated by this method, except in the case of incompressibility.

A complete elimination of p' can be attained in all cases by multiplying the equations of motion by the integrating factor $e^{\int_0^z \gamma g\,dz}$ before taking the line integral. There results then

$$\nu^2\oint\bar{q}\,e^{\int_0^z\gamma g\,dz}(\xi\,dx + \zeta\,dz) = \oint(\Gamma-\gamma)\,g^2\bar{q}\,e^{\int_0^z\gamma g\,dz}\,\zeta\,dz, \tag{22.4}$$

where ν denotes the frequency (or $i\nu$ the "flight frequency" in the case of instability). This equation may be considered as an improved form of Høiland's frequency formula, which determines the frequency when the kinematics of the motion is known.

The left-hand side of the equation represents the inertia forces, and the right-hand side is the stabilizing effect of gravity. When the line of integration intersects a surface of discontinuity, then a term representing the stabilizing effect of this surface must be added on the right-hand side. We may, however, avoid such additional terms by considering a surface of discontinuity as a thin layer of rapid transition; then the total stabilizing effect is represented by the right side of (22.4).

The quasi-static approximation ignores the vertical accelerations. The corresponding quasi-static frequency formula is therefore

$$(22.5) \qquad \nu^2 \oint q\, e^{\int_0^z \gamma g\, dz} \xi\, dx = \oint (\Gamma - \gamma)\, g^2 \bar{q}\, e^{\int_0^z \gamma g\, dz} \zeta\, dz.$$

By similar kinematics, the quasi-static method will give the same frequency as the "exact" method (22.4) if

$$(22.6) \qquad \left| \oint \bar{q}\, e^{\int_0^z \gamma g\, dz} \zeta\, dz \right| \ll \left| \oint \bar{q}\, e^{\int_0^z \gamma g\, dz} \xi\, dx \right|,$$

which may thus be considered as a criterion for the legitimacy of the quasi-static method.

It should be noted that if the stability factor $(\Gamma - \gamma)$ varies with z, the above criterion does not apply to an arbitrary closed curve. The criterion applies to a streamline, or to a curve intersecting the layers where the stability factor $(\Gamma - \gamma)$ is greatest. When applied to a curve which contains indifferent (or nearly indifferent) layers only, the criterion is obviously erroneous. Such a curve will also be unfit for frequency determination from (22.4), because a slight variation in the kinematics will greatly affect the value of the computed frequency.

23. The perturbation equations.

Quasi-static wave motions in autobarotropic layers have been treated by V. Bjerknes [4] and in Physikalische Hydrodynamik [3], in the latter case by using pressure as an independent variable. As an illustration of the methods developed in chapter II, we will now deal with quasi-static waves in stable layers, using pressure as an independent variable. The results thus obtained are of course not new, since the quasi-static waves are a special case of the more general non-quasistatic waves which have been thoroughly treated by many writers.

It should be noted that in the case of small perturbations of the equilibrium state, it is not necessary to distinguish between the operations $\left(\frac{\partial}{\partial x}\right)_p$ and $\left(\frac{\partial}{\partial x}\right)_z$ or between $\left(\frac{\partial}{\partial t}\right)_p$ and $\left(\frac{\partial}{\partial t}\right)_z$, since the difference between them will be small of the second order. As independent variable, p may therefore in this case be considered as a function of z only. For this reason, the derivatives with respect to x and t will in this chapter be written without the suffix p.

The equation for the equilibrium state in the system p is

$$(23.1) \qquad \frac{\partial \bar{\Phi}}{\partial p} = -\frac{1}{\bar{q}}, \quad \frac{\partial \bar{q}}{\partial p} = \Gamma,$$

where Γ is the coefficient of barotropy, and $\bar{\Phi}$ and $\bar{q}$ denote the values of Φ and q in equilibrium.

Let $\xi(x, p, t)$ be the horizontal displacement, and $\pi'(x, p, t)$ the individual pressure perturbation in the perturbed state. Then ξ and π' may be interpreted as the components of displacements in the xp-plane. These displacements must fulfil an equation of continuity of the same form as eq. (15.2):

$$(23.2) \qquad \frac{\partial \xi}{\partial x} + \frac{\partial \pi'}{\partial p} = 0.$$

This equation will be identically satisfied when ξ and π' are expressed by a stream function F,

$$(23.3) \qquad \xi = \frac{\partial F}{\partial p}, \quad \pi' = -\frac{\partial F}{\partial x}.$$

The horizontal acceleration is seen to be $\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial p}\frac{\partial^2 F}{\partial t^2}$, when second order terms are omitted. Further we denote by Φ' the perturbation of geopotential in a fixed point in the xp-plane; this will in the following be called the "local" perturbation. For $\Omega = 0$, the horizontal equation of motion (11.1) then assumes the form

$$(23.4) \qquad \frac{\partial}{\partial p}\frac{\partial^2 F}{\partial t^2} = -\frac{\partial \Phi'}{\partial x}.$$

The hydrostatic equation for the perturbed state is

$$(23.5) \qquad \frac{q'}{\bar{q}^2} = \frac{\partial \Phi'}{\partial p}$$

where q' denotes the local density perturbation in the xp-plane. Since the individual density perturbation is $q' + \Gamma\pi'$, the equation of piezotropy may be written in the form

$$(23.6) \qquad q' = -(\Gamma - \gamma)\, \pi' = (\Gamma - \gamma)\frac{\partial F}{\partial x},$$

in virtue of (23.3). Here γ denotes the coefficient of piezotropy. Elimination of q' between the last two equations gives

$$\frac{\Gamma-\gamma}{\bar{q}^2}\frac{\partial F}{\partial x}=\frac{\partial \Phi'}{\partial p}. \tag{23.7}$$

Eqs. (23.4) and (23.7) permit a solution of the form

$$\begin{cases} F=\hat{F}(p)\,A(x-ct), \\ \Phi'=\hat{\Phi}(p)\,A'(x-ct), \end{cases} \tag{23.8}$$

where A represents an arbitrary wave profile and c is the constant wave velocity; A' is the first derivative of A. This form of the solution is possible owing to the non-dispersive character of the quasi-static waves; sinusoidal waves of all wave-lengths travel with the same speed, and may therefore be added to form a wave of arbitrary profile.

Inserting (23.8) into (23.4) and (23.7), we find

$$\begin{cases} c^2\dfrac{d\hat{F}}{dp}=-\hat{\Phi} \\ \dfrac{\Gamma-\gamma}{\bar{q}^2}\hat{F}=\dfrac{d\hat{\Phi}}{dp}. \end{cases} \tag{23.9}$$

Elimination of $\hat{\Phi}$ gives the differential equation of the stream function,

$$\frac{d^2\hat{F}}{dp^2}+\frac{\Gamma-\gamma}{c^2\bar{q}^2}\hat{F}=0. \tag{23.10}$$

To formulate the boundary conditions, we introduce the individual geopotential perturbation $\Phi_i'=\Phi'-\frac{1}{\bar{q}}\pi'$. Then, from (23.3), (23.8) and (23.9),

$$\begin{cases} \pi'=\hat{\pi}(p)\,A'(x-ct) \\ \Phi_i'=\hat{\Phi}_i(p)\,A'(x-ct), \end{cases} \tag{23.11}$$

where

$$\begin{cases} \hat{\pi}=-\hat{F} \\ \hat{\Phi}_i=\dfrac{\hat{F}}{\bar{q}}-c^2\dfrac{d\hat{F}}{dp}. \end{cases} \tag{23.12}$$

$\hat{\pi}$ and $\hat{\Phi}_i$ must be continous at an internal surface of discontinuity; $\hat{\pi}$ must vanish at a free surface, and $\hat{\Phi}_i$ must vanish at a rigid horizontal boundary plane.

The solution (23.8) is not permissible for all forms of the function A. To see this, suppose first that A is a sine function with wave length L. The travelling wave may then be split up into two standing oscillations, the kinematics of which is given by

$$\begin{cases} \xi=\dfrac{d\hat{F}}{dp}\sin 2\pi\dfrac{x}{L}\cos 2\pi\dfrac{ct}{L} \\ \Phi_i'=\dfrac{2\pi}{L}\hat{\Phi}_i\cos 2\pi\dfrac{x}{L}\cos 2\pi\dfrac{ct}{L}, \end{cases} \tag{23.13}$$

in virtue of (23.8) and (23.11). Suppose that the fluid is bounded by two rigid planes at heights corresponding to the equilibrium pressure values p_1 and p_2. Then there will be one rectangular streamline, consisting of the horizontal lines $p=p_1$ and $p=p_2$ and the vertical lines $x=0$ and $x=\frac{1}{2}L$. Applying the condition (22.6) to this streamline, we obtain the result that the quasi-static solution is valid only for those wave-lengths, which are great compared to the quantity L_0, given by

$$L_0^2=\frac{4\pi^2}{g^2}\left|\frac{\displaystyle\int_{p_1}^{p_2} e^{-\int_{p_1}^{p}\frac{\gamma}{\bar{q}}dp}\left(\frac{\hat{F}}{\bar{q}}-c^2\frac{d\hat{F}}{dp}\right)dp}{\left[\bar{q}\,e^{-\int_{p_1}^{p}\frac{\gamma}{q}dp}\dfrac{d\hat{F}}{dp}\right]_{p_1}^{p_2}}\right| \tag{23.14}$$

This formula holds in the case of a free surface also, provided that the upper horizontal line is chosen above the free surface.

For wave lengths comparable with, or smaller than L_0, the vertical accelerations become important. L_0 can be evaluated if $\hat{F}$ is known as a function of p.

Since the travelling sinusoidal wave may be considered as the sum of two standing oscillations, it follows that this condition applies to a travelling wave as well.

If not a sine function, A may be decomposed by Fourier analysis into sine functions, the wave lengths of which must all be great compared to L_0.

24. Solution for a quasi-homogeneous layer.

In the derivation of eq. (23.10), nothing was assumed concerning the variation of stability and density along the vertical. The validity of this differential equation is therefore quite general.

In the case of autobarotropy ($\Gamma-\gamma=0$) it followsfrom (23.3) and (23.10) that $F$and π'

are linear functions of p, whereas ξ is independent of p.

If $\Gamma-\gamma$ is constant, it will be suitable to return to height as an independent variable, whereby (23.10) transforms into

$$\frac{d^2\hat{F}}{dz^2}+\Gamma g\frac{d\hat{F}}{dz}+\frac{(\Gamma-\gamma)g^2}{c^2}\hat{F}=0. \tag{24.1}$$

This equation has constant coefficients and can easily be solved.

