APPLICATION OF INTEGRAL THEOREMS IN DERIVING CRITERIA OF STABILITY FOR LAMINAR FLOWS AND FOR THE BAROCLINIC CIRCULAR VORTEX

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(Manuscript received April 22nd, 1949.)

CHAPTER I.

THE STABILITY PROPERTIES OF THE STATIONARY CIRCULAR VORTEX FOR VORTEX RING PERTURBATIONS.

1. A theorem by variational problems for an incompressible fluid.

In this paper certain hydrodynamical problems for an incompressible and inviscid fluid will be formulated as variational problems. We will then meet with identities of the form

$$(1.1) \qquad \int_\tau \boldsymbol{A}\cdot\boldsymbol{S}d\tau \equiv 0 \qquad (\boldsymbol{S} = \text{arbitrary})$$

Here, τ is a space domain of integration. $\boldsymbol{A}$ is a vector function of space to be determined closer from the considered variational problem. $\boldsymbol{S}$ is any arbitrary vector function of space satisfying the solenoidal condition

$$(1.2) \qquad \nabla\cdot\boldsymbol{S} = 0$$

and the following conditions at the boundaries of the fluid, S_n denoting the components of $\boldsymbol{S}$ normal to the boundary:

a) At a rigid boundary: $S_n = 0$.
(1.3) b) At a free surface: $S_n =$ arbitrary.
c) At inner surfaces of discontinuity: $S_n =$ arbitrary, $\triangle S_n = 0$.

In the cases to be considered $\boldsymbol{S}$ will be identical with or proportional to infinitely small displacements of the fluid particles. Conditions (1.2), (1.3) are therefore the kinematical conditions which the incompressible fluid must satisfy in the inner and at the boundaries. It should be noted that in the conditions (1.3) b) and c) $S_n =$ arbitrary only so far as this is consistent with the condition of incompressibility. For by integrating (1.2) over the fluid, we obtain, using (1.3):

At a free surface: $\int_{F_f} S_n\,dF = 0$

At surfaces of discontinuity: $\int_{F_d} S_n\,dF = 0$.

We shall now show that the sufficient and necessary conditions for the identity (1.1) with the side conditions (1.2), (1.3) are

a) $\boldsymbol{A} =$ laminar $= -\nabla\lambda$,
(1.4) b) $\lambda =$ constant at the free surface,
c) $\triangle\lambda =$ constant at the surfaces of discontinuity.

It is easily seen that these conditions are sufficient. By substituting for $\boldsymbol{A}$ in the identity (1.1) the laminar vector $-\nabla\lambda$ and thereafter transforming the volume integral to surface integrals by means of the theorem of Gauss, we obtain

$$(1.5) \qquad \int_\tau \boldsymbol{A}\cdot\boldsymbol{S}d\tau = \int_{F_f}\lambda S_n dF + \int_{F_d}\triangle\lambda\cdot S_n dF,$$

having used the boundary condition (1.3) a). According to (1.4) b) and c) the last equation may be written

$$\int_\tau \boldsymbol{A}\cdot\boldsymbol{S}\,d\tau = \lambda\int_{F_f} S_n\,dF + \triangle\lambda\int_{F_d} S_n\,dF.$$

As pointed out above each of the right-hand-side integrals vanish. Therefore it appears that $\int_\tau \boldsymbol{A}\cdot\boldsymbol{S}\,d\tau$ will vanish identically if $\boldsymbol{A}$ satisfies conditions (1.4).

To prove that these conditions are necessary as well, we start proving that the first of these conditions, $\boldsymbol{A} =$ laminar, is necessary. For this purpose we choose the fields of $\boldsymbol{S}$ such that the components of $\boldsymbol{S}$ vanish normal to a contingent free surface. The identity (1.1) will then be one of less generality. This, however makes no difference as long as the problem is to find n e c e s s a r y conditions only for the identity (1.1). With this limitation as to the vector fields of $\boldsymbol{S}$, the solenoidal condition requires that the vector lines for $\boldsymbol{S}$ must be closed curves. We divide now the volume τ into infinitesimal volumes by a system of surfaces, $f(x, y, z) = $ const and the elementary vector tubes for $\boldsymbol{S}$. We may then write $d\tau = |\boldsymbol{df} \cdot \boldsymbol{dl}|$, $\boldsymbol{df}$ denoting the infinitesimal vector surfaces which the vector tubes cut off from the surfaces $f(x, y, z) = $ const, and $\boldsymbol{dl}$ the line elements along $\boldsymbol{S}$. The identity (1.1) may now be written, interchanging the parallel vectors $\boldsymbol{S}$ and $\boldsymbol{dl}$

$$\int_\tau \boldsymbol{A} \cdot \boldsymbol{dl} \, |\boldsymbol{S} \cdot \boldsymbol{df}| \equiv 0.$$