On the other hand, pressure will be the suitable independent variable, if

$$\frac{\Gamma-\gamma}{\bar{q}^2}=\text{constant}. \tag{24.2}$$

The solution of (23.10) is then

$$\hat{F}=K\sin\alpha(p-p_0), \tag{24.3}$$

where

$$\alpha^2=\frac{\Gamma-\gamma}{c^2\bar{q}^2}, \tag{24.4}$$

and K and p_0 are constants of integration. The corresponding solutions for $\hat{\pi}$ and $\hat{\Phi}_i$ are obtained from (23.12):

$$\left\{\begin{array}{l}\hat{\pi}=-K\sin\alpha(p-p_0)\\ \hat{\Phi}_i=\dfrac{K}{\bar{q}}\left(\sin\alpha(p-p_0)-\dfrac{\Gamma-\gamma}{\alpha\bar{q}}\cos\alpha(p-p_0)\right).\end{array}\right. \tag{24.5}$$

When α is real, $\hat{\pi}$ will vanish for the equidistant values of p

$$p=p_0+\frac{n\pi}{\alpha}\qquad (n \text{ an integer}), \tag{24.6}$$

corresponding to a cellular motion in the xp-plane. $\hat{\Phi}_i$ will vanish for the values of p which satisfy the equation

$$\operatorname{tg}\alpha(p-p_0)=\frac{\Gamma-\gamma}{\alpha\bar{q}}. \tag{24.7}$$

It will be seen that there will always be one zero of $\hat{\Phi}_i$ between two successive zeros of $\hat{\pi}$. The motion is therefore cellular also in the xz-plane, when α is real; but the cells in the xz-plane are displaced and distorted relatively to the cells in the xp-plane.

When α is imaginary, π will have one zero only, or no zeros at all, and the same applies to $\hat{\Phi}_i$. The motion is then non-cellular.

From (24.4) it follows that in the case of cellular motion, the waves will be stable ($c^2>0$) when $\Gamma-\gamma$ is positive, and unstable ($c^2<0$) when $\Gamma-\gamma$ is negative. For non-cellular motion, the opposite is seen to be true.

The supposition (24.2) generally involves a rather artificial distribution of density along the vertical. The solution (24.3) is therefore of limited interest, except when the density variation within the layer considered is slight in comparison with $\bar{q}$. In this "quasi-homogeneous" case, we may consider $\bar{q}$ approximately as a constant, and the condition (24.2) may as well be written $\Gamma-\gamma=$ constant. Then it follows from (24.7) that two successive zeros of $\hat{\Phi}_i$ will have the constant pressure difference $\frac{\pi}{\alpha}$. Hence we obtain for a layer bounded by two rigid horizontal planes, when the pressure difference between these planes is denoted by $\triangle p$,

$$\triangle p=\frac{n\pi}{\alpha}=n\pi\frac{\bar{q}}{\sqrt{\Gamma-\gamma}}c\qquad (n \text{ a positive integer}) \text{ or}$$

$$c=\frac{\sqrt{\Gamma-\gamma}}{n\pi}\frac{\triangle p}{\bar{q}}. \tag{24.8}$$

Since $\bar{q}$ is assumed to be nearly constant, this formula may also be written:

$$c=\frac{\sqrt{\Gamma-\gamma}}{n\pi}gH, \tag{24.9}$$

where H is the depth of the layer. Inserting this value of c and the corresponding formulae for α and $\hat{F}$ into (23.14), we find

$$L_0=\frac{2H}{n}, \tag{24.10}$$

showing that the solution is valid for wave lengths which are great compared to the height of the cells.

Now suppose that the layer is bounded above by a free surface. In virtue of (24.6), the boundary condition at this surface will be satisfied if we take p_0 equal to the pressure at this surface. The boundary condition for a plane bottom at $p=p_0+\triangle p$ then is, according to (24.7),

$$(\alpha\triangle p)\operatorname{tg}(\alpha\triangle p)=\frac{(\Gamma-\gamma)\triangle p}{\bar{q}}. \tag{24.11}$$

When $(a \triangle p)$ is considered as the unknown quantity, this equation has an infinite number of roots. The right side of the equation will be a small quantity in the quasi-homogeneous case, provided that γ is not of greater order of magnitude than Γ. One of the roots will then be small, and this root can be determined approximately by replacing tg $(a \triangle p)$ by $a \triangle p$; this gives

$$a^2 = \frac{\Gamma - \gamma}{\bar{q} \triangle p},$$

or

$$c = \sqrt{\frac{\triangle p}{\bar{q}}} = \sqrt{gH}. \tag{24.12}$$

This is the Lagrangian wave velocity for long waves in a shallow layer. The kinematics of this kind of wave motion is such that the great stability of the free surface will dominate against the slight internal stability. These waves are therefore, in first approximation, independent of the magnitude and sign of the quantity $\Gamma - \gamma$. When applied to these waves, formula (23.14) gives

$$L_0 = \pi \sqrt{2}\, H. \tag{24.13}$$

The quasi-static solution is therefore valid for wavelengths which are great compared to the depth of the layer.

The further roots of (24.11) correspond to internal cellular waves, the velocity of which are approximately given by (24.8).

25. Solution for an isothermal gaseous layer.

Høiland [14] has pointed out that one may replace a non-homogeneous fluid by a homogeneous fluid without losing the main features of the motion, provided that the correct stability effect is taken into account. A slight change in the density distribution, which does not affect the stability, will therefore be of very little importance to the motion.

The density distribution in the atmosphere may be characterized as quasi-isothermal, since $\frac{\partial \ln T}{\partial \ln p}$ is a small quantity (about 0.2 in the troposphere). In dealing with wave motion, we are therefore justified in replacing the actual atmosphere by an isothermal atmosphere, provided only that we reckon with a static stability which corresponds to that of the actual atmosphere.

Gravitational waves in an isothermal layer are thoroughly treated in Physikalische Hydrodynamik [3]. We will here only show that the stream function introduced in section (23) can be applied in this case also, by introducing

$$P = -\ln p, \quad p = e^{-P} \tag{25.1}$$

as a new vertical coordinate (compare section 18). With the notation

$$\sigma = \frac{\partial \ln \bar{\vartheta}}{\partial P} \tag{25.2}$$

where $\bar{\vartheta}$ is the potential temperature in the equilibrium state, we find from (2.2)

$$\Gamma - \gamma = \frac{\sigma}{R\bar{T}} \tag{25.3}$$

where $\bar{T}$ is the equilibrium temperature. When $\bar{T}$ is assumed to be constant, it follows from (2.1) and (2.2) that

$$\sigma = 1 - \frac{1}{\varkappa}, \tag{25.4}$$

so that we may obtain any stability wanted by assuming an appropriate value for $\varkappa$.

Instead of the pressure perturbation π' considered in section 23, we introduce the individual perturbation of P, which will be denoted by Π':

$$\pi' = -p\Pi' = -e^{-P}\Pi'. \tag{25.5}$$

Using (18.2), the variable P may be introduced into the equations of section 23. Eqs. (23.3) become

$$\xi = -e^P \frac{\partial F}{\partial P}, \quad \Pi' = e^P \frac{\partial F}{\partial x}. \tag{25.6}$$

For a solution of the form

$$\begin{cases} F = \hat{F}(P)\, A(x - ct) \\ \Phi' = \hat{\Phi}(P)\, A'(x - ct) \\ \Pi' = \hat{\Pi}(P)\, A'(x - ct) \\ \Phi_i' = \hat{\Phi}_i(P)\, A'(x - ct), \end{cases} \tag{25.7}$$

the equation of the stream function (23.10) assumes the form

$$\frac{d^2\hat{F}}{dP^2} + \frac{d\hat{F}}{dP} + \frac{R\bar{T}\sigma}{c^2}\hat{F} = 0. \tag{25.8}$$

The other variables in the solution are given by

$$(25.9)\quad \begin{cases} \hat{\Phi} = c^2 e^P \dfrac{d\hat{F}}{dP} \\ \hat{\Pi} = e^P \hat{F} \\ \hat{\Phi}_i = e^P \left(R\bar{T}\hat{F} + c^2 \dfrac{d\hat{F}}{dP}\right), \end{cases}$$

in virtue of (23.9) and (23.12).

Eq. (25.8) has constant coefficients when σ is assumed to be constant. The solution may be written as

$$(25.10)\quad \begin{cases} \hat{F} = K e^{-\frac{1}{2}P} \sin a(P - P_0) \\ \hat{\Pi} = K e^{\frac{1}{2}P} \sin a(P - P_0) \\ \hat{\Phi}_i = K e^{\frac{1}{2}P} [(R\bar{T} - \frac{1}{2}c^2) \sin a(P - P_0) \\ \qquad\qquad + ac^2 \cos a(P - P_0)] \end{cases}$$

in the cellular case, and

$$(25.11)\quad \begin{cases} \hat{F} = e^{-\frac{1}{2}P} (K_1 e^{\beta P} + K_2 e^{-\beta P}) \\ \hat{\Pi} = e^{\frac{1}{2}P} (K_1 e^{\beta P} + K_2 e^{-\beta P}) \\ \hat{\Phi}_i = e^{\frac{1}{2}P} [(R\bar{T} - c^2(\frac{1}{2} - \beta)) K_1 e^{\beta P} \\ \qquad\qquad + (R\bar{T} - c^2(\frac{1}{2} + \beta)) K_2 e^{-\beta P}] \end{cases}$$

for non-cellular motion. In these expressions, K, P_0, K_1 and K_2 are constants of integration, and a and β are given by

$$(25.12)\quad a^2 = -\beta^2 = \frac{\sigma R\bar{T}}{c^2} - \frac{1}{4}.$$

For stable waves, we have

cellular motion when $c^2 < 4\sigma R\bar{T}$

non-cellular motion when $c^2 > 4\sigma R\bar{T}$

In the case of cellular motion, $\hat{\Phi}_i$ vanishes at equidistant values of P. The distance $\triangle P$ between two successive zeros of $\hat{\Phi}_i$ is given by $a\triangle P = \pi$. Hence, from (25.12)

$$(25.13)\quad c = \sqrt{\frac{\sigma}{R\bar{T}}} \frac{R\bar{T}\triangle P}{\pi\sqrt{1 + \left(\dfrac{\triangle P}{2\pi}\right)^2}} = \sqrt{\frac{\sigma}{R\bar{T}}} \frac{gH}{\pi\sqrt{1 + \left(\dfrac{gH}{2\pi R\bar{T}}\right)^2}},$$

where $H = R\bar{T}\triangle P$ is the geometrical height of the cells. This formula is in agreement with the formula for "long cellular waves", Physikalische Hydrodynamik [3], eq. (20) p. 341.

Introduction of P into formula (23.14) gives

$$(25.14)\quad L_0^2 = \frac{4\pi^2}{g^2} R\bar{T} \left| \frac{\displaystyle\int_{P_1}^{P_2} e^{(1-\sigma)P} \left(R\bar{T}\hat{F} + c^2 \frac{d\hat{F}}{dP}\right) dP}{\left[e^{(1-\sigma)P} \dfrac{d\hat{F}}{dP}\right]_{P_1}^{P_2}} \right|.$$

For the cellular solution (25.10), this formula becomes, when the velocity formula (25.13) is utilized,

$$(25.15)\quad L_0 = \frac{2\pi c}{g} \sqrt{\frac{R\bar{T}}{\sigma}} = \frac{2H}{\sqrt{1 + \left(\dfrac{gH}{2\pi R\bar{T}}\right)^2}}$$

When H is small compared to $\dfrac{R\bar{T}}{g}$, we obtain $L_0 \approx 2H$; therefore, these waves may be considered as quasi-static when the wave length is great compared to the height of the cells. When H increases, L_0 will increase at a slower rate, and as $H \to \infty$, L_0 approaches the limit

$$(25.16)\quad L_{0_{\max}} = \frac{4\pi R\bar{T}}{g},$$

which for atmospheric conditions will amount to about 100 km. Cellular waves with wavelengths considerably longer than 100 km. will therefore always be quasi-static.

CHAPTER V. ON THE THEORY OF PERTURBATIONS OF A WESTERLY AIR CURRENT.

26. The basic current.

In this chapter we shall deal with some problems concerning perturbations of a westerly current, on the basis of the quasi-static equations. The variation of the Coriolis parameter with latitude will be taken into consideration, but the further effects of the earth's curvature will be neglected.

In the horizontal plane we will use a Cartesian coordinate system with the x-axis eastward and the y-axis northward. $P = -\ln p$ is used as a vertical coordinate.

The fundamental state is chosen as a steady linear westerly current defined by

(26.1) $\bar{v} = \bar{u}(y, P)\,\mathbf{i},\ \psi = 0,\ \bar{\Phi} = \bar{\Phi}(y, P),$
$\bar{T} = \bar{T}(y).$

The potential temperature $\bar{\vartheta}$ then becomes a function of y and P. The following abbreviations will be used:

$$\left(\frac{\partial \ln \bar{\vartheta}}{\partial y}\right)_p = \frac{d \ln \bar{T}}{dy} = -b\,, \qquad \frac{\partial \ln \bar{\vartheta}}{\partial P} = 1 - \frac{1}{\varkappa} = \sigma\,. \tag{26.2}$$

The atmosphere may be given any desired static stability by choosing an appropriate value of $\varkappa$. The quantities b and σ will be considered as constants; $\ln \bar{T}$ is then a linear function of y, and $\ln \bar{\vartheta}$ is a linear function of y and P. The state is baroclinic when $b \neq 0$, barotropic when $b = 0$, and autobarotropic when $b = 0$ and $\sigma = 0$.

For the basic current, the equation of motion and the hydrostatic equation assume the form

$$\left(\frac{\partial \bar{\Phi}}{\partial y}\right)_p = -f\bar{u}\,, \quad \frac{\partial \bar{\Phi}}{\partial P} = R\bar{T}\,, \tag{26.3}$$

where $f = 2\Omega_z$ is the Coriolis parameter. The flow is geostrophic, and by eliminating $\bar{\Phi}$ we find the thermal wind equation

$$f\frac{\partial \bar{u}}{\partial P} = -R\frac{d\bar{T}}{dy} = R\bar{T}b. \tag{26.4}$$

27. The perturbation equations.

In the perturbed motion we denote the deviations from the basic current by primed letters:

$$\begin{cases} u = \bar{u} + u' & \psi = \psi' \\ v = v' & \Phi = \bar{\Phi} + \Phi'. \end{cases} \tag{27.1}$$

Assuming adiabatic changes of state, we find the perturbation equations from eqs. (11.2), (18.9), and (18.8) by utilizing (18.4), (18.5) and (26.4) and neglecting terms which are small of the second or higher orders,

$$\begin{cases} \left(\frac{\partial u'}{\partial t}\right)_p + \bar{u}\left(\frac{\partial u'}{\partial x}\right)_p - \left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right) v' + \frac{R\bar{T}b}{f}\psi' + \left(\frac{\partial \Phi'}{\partial x}\right)_p = 0 \\ \left(\frac{\partial v'}{\partial t}\right)_p + \bar{u}\left(\frac{\partial v'}{\partial x}\right)_p + f u' + \left(\frac{\partial \Phi'}{\partial y}\right)_p = 0 \\ \left(\frac{\partial}{\partial t}\frac{\partial \Phi'}{\partial P}\right)_p + \bar{u}\left(\frac{\partial}{\partial x}\frac{\partial \Phi'}{\partial P}\right)_p - R\bar{T}bv' + R\bar{T}\sigma\psi' = 0 \\ \left(\frac{\partial u'}{\partial x}\right)_p + \left(\frac{\partial v'}{\partial y}\right)_p + \frac{\partial \psi'}{\partial P} - \psi' = 0. \end{cases} \tag{27.2}$$

The boundary condition at the ground is

$\frac{D}{dt}(\bar{\Phi} + \Phi') = 0$ when $\bar{\Phi} = 0$, or

$$\left(\frac{\partial \Phi'}{\partial t}\right)_p + \bar{u}\left(\frac{\partial \Phi'}{\partial x}\right)_p - f\bar{u}v' + R\bar{T}\psi' = 0 \quad \text{when } P = P_0\,, \tag{27.3}$$

P_0 being the value of P at the ground. In the first approximation, P_0 may here be replaced by a constant.

The boundary condition at the upper limit of the atmosphere becomes

$$\lim_{P \to \infty} (e^{-P}\psi') = 0. \tag{27.4}$$

28. Baroclinic current, perturbations independent of x. Stability criteria of Solberg—Høiland.

Consider a perturbed motion which is independent of x,

$$\left(\frac{\partial}{\partial x}\right)_p = 0\,. \tag{28.1}$$

The perturbation equations (27.2) then become

$$\begin{cases} \left(\frac{\partial u'}{\partial t}\right)_p - \left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right) v' + \frac{R\bar{T}b}{f}\psi' = 0 \\ \left(\frac{\partial v'}{\partial t}\right)_p + fu' + \left(\frac{\partial \Phi'}{\partial y}\right)_p = 0 \\ \frac{\partial}{\partial P}\left(\frac{\partial \Phi'}{\partial t}\right)_p - R\bar{T}bv' + R\bar{T}\sigma\psi' = 0 \\ \left(\frac{\partial v'}{\partial y}\right)_p + \frac{\partial \psi'}{\partial P} - \psi' = 0. \end{cases} \tag{28.2}$$

Elimination of u' between the two first equations gives

$$(28.3)\quad \left(\frac{\partial^2 v'}{\partial t^2}\right)_p = -\left(\frac{\partial}{\partial y}\frac{\partial \Phi'}{\partial t}\right)_p - f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right) v' + R\bar{T}\, b\, \psi'.$$

We write the third equation of (28.2) in the following way

$$(28.4)\quad 0 = -\frac{\partial}{\partial P}\left(\frac{\partial \Phi'}{\partial t}\right)_p + R\bar{T} b v' - R\bar{T}\sigma \psi'.$$

From these equations, the stability criteria can be derived by a method analogous to the method introduced by Høiland [12], [13]. Multiplying (28.3) by dy, (28.4) by dP, adding and integrating along a closed curve in the yP-plane, Φ' drops out[1]), and we find

$$(28.5)\quad \oint \left(\frac{\partial^2 v'}{\partial t^2}\right)_p dy = -\oint \left[f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right) v'\, dy - R\bar{T} b\,(\psi'\, dy + v'\, dP) + R\bar{T}\sigma \psi'\, dP\right].$$

Now suppose that the motion takes place along rigid, closed streamlines in the xP-plane. Choosing a streamline as the curve of integration, we may write

$$(28.6)\quad \frac{\psi'}{v'} = \frac{dP}{dy} = k,$$

where k represents the slope of the tangent of the streamline in the yP-plane (in the following denoted by "P-slope"). Further we assume a sinusoidal or exponential time dependency,

$$(28.7)\quad \left(\frac{\partial^2 v'}{\partial t^2}\right)_p = -\nu^2 v',$$

where ν is the frequency in the case of stable oscillations ($\nu^2 > 0$). In the case of instability ($\nu^2 < 0$), $i\nu$ is the "flight frequency".

Hence, (28.5) may be written

$$(28.8)\quad \nu^2 \oint v'\, dy = \oint \left[f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right) - 2R\bar{T} b k + R\bar{T}\sigma k^2 \right] v'\, dy,$$

or

$$(28.9)\quad \nu^2 = \left[f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right) - 2R\bar{T} b k + R\bar{T}\sigma k^2 \right]_m,$$

where the subscript m denotes a certain mean value along the streamline considered.

The stability conditions can be deduced by considering the expression

$$(28.10)\quad \nu_k^2 = f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right) - 2R\bar{T} b k + R\bar{T}\sigma k^2.$$

If ν_k^2 is positive for all values of k, then ν^2 will be positive for an arbitrary closed streamline, and the basic current is stable for perturbations independent of x. If ν_k^2 is zero for one value of k, ($k = k_0$), and positive for all other values of k, then ν^2 will be positive for all closed curves; but ν^2 will approach zero if the shape of the streamline is such that the motion takes place mainly in the direction k_0. The basic current may be said to be indifferent. If ν_k^2 is positive for some values of k and negative for others, then the sign of ν^2 depends upon the shape of the streamline. ν^2 will be negative if the shape of the streamline is such that the motion is directed mainly in the sector where ν_k^2 is negative. The basic current is said to be unstable.

Assuming static stability:

$$(28.11)\quad \sigma > 0,$$

we find from (28.9) the stability criterion:

$$(28.12)\quad f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)\sigma \gtreqless R\bar{T} b^2 \quad \begin{matrix}\text{stable}\\ \text{indifferent}\\ \text{unstable.}\end{matrix}$$

Hence, great meridional temperature gradient, slight static stability and great anticyclonic wind shear in an isobaric surface are destabilizing factors.

In virtue of (26.4) the stability criterion may be rewritten as

$$\left(\frac{\partial \bar{u}}{\partial y}\right)_p + \frac{b}{\sigma}\left(\frac{\partial \bar{u}}{\partial P}\right) \lesseqgtr f \quad \begin{matrix}\text{stable}\\ \text{indifferent}\\ \text{unstable.}\end{matrix}$$

Here the left-hand side represents the variation of $\bar{u}$ along an isentropic surface. Denoting this variation by $\left(\frac{\partial \bar{u}}{\partial y}\right)_\vartheta$, we find the criterion:

$$(28.13)\quad \left(\frac{\partial \bar{u}}{\partial y}\right)_\vartheta \lesseqgtr f \quad \begin{matrix}\text{stable}\\ \text{indifferent}\\ \text{unstable.}\end{matrix}$$

The fundamental state is therefore unstable when the wind shear, measured on an isentropic chart, is anticyclonic with a numerical value greater than f.