Owing to the solenoidal character of $\boldsymbol{S}$, $\boldsymbol{S} \cdot \boldsymbol{df}$ is a constant along each tube. If therefore f_0 is the one among the surfaces $f(x, y, z) = $ const which crosses a l l vector tubes of $\boldsymbol{S}$, and if $\boldsymbol{S}_0$ represents $\boldsymbol{S}$ in this surface, we may write the last identity

$$\int_F \left[\oint \boldsymbol{A} \cdot \boldsymbol{dl}\right] |\boldsymbol{S}_0 \cdot \boldsymbol{df}_0| \equiv 0,$$

where the surface F to be integrated over consists of all surface elements df_0 cut off once by the vector tubes of $\boldsymbol{S}$. On account of the arbitrariness in the choice of $\boldsymbol{S}$ at the surface F, this identity will be satisfied if and only if

$$\oint \boldsymbol{A} \cdot \boldsymbol{dl} = 0.$$

This equation must be fulfilled for arbitrary closed curves, since it will always be possible to choose the vector function $\boldsymbol{S}$ such that a prescribed closed curve becomes a vector line for $\boldsymbol{S}$. The above equation is therefore itself an identity, and is satisfied as is well known, if and only if

$$\boldsymbol{A} = \text{laminar} = -\nabla\lambda.$$

Having thus proved that $\boldsymbol{A}$ necessarily must be laminar, the two remaining conditions (1.4) b) and c) are proved to be necessary almost immediately from the identity

$$\int_{F_f} \lambda S_n \, dF + \int_{F_d} \triangle\lambda \cdot S_n \, dF \equiv 0.$$

2. The equations of motion by axial symmetry in an incompressible and inviscid fluid.

Suppose now that we have an axis, z, of symmetry for the potential φ of the external force and that the distributions of velocity and density at a certain instant are also symmetric with respect to this axis. It follows then already from reasons of symmetry that the motion must maintain its original symmetric character at all later instants. The meridional and zonal equations of motion are[1])

$$q\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) = -\nabla p - q \nabla \varphi + q \frac{u^2}{R} \boldsymbol{R}_1$$

$$\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u + \frac{u v_R}{R} = 0.$$

Integrated along zonal circles, $R = $ const, the latter equation may be written

$$\frac{\partial c}{\partial t} + \boldsymbol{v} \cdot \nabla c = 0. \tag{2.1}$$

Here c denotes the velocity circulation along zonal circles,

$$c = \int_0^{2\pi} u R \, d\psi. \tag{2.2}$$

The equation of continuity is

$$\frac{\partial q}{\partial t} + \boldsymbol{v} \cdot \nabla q = 0. \tag{2.3}$$

Multiplying (2.1) by $2q \cdot c$ and (2.3) by c^2 and adding we obtain

$$\frac{\partial (qc^2)}{\partial t} + \boldsymbol{v} \cdot \nabla qc^2 = 0. \tag{2.4}$$

From (2.2) we get $u = \dfrac{c}{2\pi R}$. With this expression for u substituted in the last term of the meridional equation of motion, this may be written

$$q\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) + q\nabla\varphi + qc^2 \nabla\varphi_c = -\nabla p \tag{2.5}$$

where φ_c is defined by

$$\varphi_c = \frac{1}{8\pi^2 R^2}.$$

It is seen that $-qc^2\nabla\varphi_c$ is to be considered as an additional force per unit volume for the meridional motion, and that this force obeys a law, as to its local time variations, which is analogous to that for the external force.

To get the complete hydrodynamic equations we have to add the dynamic boundary conditions

(2.6) a) $p = 0$ at a free surface
b) $\triangle p = 0$ at surfaces of discontinuity,

and the condition of incompressibility, which in the axial-symmetric motion can be written

$$\nabla \cdot \boldsymbol{v} = 0. \tag{2.7}$$

[1]) Throughout this chapter $\boldsymbol{v}$ denotes the meridional velocity, and ∇ the corresponding nabla operator. As to other notations, see the beginning of Chapter II.