[1]) It is interesting to note that one here attains a complete elimination of Φ'. When the corresponding method is used in the system z, however, the pressure tendency is not completely eliminated, except in the incompressible case. Compare Høiland [12], [13].

Consider the quantity

$$U = 2u_e + u\,, \tag{28.14}$$

where u_e is the absolute velocity due to the earth's rotation. The value of U in the fundamental motion is $\bar{U} = 2u_e + \bar{u}$. We may consider u_e as a function of y, and find $\left(\frac{du_e}{dy}\right)_p = -\frac{1}{2}f$. Hence:

$$\left(\frac{\partial \bar{U}}{\partial y}\right)_p = -f + \left(\frac{\partial \bar{u}}{\partial y}\right)_p, \quad \frac{\partial \bar{U}}{\partial P} = \frac{\partial \bar{u}}{\partial P} = \frac{R\bar{T}b}{f}. \tag{28.15}$$

The first equation in the system (28.2) is seen to express that U is a conservative property; but U has generally different values for the different particles. In the basic current, the surfaces $\bar{U} = \text{const.}$ will be parallel to the x-axis; the P-slope of these surfaces is represented by

$$k_U = \left(\frac{\partial P}{\partial y}\right)_{\bar{U}} = -\frac{\left(\frac{\partial \bar{U}}{\partial y}\right)_p}{\frac{\partial \bar{U}}{\partial P}} = \frac{f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)}{R\bar{T}b}. \tag{28.16}$$

In like manner, the P-slope of the isentropic surface is given by:

$$k_\vartheta = \left(\frac{\partial P}{\partial y}\right)_{\bar{\vartheta}} = -\frac{\left(\frac{\partial \bar{\vartheta}}{\partial y}\right)_p}{\frac{\partial \bar{\vartheta}}{\partial P}} = \frac{b}{\sigma}. \tag{28.17}$$

Utilizing these expressions, (28.10) may be written in the form

$$\nu_k^2 = R\bar{T}\sigma\,(k^2 - 2\,k_\vartheta k + k_\vartheta k_U)\,, \tag{28.18}$$

and the stability criterion may accordingly be written:

$$k_\vartheta\,(k_U - k_\vartheta) \begin{array}{c} > \\ = \\ < \end{array} 0 \quad \begin{array}{l} \text{stable} \\ \text{indifferent} \\ \text{unstable} \end{array} \tag{28.19}$$

Under normal atmospheric conditions, we have $k_\vartheta > 0$, and the condition for instability is that the surface $\bar{U} = \text{constant}$ is less steep than the isentropic surface.

The direction (k_m) of minimum stability, or maximum instability, is given by:

$$\frac{d(\nu_k^2)}{dk} = 0\,, \text{ or } k_m = k_\vartheta\,, \tag{28.20}$$

and is thus represented by the slope of the isentropic surfaces.

In the unstable case, the directions ($k = k_0$) which separate the unstable sectors from the stable ones are given by

$$\nu_k^2 = 0\,, \text{ or } k_0 = k_\vartheta \pm \sqrt{k_\vartheta\,(k_\vartheta - k_U)}\,, \tag{28.21}$$

showing that the isentropic surface lies within the unstable sector, provided the atmosphere is statically stable. In the indifferent case, the isentropic surface represents the indifferent direction.

The stability criteria for a baroclinic circular vortex subjected to vortex ring perturbations have been derived, in general form, by Solberg [20] and Høiland [12], [13]. Fjørtoft [8], [9] has applied the theory to atmospheric conditions and thus derived the stability criteria of a zonal current. The corresponding stability criteria for a linear current on the rotating flat earth have been derived by Kleinschmidt [15]. The stability criteria for a linear current prove to be nearly the same as the stability criteria of a curved zonal current, except near the pole, where the curvature of the zonal current is great. The analysis given in this section shows that the stability criteria for a linear current are arrived at even by using the quasi-static approximation.

The theory of Solberg and Høiland is based upon the conservation of circulation (or angular momentum) of every individual zonal circle. Applied to a statically stable atmosphere, whose temperature decreases northwards, the theory shows that the condition for instability is that the surfaces of constant circulation are less steep than the isentropic surfaces. This is almost the same criterion as (28.19), the only difference being that the slope of the surface of constant circulation is replaced by the slope of the surface $\bar{U} = \text{constant}$. This difference is due to the neglect of the curvature of the zonal current and of the vertical Coriolis force. Provided that the surface of constant circulation are quasi-horizontal, and the region near the pole is excluded, the difference is, however, unimportant, because the surfaces $\bar{U} = \text{constant}$ are nearly coincident with the surfaces of constant circulation, as will be easily verified. It is also easy to show that for small meridional displacements, it is of no consequence whether U or the circulation is considered as conservative. Hence, the stability criteria found by the quasi-static method for a linear westerly current are approximately the same as the stability criteria found by the "exact" method for a curved, zonal current, provided that the surfaces $\bar{U}$ = constant are quasi-horizontal; and this is normally the case in the atmosphere when instability of this kind occurs.

The frequencies computed by the quasi-static method from eq. (28.9) will not be strictly correct, but the error will be slight when the motion is mainly quasi-horizontal.

Formulae (28.20) and (28.21), which define the unstable sector and the direction of maximum instability, will be nearly correct when the surface $\bar{U}$ = constant and the isentropic surface are quasi-horizontal.

We will now consider some simple solutions of the equations (28.2). The equation of continuity, which is the last equation of the system (28.2), will be identically satisfied by putting:

$$(28.22)\qquad v' = -e^P \frac{\partial F}{\partial P}, \quad \psi' = e^P \left(\frac{\partial F}{\partial y}\right)_p,$$

F being the stream function in the yp-plane. Substituting this into eqs. (28.3) and (28.4), and assuming a time dependency as expressed by (28.7), we obtain:

$$(28.23)\qquad \begin{cases} -\left(\dfrac{\partial}{\partial y}\dfrac{\partial \Phi'}{\partial t}\right)_p + \left[f\left(f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right) - \nu^2\right] e^P \dfrac{\partial F}{\partial P} \\ \qquad\qquad + R\bar{T}be^P \left(\dfrac{\partial F}{\partial y}\right)_p = 0 \\ -\dfrac{\partial}{\partial P}\left(\dfrac{\partial \Phi'}{\partial t}\right)_p - R\bar{T}be^P \dfrac{\partial F}{\partial P} - R\bar{T}\sigma e^P \left(\dfrac{\partial F}{\partial y}\right)_p = 0. \end{cases}$$

For simplicity, the coefficients of these equations will now be considered as constants. This of course is not strictly correct, since $\bar{T}$ is assumed to vary in the y-direction; but the main effect of this variation has already been taken into consideration.

By eliminating Φ' between the above equations, we find the differential equation for the stream function:

$$(28.24)\qquad \frac{\partial^2 F}{\partial P^2} + \frac{2}{k'}\left(\frac{\partial^2 F}{\partial P \partial y}\right)_p + \frac{1}{k' k_\vartheta}\left(\frac{\partial^2 F}{\partial y^2}\right)_p + \frac{\partial F}{\partial P} + \frac{1}{k'}\left(\frac{\partial F}{\partial y}\right)_p = 0,$$

with the abbreviation

$$(28.25)\qquad k' = \frac{f\left(f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right) - \nu^2}{R\bar{T}b} = k_U - \frac{\nu^2}{R\bar{T}b}.$$

Eq. (28.24) can be simplified by introducing a new variable Y, defined by:

$$(28.26)\qquad y = Y + \frac{P}{k'}.$$

This gives a skew coordinate system (Y, P), where the "P-slope" of the lines Y = constant is equal to k'. When transformed into this system, the differential equation (28.24) becomes

$$(28.27)\qquad \frac{\partial^2 F}{\partial P^2} + \frac{\partial F}{\partial P} + \frac{k' - k_\vartheta}{k'^2 k_\vartheta}\frac{\partial^2 F}{\partial Y^2} = 0.$$

By separation we find solutions of the form

$$(28.28)\qquad F = Ke^{-\frac{1}{2}P} \sin \alpha (P - P_0) \sin \beta (Y - Y_0),$$

where K, P_0 and Y_0 are constants of integration, and

$$(28.29)\qquad \alpha^2 + \tfrac{1}{4} + \frac{k' - k_\vartheta}{k'^2 k_\vartheta}\beta^2 = 0.$$

We assume that β is real. When α is real too, the motion will be cellular, taking place within skew parallelogrammatic cells in the yP-plane. The "P-slope" of these cells is k', and thus depends upon the frequency. The breadth of the cells is $B = \dfrac{\pi}{\beta}$, and the depth, measured in the coordinate P, is $A = \dfrac{\pi}{\alpha}$.

On the other hand, if α is imaginary, the motion is non-cellular; such a motion is possible only when there is an exchange of energy at the boundaries of the system considered.

If the dimensions of the cells, and hence α and β, are given, then the frequency can be computed from (28.29), by inserting the expression (28.25) for k'. This gives an equation of the second degree in ν^2. To either of the roots, there corresponds a certain slope of the cells, computable from (28.25).

With a view to applications to the atmosphere, we are most interested in the case of a slight stability or instability, in the sense that the surfaces $\bar{U}$ = constant and $\bar{\vartheta}$ = constant are nearly coinciding, so that $|k_U - k_\vartheta|$ is small. The frequency equation will then have one root which is small compared to $f\left(f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right)$. In the following we will consider this root only, since it is this root which gives the instability.

The small root can be approximately determined by putting $k'^2 = k_U^2$ in the denominator in the third term of (28.29), whereas the formula (28.25) is used in the numerator. This gives

$$\alpha^2 + \tfrac{1}{4} + \left(1 - \left(\frac{\nu}{\nu_m}\right)^2\right) S\beta^2 = 0, \tag{28.30}$$

where

$$S = \frac{k_U - k_\vartheta}{k_U^2 k_\vartheta} = \frac{R\bar{T}}{f^2\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)^2}\left[f\left(f - \left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)\sigma - R\bar{T}\,b^2\right], \tag{28.31}$$

and

$$\nu_m^2 = R\bar{T}b\,(k_U - k_\vartheta) = R\bar{T}\sigma k_\vartheta\,(k_U - k_\vartheta). \tag{28.32}$$

It will be seen from (28.19) that S and ν_m^2 will both be positive for a stable current, negative for an unstable current and zero in the indifferent case. ν_m is the limiting frequency for infinitely narrow cells ($\beta = \infty$), and ν_m^2 is the smallest possible value of ν^2 by real α. The

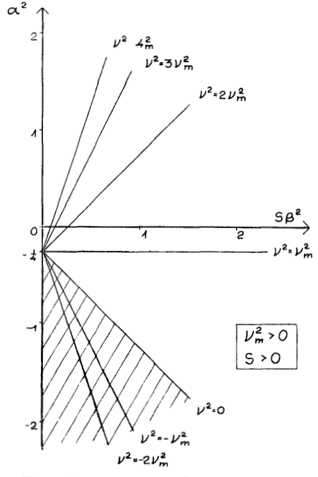

Fig. 1. Lines ν^2 = constant for a stable current. Hatched area: unstable motion.

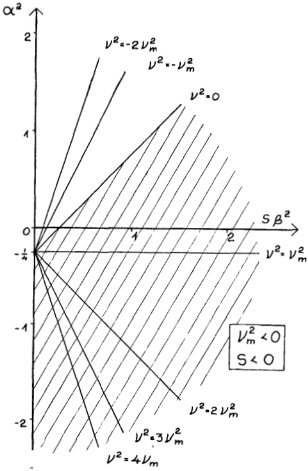

Fig. 2. Lines ν^2 = constant for an unstable current. Hatched area: unstable motion.

slope of the cells will in this case be approximately given by the slope of the surfaces $\bar{\vartheta}$ = constant.

The relation (28.30) between ν^2, α^2 and β^2 for different values of S and ν_m^2 can be represented graphically. In figs. 1 and 2, the dimensionless quantities $(S\beta^2)$ and α^2 are used as coordinates, and lines $\left(\frac{\nu}{\nu_m}\right)^2$ = constant are drawn. These are straight lines running through the point $S\beta^2 = 0$, $\alpha^2 = -\frac{1}{4}$. Fig. 1 applies in case of stability ($S > 0$, $\nu_m^2 > 0$), fig. 2 in case of instability ($S < 0$, $\nu_m^2 < 0$). In the stable case, ν^2 is seen to be positive for all positive values of α^2. In the case of instability, ν^2 is seen to be negative for cells which are sufficiently narrow compared to their depth. The line $\nu^2 = 0$ separates the region of stable oscillation from the region of unstable motion.

29. Wave perturbations in an autobarotropic current.

Consider now wave perturbations of the form

$$(29.1)\qquad \begin{cases} u' = \hat{u}\,(y, P)\, e^{i\mu(x-ct)} \\ v' = \hat{v}\,(y, P)\, e^{i\mu(x-ct)} \\ \psi' = \hat{\psi}\,(y, P)\, e^{i\mu(x-ct)} \\ \Phi' = \hat{\Phi}\,(y, P)\, e^{i\mu(x-ct)} \end{cases}$$

where μ denotes the wave number and c the wave velocity. Inserting these expressions into the perturbation equations (27.2), we obtain

$$(29.2)\qquad \begin{cases} i\mu\,(\bar{u}-c)\,\hat{u} - \left(f - \left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right)\hat{v} + \dfrac{R\overline{T}\,b}{f}\,\hat{\psi} + i\,\mu\hat{\Phi} = 0 \\ f\hat{u} + i\mu\,(\bar{u}-c)\,\hat{v} + \left(\dfrac{\partial\hat{\Phi}}{\partial y}\right)_p = 0 \\ i\,\mu\,(\bar{u}-c)\,\dfrac{\partial\hat{\Phi}}{\partial P} - R\,\overline{T}\,b\,\hat{v} + R\,\overline{T}\,\sigma\,\hat{\psi} = 0 \\ i\,\mu\,\hat{u} + \left(\dfrac{\partial\hat{v}}{\partial y}\right)_p + \dfrac{\partial\hat{\psi}}{\partial P} - \hat{\psi} = 0. \end{cases}$$

The boundary conditions (27.3) and (27.4) become

$$(29.3)\qquad i\mu\,(\bar{u}-c)\,\hat{\Phi} - f\bar{u}\,\hat{v} + R\overline{T}\hat{\psi} = 0 \text{ when } P = P_0$$

$$(29.4)\qquad \lim_{P\to\infty}\,(e^{-P}\hat{\psi}) = 0.$$

The system (29.2) shows that u' and Φ' have the same phase, and also that v' and ψ' have the same phase, whereas there is a phase difference of $\frac{1}{2}\pi$ between u' and v'.

In this section, we will only deal with the case of autobarotropy

$$(29.5)\qquad b = 0,\quad \sigma = 0,\quad \frac{\partial\bar{u}}{\partial P} = 0,\ \bar{u} = \bar{u}\,(y).$$

The perturbation equations (29.2) then become

$$(29.6)\qquad \begin{cases} i\,\mu\,(\bar{u}-c)\,\hat{u} - \left(f - \dfrac{d\bar{u}}{dy}\right)\hat{v} + i\mu\hat{\Phi} = 0 \\ f\hat{u} + i\mu\,(\bar{u}-c)\,\hat{v} + \left(\dfrac{\partial\hat{\Phi}}{\partial y}\right)_p = 0 \\ i\mu\,(\bar{u}-c)\,\dfrac{\partial\hat{\Phi}}{\partial P} = 0 \\ i\mu\hat{u} + \left(\dfrac{\partial\hat{v}}{\partial y}\right)_p + \dfrac{\partial\hat{\psi}}{\partial P} - \hat{\psi} = 0. \end{cases}$$

If there is to be an autobarotropic transition from the basic motion to the perturbed motion, then the local[1]) perturbation of specific volume must vanish; this gives

$$(29.7)\qquad \frac{\partial\hat{\Phi}}{\partial P} = 0.$$

Differentiating the two first equations of (29.6) with respect to P, we find, in virtue of (29.5) and (29.7)

$$(29.8)\qquad i\mu\,(\bar{u}-c)\,\frac{\partial\hat{u}}{\partial P} - \left(f - \frac{\partial\bar{u}}{\partial y}\right)\frac{\partial\hat{v}}{\partial P} = 0$$

$$f\,\frac{\partial\hat{u}}{\partial P} + i\mu\,(\bar{u}-c)\,\frac{\partial\hat{v}}{\partial P} = 0.$$

Hence, if $\hat{u}$ and $\hat{v}$ vary with P,

$$(29.9)\qquad \mu^2\,(\bar{u}-c)^2 = f\left(f - \frac{d\bar{u}}{dy}\right).$$

Here $\left(f - \dfrac{d\bar{u}}{dy}\right)$ is the absolute vorticity of the basic current, and the right-hand side equals the square of the limiting frequency found by V. Bjerknes [5] for pure inertial oscillations in a circular vortex rotating with the angular velocity $\frac{1}{2}f$. The orbital frequency $\mu\,(\bar{u}-c)$ in the wave motion is thus determined merely by the inertia stability.

The inertia stability becomes inactive when there is no circulation in the meridional plane; in the quasi-static theory, this means that $\dfrac{\partial\hat{v}}{\partial P}$ is zero. This case has been investigated by Rossby and Coll. [19], J. Bjerknes and Holmboe [1], and Charney [7]. It will now be shown that their results can be deduced from the system (29.6) together with the boundary conditions.

When $\hat{v}$ is assumed to be independent of P, it follows from (29.8) that $\hat{u}$ is also independent of P. The last equation of (29.6), together with the boundary condition (29.4) then shows that $\hat{\psi}$ is independent of P. This entails that the condition (29.3) for horizontal motion at the ground will be valid for all values of P and the motion is, therefore, strictly horizontal. It follows that the dependent variables, which are now functions of y only, must satisfy the following system:

[1]) i. e. by constant x, y and P.

$$(29.10)\quad \begin{cases} i\mu(\bar{u}-c)\,\hat{u}-\left(f-\dfrac{d\bar{u}}{dy}\right)\hat{v}+i\mu\hat{\Phi}=0 \\ f\hat{u}+i\mu(\bar{u}-c)\,\hat{v}+\dfrac{d\hat{\Phi}}{dy}=0 \\ i\mu\hat{u}+\dfrac{d\hat{v}}{dy}-\hat{\psi}=0 \\ i\mu\,(\bar{u}-c)\,\hat{\Phi}-f\bar{u}\hat{v}+R\bar{T}\hat{\psi}=0. \end{cases}$$

Eliminating $\hat{\psi}$ between the two last equations, we obtain the tendency equation for this particular kind of motion,

$$(29.11)\quad i\mu(\bar{u}-c)\,\hat{\Phi}=f\bar{u}\hat{v}-R\bar{T}\left(i\mu\hat{u}+\frac{d\hat{v}}{dy}\right).$$

Here the parenthesis in the last term on the right represents the horizontal p-divergence.

Eliminating $\hat{\Phi}$ between the two first equations of (29.10), we find the equation for the vertical p-vorticity,

$$(29.12)\quad \left(f-\frac{d\bar{u}}{dy}\right)\left(i\mu\hat{u}+\frac{d\hat{v}}{dy}\right)-i\mu(\bar{u}-c)\frac{d\hat{u}}{dy} -\mu^2\left[\bar{u}-c-u_c+\frac{1}{\mu^2}\frac{d^2\bar{u}}{dy^2}\right]\hat{v}=0,$$

where

$$(29.13)\quad u_c=\frac{1}{\mu^2}\frac{df}{dy}$$

is called the "critical velocity" by J. Bjerknes and Holmboe.

Rossby's result is obtained by making the following assumptions:

(i) the horizontal p-divergence $\left(i\mu\hat{u}+\dfrac{d\hat{v}}{dy}\right)$ is equal to zero;

(ii) $\hat{u}$ is independent of y;

(iii) the basic current has a constant vorticity, so that $\dfrac{d^2\bar{u}}{dy^2}=0$.

The vorticity equation then simply becomes

$$\mu^2\,[\bar{u}-c-u_c]\,\hat{v}=0,$$

and so we find

$$(29.14)\quad c=\bar{u}-u_c,$$

which is Rossby's formula. It was obtained by Rossby from the equation of vertical vorticity by use of similar assumptions. The equation states that the wave is stationary ($c=0$) for a certain "stationary wave length"

$$(29.15)\quad L_s=2\pi\sqrt{\frac{\bar{u}}{\dfrac{df}{dy}}}.$$

Waves with wave lengths smaller than L_s are propagated eastwards, whereas those with wave lengths greater than L_s travel westwards.

Rossby has also discussed the effect of a variable vorticity in the basic current. From the mean wind distribution over North America he finds that $\left|\dfrac{d^2\bar{u}}{dy^2}\right|$ is about 25 % of $\dfrac{df}{dy}$; this gives in the formula (29.14) a correction term, which is about 25 % of u_c.

In this case, however, it seems to be more correct to base the considerations on the qualities of the instantaneous wind distribution; and by inspection of actual aerological charts, one finds that $\left|\dfrac{d^2\bar{u}}{dy^2}\right|$ is very often of the same order of magnitude as $\dfrac{df}{dy}$. This is especially true in the vicinity of a strong west-wind maximum. The correction term in (29.14) due to $\dfrac{d^2\bar{u}}{dy^2}$ will then be of the same order of magnitude as u_c. In this case, we cannot obtain a correct estimate of the wave velocity by assuming that $\hat{u}$ is independent of y. This assumption means that at every point the zonal shear $\dfrac{\partial u}{\partial y}$ has the same value in the perturbed motion as in the basic current. For instance, if the zonal wind in the basic current has a maximum along a line $y=y_{max}$, then this line will also represent the maximum zonal wind speed in the perturbed motion. This does not agree with the observed wave patterns; on the contrary, the observations show that the lines of constant zonal shear, e. g. the line of maximum zonal speed, assume a waveshaped pattern in the same way as the contour lines, or the isotherms. Thus, the observations contradict the assumption of local conservation of the zonal shear. From the observations, one should rather suggest that there is a tendency for the zonal shear to be individually conserved; i. e.

$$(29.16)\quad \frac{D}{dt}\frac{\partial}{\partial y}(\bar{u}+u')\approx 0.$$

This may also be written:

$$(29.17)\quad i\mu(\bar{u}-c)\frac{d\hat{u}}{dy}+\frac{d^2\bar{u}}{dy^2}\hat{v}\approx 0.$$

This means that in the vorticity equation (29.12), the term involving $\frac{d^2\bar{u}}{dy^2}$ will have a tendency to be balanced by the term involving $\frac{d\hat{u}}{dy}$. The assumptions $\frac{d\hat{u}}{dy} = 0$ and $\frac{d^2\bar{u}}{dy^2} = 0$, which were made at the outset, may therefore be replaced by the single assumption that the zonal shear is individually conserved; and this assumption seems to be more justified than the former ones. There is therefore reason to believe that Rossby's formula (29.14) may be approximately correct, even in cases when $\left|\frac{d^2\bar{u}}{dy^2}\right|$ is of the same order of magnitude as $\frac{df}{dy}$. Strictly speaking, since c is considered as a constant, the formula (29.14) can be valid only when $\bar{u}$ varies with y in the same way as u_c. It is, however, reasonable to suppose that the formula will be approximately true even when this condition is not strictly fulfilled.

Rossby [19] has generalized the formula (29.14) by taking into consideration the horizontal divergence, assuming the air to be homogeneous and incompressible. Holmboe [11] has derived the corresponding formula for an autobarotropic compressible atmosphere, by making the assumptions

(i) $\hat{u}$ and $\hat{v}$ are independent of y
(ii) $\bar{u}$ is constant.

With these assumptions, the vorticity equation (29.12) becomes

$$f\, i\, \mu\, \hat{u} - \mu^2 (\bar{u} - c - u_c)\, \hat{v} = 0. \tag{29.18}$$

By eliminating $\hat{\Phi}$ between (29.11) and the first equation of (29.10), we find

$$i\, \mu\, [R\bar{T} - (\bar{u} - c)^2]\, \hat{u} - f\, c\, \hat{v} = 0. \tag{29.19}$$

These two equations form a linear and homogeneous system, whose determinant must vanish. This gives

$$\bar{u} - c - u_c = \frac{f^2}{\mu^2} \frac{c}{R\bar{T} - (\bar{u} - c)^2}. \tag{29.20}$$

This equation is identic with the formula for autobarotropic waves found by Charney [7], eq. 31, and approximately identic with the equation found by Holmboe [11], eq. 12.07,6. Eq. (29,20) is of the third degree in c, but only the root which is small compared with $\sqrt{R\bar{T}}$ is of meteorological interest. As pointed out by Charney, this root can be approximately determined by disregarding $(\bar{u} - c)^2$ versus $R\bar{T}$; this gives

$$c = \frac{\bar{u} - u_c}{1 + \frac{f^2}{\mu^2 R\bar{T}}}. \tag{29.21}$$

This formula is discussed by Holmboe; he finds that the difference between this formula and Rossby's formula (29.14) is slight, except for extremely long waves.

Charney [7] has shown that the velocity formula (29.21) implies that the meridional wind component is nearly geostrophic, and conversely that the approximate solution (29.21) of (29.20) is arrived at directly by assuming geostrophic wind in the y-direction. This assumption therefore automatically eliminates the great roots of (29.20), which correspond to fast-moving gravitational waves of little meteorological importance, so that only the meteorological important solution remains.

It will now be shown that eq. (29.21) holds also when $\hat{u}$ and $\hat{v}$ vary in the y-direction, provided that the meridional wind component can be considered as geostrophic, and the zonal shear is individually conserved.

The condition for geostrophic wind in the y-direction is

$$i\, \mu\, \hat{\Phi} = f\, \hat{v}. \tag{29.22}$$

If there is no horizontal p-divergence, the tendency equation (29.11) becomes

$$i\, \mu\, (\bar{u} - c)\, \hat{\Phi} = f\, \bar{u}\, \hat{v}. \tag{29.23}$$

Comparison with (29.22) shows that the meridional wind will be geostrophic for the stationary wave, but for wave-lengths considerably smaller, or greater, than the stationary wave-length, the meridional wind will differ considerably from the geostrophic wind.

By combining (29.11) and (29.22) we may compute the horizontal p-divergence necessary to secure that the meridional wind component be geostrophic for all wave velocities. We then obtain

$$i\, \mu\, \hat{u} + \frac{d\hat{v}}{dy} = \frac{i\, \mu\, c}{R\bar{T}} \hat{\Phi} = \frac{f\, c}{R\bar{T}} \hat{v}. \tag{29.24}$$

Inserting this in the vorticity equation (29.12), and assuming further that the zonal shear is nearly conservative, so that the term involving $\frac{d\bar{u}}{dy}$ is nearly neutralized by the term involving $\frac{d^2\bar{u}}{dy^2}$, we find

$$\left[\frac{f\left(f-\frac{d\bar{u}}{dy}\right)}{R\bar{T}} c - \mu^2 (\bar{u} - c - u_c)\right] \hat{v} = 0.$$

Hence

$$c = \frac{\bar{u} - u_c}{1 + \frac{f\left(f-\frac{d\bar{u}}{dy}\right)}{\mu^2 R \bar{T}}}. \tag{29.25}$$

which for $\bar{u}$ = constant is identical with Holmboe's equation. It is very interesting to note that the potential field (pressure field) and the tendencies in this diverging wave are very different from the potential field and the tendencies in the non-diverging wave considered by Rossby, although their speed of propagation are approximately the same (except for extremely long waves). This illustrates the important fact that a horizontal p-divergence which is small compared to the other terms in the vorticity equation, may be one of the main terms in the tendency equation. As to the speed of propagation, it is therefore of very little importance whether eq. (29.22) is considered as strictly fulfilled, or as a mere approximation.

The solutions considered in this section are based upon more or less artificial assumptions concerning the kinematics of the motion. A physically satisfying theory is possible only by formulating boundary conditions also at the lateral boundaries of the stream, and solving the boundary problem thus arising.

CHAPTER VI. ON THE THEORY OF QUASI-GEOSTROPHIC MOTION.

30. The quasi-geostrophic wind formula.

In this chapter we shall consider the simplifications gained by assuming the wind to be approximately geostrophic.

The equation of motion (11.2) may be written in the form

$$\boldsymbol{v} = \boldsymbol{v}_g + \frac{\boldsymbol{k}}{f} \times \frac{D\boldsymbol{v}}{dt}, \tag{30.1}$$

where

$$\boldsymbol{v}_g = \frac{\boldsymbol{k}}{f} \times \nabla_p \Phi \tag{30.2}$$

is the geostrophic wind. We assume that the horizontal acceleration can be approximately written (compare eq. 18.4)

$$\frac{D\boldsymbol{v}}{dt} = \left(\frac{\partial \boldsymbol{v}_g}{\partial t}\right)_p + \boldsymbol{v}_g \cdot \nabla_p \boldsymbol{v}_g + \psi \frac{\partial \boldsymbol{v}_g}{\partial P}, \tag{30.3}$$

i. e. the horizontal acceleration is computed as if the horizontal wind were geostrophic. Using the notation

$$\boldsymbol{b} = -\nabla_p \ln \vartheta = -\nabla_p \ln T = -\frac{1}{RT}\frac{\partial}{\partial P} \nabla_p \Phi, \tag{30.4}$$

and neglecting the variation of f with latitude in the term $\boldsymbol{v}_g \cdot \nabla_p \boldsymbol{v}_g$, eq. (30.3) may be written, in virtue of (18.7),

$$\frac{D\boldsymbol{v}}{dt} = \frac{\boldsymbol{k}}{f} \times \left[\nabla_p \left(\frac{\partial \Phi}{\partial t}\right)_p + \boldsymbol{v}_g \cdot \nabla_p \nabla_p \Phi - RT\, \psi\, \boldsymbol{b}\right]. \tag{30.5}$$

Inserting this into (30.1), we obtain the quasi-geostrophic wind formula

$$\boldsymbol{v} = \boldsymbol{v}_g - \left[\frac{1}{f^2}\left(\nabla_p \left(\frac{\partial \Phi}{\partial t}\right)_p + \boldsymbol{v}_g \cdot \nabla_p \nabla_p \Phi - RT\, \psi \boldsymbol{b}\right)\right], \tag{30.6}$$

by which the velocity is expressed in terms of the potential field (or pressure field) and its variation with time, the temperature field and the "vertical P-velocity" ψ. The term within the brackets corresponds to the geostrophic departure. The first term in the parenthesis is the "isallobaric wind". Hesselberg [10] was the first to realize this effect, which was later discussed by Brunt and Douglas [6]. The second term represents the effects of curvature and divergence of stationary contour lines. This term gives a contribution in the direction of $\boldsymbol{v}_g$ for anticyclonically curved contour lines, and in the opposite direction for cyclonically curved contour lines; it also gives a contribution along $\nabla_p \Phi$ when the contour lines diverge in the direction of $\boldsymbol{v}_g$, and a contribution along $-\nabla_p \Phi$ when the contour lines converge in the direction of $\boldsymbol{v}_g$. The third term in the parenthesis corresponds to the term introduced by

Sutcliffe [21]. This term is directed along $\boldsymbol{b}$ (towards colder air) for ascending motion (relatively to the isobaric surfaces), and along $-\boldsymbol{b}$ (towards warmer air) for descending motion.

It is possible to attain a still better approximation by writing, instead of (30.3)

$$(30.7)\quad \frac{D\boldsymbol{v}}{dt}=\frac{D\boldsymbol{v}_g}{dt}=\left(\frac{\partial \boldsymbol{v}_g}{\partial t}\right)_p+\boldsymbol{v}\cdot\nabla_p\boldsymbol{v}_g+\psi\frac{\partial \boldsymbol{v}_g}{\partial P}$$
$$=\frac{\boldsymbol{k}}{f}\times\left[\nabla_p\left(\frac{\partial\Phi}{\partial t}\right)_p+\boldsymbol{v}\cdot\nabla_p\nabla_p\Phi-RT\psi\boldsymbol{b}\right].$$

This approximation has the advantage compared to (30.3) that the convective acceleration due to a horizontal shear of the geostrophic wind is taken into consideration. This acceleration term is important when the inertial stability of the wind field is examined. For instance, suppose we are dealing with the convective acceleration in a frontal zone parallel to the contour lines, due to the upsliding motion along the frontal zone. In this case, formula (30.3) gives the convective acceleration due to the vertical motion only. The result is obviously wrong, because a parcel moving vertically through the frontal zone will have a convective acceleration completely different from that of a parcel moving along the frontal slope. This difficulty is avoided by using the acceleration formula (30.7), which must therefore be considered as a much better approximation.

When (30.7) is inserted into (30.1), we find

$$(30.8)\quad (\boldsymbol{v}-\boldsymbol{v}_g)\cdot(f\varepsilon-\nabla_p(\boldsymbol{k}\times\boldsymbol{v}_g))$$
$$=-\frac{1}{f}\left[\nabla_p\left(\frac{\partial\Phi}{\partial t}\right)_p+\boldsymbol{v}_g\cdot\nabla_p\nabla_p\Phi-RT\psi\boldsymbol{b}\right],$$

where ε is the unit tensor. This vector equation involves two linear equations for the components of $\boldsymbol{v}-\boldsymbol{v}_g$. Solving with respect to $\boldsymbol{v}$, we obtain

$$(30.9)\quad \boldsymbol{v}=\boldsymbol{v}_g-\frac{1}{f}\frac{f\varepsilon+\boldsymbol{k}\times\nabla_p\boldsymbol{v}_g}{|f\varepsilon+\boldsymbol{k}\times\nabla_p\boldsymbol{v}_g|}$$
$$\cdot\left[\nabla_p\left(\frac{\partial\Phi}{\partial t}\right)_p+\boldsymbol{v}_g\cdot\nabla_p\nabla_p\Phi-RT\psi\boldsymbol{b}\right],$$

where

$$|f\varepsilon+\boldsymbol{k}\times\nabla_p\boldsymbol{v}_g|$$
$$=\left(f+\left(\frac{\partial v_g}{\partial x}\right)_p\right)\left(f-\left(\frac{\partial u_g}{\partial y}\right)_p\right)+\left(\frac{\partial u_g}{\partial x}\right)_p\left(\frac{\partial v_g}{\partial y}\right)_p$$

is the determinant corresponding to the tensor $f\varepsilon+\boldsymbol{k}\times\nabla_p\boldsymbol{v}_g$. The formula (30.9) becomes unserviceable when this determinant is zero.

Formula (30.9) will probably give a better approximation to the true wind than (30.6), but owing to the intricate form of (30.9), calculations with this formula are often very cumbersome. In cases where the convective acceleration due to the geostrophic deviation is slight (i. e. $(\boldsymbol{v}-\boldsymbol{v}_g)\cdot\nabla_p\boldsymbol{v}_g$ is slight) the two formulae (30.6) and (30.9) will give nearly the same result.

31. Quasi-geostrophic motion with rectilinear contour lines.

Suppose that the potential is independent of x,

$$(31.1)\qquad \Phi=\Phi(y,P,t).$$

The contour lines are then straight lines parallel to the x-axis. From (30.4) it follows that

$$(31.2)\qquad \boldsymbol{b}=b\boldsymbol{j},\; b=-\left(\frac{\partial}{\partial y}\ln\vartheta\right)_p.$$

The geostrophic wind is

$$(31.3)\qquad \boldsymbol{v}_g=u_g\boldsymbol{i},\; u_g=-\frac{1}{f}\left(\frac{\partial\Phi}{\partial y}\right)_p.$$

The motion will be essentially the same as that treated in section 28; but the motion will now be considered by means of the quasi-geostrophic wind formula (30.9), which assumes the form

$$(31.4)\quad u=u_g$$
$$v=\frac{1}{f\left(f-\left(\frac{\partial u_g}{\partial y}\right)_p\right)}\left[-\left(\frac{\partial}{\partial y}\frac{\partial\Phi}{\partial t}\right)_p+RTb\psi\right].$$

Here we have assumed that f is independent of x, so that $\left(\frac{\partial u_g}{\partial x}\right)_p=0$; this is strictly true only when the contour lines are zonal.

The last equation of (31.4) may be written

$$(31.5)\quad -\left(\frac{\partial}{\partial y}\frac{\partial\Phi}{\partial t}\right)_p-f\left(f-\left(\frac{\partial u_g}{\partial y}\right)_p\right)v+RTb\psi=0.$$

Comparison with eq. (28.3) shows that the quasi-geostrophic approximation in this case means to neglect the acceleration in the y-direction. It follows that considerations concerning the stability of the motion for perturbations independent of x can not be based upon the quasi-geostrophic approximation. From this approximation one can only find the kinematics of the motion when $\nu^2 = 0$, corresponding to the line $\nu^2 = 0$ in the diagrams figs. 1 and 2.

32. On quasi-geostrophic treatment of waves in a baroclinic westerly current.

Consider wave perturbations of the basic current defined in section 26. In the perturbation equations (29.2) for a baroclinic current, the equations of motion will now be replaced by the quasi-geostrophic equations (30.9).

In the basic current, the value of the tensor $f\boldsymbol{\varepsilon}+\boldsymbol{k}\times\nabla_p\boldsymbol{v}_g$ is seen to be $\left(f-\left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)\boldsymbol{ii}+f\boldsymbol{jj}$, and the perturbation equations corresponding to (30.9) become

$$(32.1)\quad \begin{cases} u' = u'_g - \dfrac{1}{f^2}\left(\dfrac{\partial}{\partial t}+\bar{u}\dfrac{\partial}{\partial x}\right)_p\left(\dfrac{\partial \Phi'}{\partial x}\right)_p \\ v' = v'_g - \dfrac{1}{f\left(f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right)} \\ \left[\left(\dfrac{\partial}{\partial t}+\bar{u}\dfrac{\partial}{\partial x}\right)_p\left(\dfrac{\partial \Phi'}{\partial y}\right)_p - f v'_g\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p - R\bar{T}b\psi'\right], \end{cases}$$

where

$$(32.2)\qquad u'_g = -\frac{1}{f}\left(\frac{\partial \Phi'}{\partial y}\right)_p,\quad v'_g = \frac{1}{f}\left(\frac{\partial \Phi'}{\partial x}\right)_p.$$

For wave perturbations of the form (29.1) we obtain

$$(32.3)\quad \begin{cases} \hat{u} = -\dfrac{1}{f}\left(\dfrac{\partial \hat{\Phi}}{\partial y}\right)_p + \dfrac{\mu^2}{f^2}(\bar{u}-c)\hat{\Phi} \\ \hat{v} = \dfrac{i\mu}{f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p}\hat{\Phi} \\ \qquad - \dfrac{i\mu(\bar{u}-c)}{f\left(f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right)}\left(\dfrac{\partial \hat{\Phi}}{\partial y}\right)_p + \dfrac{R\bar{T}b}{f\left(f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right)}\hat{\psi}. \end{cases}$$

We may use these expressions to eliminate $\hat{u}$ and $\hat{v}$. Inserting for $\hat{u}$ and $\hat{v}$ into the third equation of (29.2) (the equation of piezotropy), we find

$$(32.4)\quad i\mu\left[(\bar{u}-c)\left(\frac{\partial \hat{\Phi}}{\partial P}+\frac{1}{k_U}\left(\frac{\partial \hat{\Phi}}{\partial y}\right)_p\right)-\frac{f}{k_U}\hat{\Phi}\right] + f\left(f-\left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)S\hat{\psi} = 0.$$

Here k_U is the P-slope of the surfaces U = constant (eq. 28.16), and S is the stability factor for perturbations independent of x, defined by (28.31).

When eqs. (32.3) are inserted into the last equation of (29.2) (the equation of continuity), it will be seen that $\left(\frac{\partial \hat{v}}{\partial y}\right)_p$ will give a great many terms, since f, $\bar{u}$ and $\bar{T}$ are all dependent on y. To simplify the analysis, we will assume the two last terms in the expression (32.3) for $\hat{v}$ to be small in comparison with the first term. We may then disregard the variation with y of the denominator $f\left(f-\left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)$ in the two last terms. Neglecting further the variation of $\bar{T}$ in the last term, we obtain

$$(32.5)\quad \frac{i\mu(\bar{u}-c)}{f\left(f-\left(\frac{\partial \bar{u}}{\partial y}\right)_p\right)}\left(\frac{\partial^2 \hat{\Phi}}{\partial y^2}\right)_p - i\mu\frac{\mu^2}{f^2}\left[\bar{u}-c-\frac{f^2}{\mu^2}\frac{\dfrac{df}{dy}-\left(\dfrac{\partial^2 \bar{u}}{\partial y^2}\right)_p}{\left(f-\left(\dfrac{\partial \bar{u}}{\partial y}\right)_p\right)^2}\right]\hat{\Phi} - \frac{\partial \hat{\psi}}{\partial P} - \frac{1}{k_U}\left(\frac{\partial \hat{\psi}}{\partial y}\right)_p + \hat{\psi} = 0.$$

This is the form assumed by the vorticity equation in the quasi-geostrophic approximation. Rossby's formula (29.14) can be deduced from this equation by putting $\hat{\psi} = 0$, $\left(\frac{\partial^2 \hat{\Phi}}{\partial y^2}\right)_p = 0$ and $\left(\frac{\partial \bar{u}}{\partial y}\right)_p = 0$.

By eliminating $\hat{\psi}$ between the equations (32.4) and (32.5), we obtain the differential equation for $\hat{\Phi}$. In doing so, we will assume, for simplicity, that the coefficients of (32.4) are constants, except the term $(\bar{u}-c)$, the variation of which is obviously of decisive importance. This assumption will not affect the coefficients of the second order terms in the differential equation for $\hat{\Phi}$, which then becomes:

$$(32.6)\quad (\bar{u}-c)\left[\frac{\partial^2\hat{\Phi}}{\partial P^2}+\frac{2}{k_U}\left(\frac{\partial^2\hat{\Phi}}{\partial P\partial y}\right)_P + \left(S+\frac{1}{k_U{}^2}\right)\left(\frac{\partial^2\hat{\Phi}}{\partial y^2}\right)_P-\frac{\partial\hat{\Phi}}{\partial P}-\frac{1}{k_U}\left(\frac{\partial\hat{\Phi}}{\partial y}\right)_P-\frac{f-\left(\frac{\partial\bar{u}}{\partial y}\right)_P}{f}\mu^2S\hat{\Phi}\right] + \left(\frac{f}{k_U}+f\,\frac{\frac{df}{dy}-\left(\frac{\partial^2\bar{u}}{\partial y^2}\right)_P}{f-\left(\frac{\partial\bar{u}}{\partial y}\right)_P}S\right)\hat{\Phi}=0.$$

When transformed into the coordinates y, z, this equation becomes identical[1]) with the differential equation for the pressure perturbation found by Solberg [20], eq. 16, provided that the following conditions are fulfilled: (i) the orbital frequency is negligible in comparison with the frequency of vertical oscillations $\left(\frac{g^2\sigma}{R\overline{T}}\right)$, and (ii) the orbital frequency is negligible in comparison with the frequency of inertial oscillations in horizontal direction $f\left(f-\left(\frac{\partial\bar{u}}{\partial y}\right)_P\right)$.

The first of these conditions is due to the quasi-static approximation, and the second is due to the quasi-geostrophic approximation. Eq. (32.6) can be used in such cases only, where both conditions are fulfilled. From this it follows that the quasi-geostrophic assumption is justified in the study of the long waves in the upper westerlies.

By constant k_U, eq. (32.6) can be simplified by introducing a new variable Y, defined by

$$(32.7)\qquad y = Y+\frac{P}{k_U}.$$

This gives a skew coordinate system (Y, P), with the surfaces $Y=$ constant coinciding with the surfaces $\overline{U}=$ constant. By this transformation, eq. (32.6) assumes the form

$$(32.8)\quad (\bar{u}-c)\left[\frac{\partial^2\hat{\Phi}}{\partial P^2}-\frac{\partial\hat{\Phi}}{\partial P}+S\left(\frac{\partial^2\hat{\Phi}}{\partial Y^2}-\frac{f-\frac{\partial\bar{u}}{\partial Y}}{f}\mu^2\hat{\Phi}\right)\right] + \left(\frac{f}{k_U}+fS\,\frac{\frac{df}{dY}-\frac{\partial^2\bar{u}}{\partial Y^2}}{f-\frac{\partial\bar{u}}{\partial Y}}\right)\hat{\Phi}=0.$$

[1]) At least as far as the second order terms are concerned.

It is interesting to note that the static stability does not occur alone in this equation, but only combined with the inertia effect in the factor S, which represents the stability for perturbations independent of x. The equation is of the elliptic, parabolic or hyperbolic type, according as S is positive, zero or negative. This shows that the stability of the current for perturbations independent of x will be a factor of decisive importance also for wave perturbations.

When $\hat{\Phi}$ and $\bar{u}$ are supposed to be independent of y, eq. (32.6) becomes

$$(32.9)\quad (\bar{u}-c)\left(\frac{\partial^2\hat{\Phi}}{\partial P^2}-\frac{\partial\hat{\Phi}}{\partial P}-\mu^2S\hat{\Phi}\right) + \left(\frac{f}{k_U}+S\,\frac{df}{dy}\right)\hat{\Phi}=0.$$

When the height z is introduced as a vertical coordinate, this equation becomes identical with the differential equation for the meridional velocity derived by Charney [7], eq. 58, except that Charney's equation involves the static stability instead of the combined stability factor S. This is justified when a smooth temperature distribution between low and high latitudes is considered, since b then will be of the order of magnitude 2.10^{-8} m^{-1}, so that the term $R\overline{T}b^2$ of eq. (28.31) becomes slight in comparison with the term involving σ. On the other hand, it is a matter of fact that these two terms are nearly equal in the frontal zones, corresponding to small positive or negative values of S. In these zones, which are closely associated with the region of maximum zonal speed, it will therefore not be justified to replace the stability factor S by the static stability. The assumption of constant stability conditions in the meridional plane thus seems to be a too rough approximation. To get a model consistent with the atmospheric westerlies, one has to consider a narrow zone where S is nearly zero, bounded on both sides by layers with great positive values of S. The mathematical treatment of this problem is of course extremely difficult.

33. A method for numerical weather prognosis based upon the quasi-geostrophic approximation.

It was mentioned in Chapter I that a prognostic utilization of the quasi-static equations without making any further simplifying assumptions, seems to be impossible, since the horizontal divergence cannot be computed with sufficient accuracy from the observational data. This difficulty may be overcome by eliminating $\boldsymbol{v}$ by means of the quasi-geostrophic wind formula. Since the wind formula (30.9) will give a very complex analysis, we will content ourselves with using the simpler formula (30.6).

From this formula, we obtain

$$\nabla_p \cdot \boldsymbol{v} = \nabla_p \cdot \left(\frac{\boldsymbol{k}}{f} \times \nabla_p \Phi\right) - \nabla_p \cdot \left[\frac{1}{f^2}\left(\nabla_p \left(\frac{\partial \Phi}{\partial t}\right)_p + \boldsymbol{v}_g \cdot \nabla_p \nabla_p \Phi - RT\psi \boldsymbol{b}\right)\right].$$

For simplicity, we will provide for the variation of f with latitude only in the first term on the right, whereas f is considered as a constant in the second term. It can be shown that this is justified when the geostrophic departure is small. Thus we find

$$(33.1)\quad \nabla_p \cdot \boldsymbol{v} = -\boldsymbol{v}_g \cdot \frac{\nabla_p f}{f} - \frac{1}{f^2}\left[\nabla_p^2 \left(\frac{\partial \Phi}{\partial t}\right)_p + \boldsymbol{v}_g \cdot \nabla_p \nabla_p^2 \Phi - RT\boldsymbol{b} \cdot \nabla_p \psi - RT(\nabla_p \cdot \boldsymbol{b} - b^2)\psi.\right].$$

Eqs. (30.6) and (33.1) will now be substituted into the first law of thermodynamics (18.9) and the equation of continuity (18.8). There results then

$$(33.2)\quad \begin{cases} \left(\frac{\partial}{\partial P} + \frac{RT}{f^2}\boldsymbol{b} \cdot \nabla_p\right)\left(\frac{\partial \Phi}{\partial t}\right)_p + RT\left(\sigma - \frac{RTb^2}{f^2}\right)\psi \\ \qquad = \frac{R}{c_p} H + RT\boldsymbol{b} \cdot \left(\boldsymbol{v}_g - \frac{\boldsymbol{v}_g}{f^2} \cdot \nabla_p \nabla_p \Phi\right), \\ -\frac{1}{f^2}\nabla_p^2 \left(\frac{\partial \Phi}{\partial t}\right)_p + \left(\frac{\partial}{\partial P} + \frac{RT}{f^2}\boldsymbol{b} \cdot \nabla_p\right)\psi \\ \qquad -\left(1 + \frac{RT}{f^2}(b^2 - \nabla_p \cdot \boldsymbol{b})\right)\psi \\ \qquad = \boldsymbol{v}_g \cdot \frac{\nabla_p f}{f} + \frac{\boldsymbol{v}_g}{f^2} \cdot \nabla_p \nabla_p^2 \Phi. \end{cases}$$

This is a system of linear differential equations for the two variables $\left(\frac{\partial \Phi}{\partial t}\right)_p$ and ψ. Together with the boundary conditions, these equations determine $\left(\frac{\partial \Phi}{\partial t}\right)_p$ and ψ when the coefficients are known. The boundary condition $\left(\frac{D\Phi}{dt} = 0\right)$ at the earth's surface ($P = P_0$) may be written, when $\boldsymbol{v}$ is eliminated by means of (30.6),

$$(33.3)\quad \left[\left(\frac{\partial \Phi}{\partial t}\right)_p - \frac{1}{f^2}\nabla_p \Phi \cdot \nabla_p \left(\frac{\partial \Phi}{\partial t}\right)_p - \frac{1}{f^2}\boldsymbol{v}_g \cdot \nabla_p \left(\tfrac{1}{2}(\nabla_p \Phi)^2\right) + RT\psi\left(1 + \frac{\boldsymbol{b}}{f^2} \cdot \nabla_p \Phi\right)\right]_{P=P_0} = 0.$$

The boundary condition at the upper limit of the atmosphere is

$$(33.4)\quad \lim_{P \to \infty} (e^{-P}\psi) = 0,$$

When H is known as a function of space and time (or as a function of space and time and temperature), then the coefficients of these equations can be determined from observations of pressure and temperature only. When these coefficients are determined from the observations at a certain initial instant, the tendency $\left(\frac{\partial \Phi}{\partial t}\right)_p$ at this instant can be computed by solving the boundary problem. From this tendency, we may compute the field of Φ a short interval of time later. The corresponding temperature field follows from the hydrostatic equation. From this new distribution of Φ and T, we can compute the new coefficients of the equations, and by solving this new boundary problem, we obtain a new tendency distribution at this later instant. Proceeding in this way, we obtain an integration in small steps of time. With each step, we also get the distribution of ψ, which determines the physical changes of state which are taking place. From a fundamental point of view, it is thus possible to use the quasi-geostrophic equations as a basis of a numerical computation of coming weather. However, it remains to be seen whether the method is sufficiently accurate for practical use.

List of the main symbols used.

x, y horizontal Cartesian coordinates (x-axis eastwards, y-axis northwards).

z vertical coordinate.

t time.

$\boldsymbol{i}$, $\boldsymbol{j}$ horizontal unit vectors (pointing eastwards, northwards).

$\boldsymbol{k}$ vertical unit vector.

$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y}$ horizontal del-operator.

∇_z, $\left(\frac{\partial}{\partial t}\right)_z$ derivatives by constant z.

∇_p, $\left(\frac{\partial}{\partial t}\right)_p$ derivatives by constant p.

System z system of independent variables x, y, z, t.

System p system of independent variables x, y, p, t.

$\frac{D}{dt}$ individual derivative.

$\boldsymbol{v}$ (with components u, v) horizontal velocity.

$\boldsymbol{v}_g$ (with components u_g, v_g) geostrophic wind.

$\bar{u}$ velocity of the basic current.

u', v' velocity perturbations.

u_e velocity due to the earth's rotation.

$U = 2u_e + u$

w vertical velocity.

Φ geopotential.

p pressure.

$P = -\ln p$

$\omega = \frac{Dp}{dt}$

$\psi = \frac{DP}{dt}$

q density.

$s = \frac{1}{q}$ specific volume.

T absolute temperature.

ϑ potential temperature.

H heat received per unit mass and unit time.

R gas constant.

c_p, c_v specific heat of air by constant pressure, volume.

$\varkappa = c_p/c_v$

Γ coefficient of barotropy.

γ coefficient of piezotropy.

$\sigma = \frac{\partial \ln \vartheta}{\partial P}$

$\boldsymbol{b} = -\nabla_p \ln \vartheta$

c wave velocity.

μ wave number.

$L = \frac{2\pi}{\mu}$ wave length.

ν frequency.

F stream function in the xp-plane.

$\boldsymbol{\Omega}$ (Ω_y, Ω_z) angular velocity of the earth.

$f = 2\,\Omega_z$ Coriolis parameter.

g acceleration of gravity.

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