The meridional equation of motion written in the form (2.5) expresses that the sum of the vectors on the left-hand side has no vorticity, while the dynamic boundary conditions on the other hand express the vanishing of the gliding vorticity of the same vector quantity at surfaces of discontinuity.[1]) These equations are therefore intimately related. This will become still more clear in the next section where the equation of motion and the dynamic boundary conditions are to be derived from a single variational principle.

3. The equations of motion by axial symmetry derived from a variational principle.

To represent the meridional positions of the fluid particles we shall use the coordinates R, z. To represent on the other hand the individual zonal rings we shall use the coordinates R_0, z_0 which give the meridional positions of the particles in some constellation of the fluid. In an actual motion R, z will, in our axial-symmetric motion be functions of R_0, z_0 and of time

$$(3.1) \qquad \begin{aligned} R &= f_1(R_0, z_0, t) \\ z &= f_2(R_0, z_0, t) \end{aligned}$$

With the vector notations $\boldsymbol{r}$ and $\boldsymbol{r}_0$ defined by $\boldsymbol{r} = R\boldsymbol{R}_I + z\boldsymbol{z}_I$ and $\boldsymbol{r}_0 = R_0\boldsymbol{R}_I + z_0\boldsymbol{z}_I$, this functional dependency may be written shorter,

$$(3.1)' \qquad \boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t).$$

The equation of continuity (2.3) expresses the individual conservation of density. Density is therefore a function of R_0, z_0 but not of time:

$$q = q(R_0, z_0) = q(\boldsymbol{r}_0).$$

The zonal equation of motion written in the form (2.1) expresses the individual conservation of velocity circulation along zonal circles, and may thus be written

$$c = c(R_0, z_0) = c(\boldsymbol{r}_0).$$

We introduce the notation Φ for the total potential energy of the fluid. Φ is given by

$$\Phi = \int_{\tau_0} q(\boldsymbol{r}_0)\, \varphi(\boldsymbol{r})\, d\tau_0,$$

$d\tau_0$ denoting a volume element in terms of the Lagrangian particle variables. Further, let K_z denote the total kinetic energy of the zonal motion. K_z may be written

$$(3.2) \qquad K_z = \int_{\tau_0} q(\boldsymbol{r}_0)\, c^2(\boldsymbol{r}_0)\, \varphi_c(\boldsymbol{r})\, d\tau_0.$$

It is understood that K_z, as Φ, only depends upon the positions of the fluid particles. Suppose $H(\boldsymbol{r})$ to be given by

$$H = \int_{t_1}^{t_2} \left[\int_{\tau_0} \tfrac{1}{2}\, q(\boldsymbol{r}_0) \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)^2 d\tau_0 - \Phi - K_z \right] dt.$$

As an introduction to the problem of the stability of the stationary circular vortex we shall derive the meridional equation of motion and the dynamic boundary conditions from the variational problem

$$(3.3) \qquad \delta H = 0$$

with the side conditions (1.2) and (1.3).[1])

Let us suppose that $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t)$ represents a quite arbitrary motion of the fluid in the meridional plane between the positions $\boldsymbol{r} = \boldsymbol{r}_1$ at time t_1 and the positions $\boldsymbol{r} = \boldsymbol{r}_2$ at time t_2, and further that $\boldsymbol{f}(\boldsymbol{r}_0, t)$ is a particular of these functions for which $H(\boldsymbol{r})$ becomes stationary: Thus

$$(3.4) \qquad \delta H(\boldsymbol{r}) = 0. \quad \boldsymbol{r} = \boldsymbol{f}$$

In order now to make possible a continious transformation from any of the arbitrary functions $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t)$ to the special one $\boldsymbol{f}(\boldsymbol{r}_0, t)$, we suppose that $\boldsymbol{r}$ is a function also of a parameter ε,

$$(3.5) \qquad \boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t; \varepsilon),$$

and that

$$(3.6) \qquad \boldsymbol{r}(\boldsymbol{r}_0, t; 0) = \boldsymbol{f}.$$

[1]) This interpretation of the dynamic boundary conditions have been utilized by Høiland (1) for the discussion of the dynamics of discontinuity surfaces, in a sence which is similar to the utilization of the circulation theorem for continious fluids.

[1]) It is rather well known that the equation of motion for a compressible, inviscid fluid is identical with the condition for stationarity of the time integral of the Lagrangian function belonging to the fluid system. Less known, it appears, is that Lagrange has derived the equation of motion for the *incompressible* and inviscid fluid from the similar variation principle. In his derivation the pressure is, save for a space constant, identical with the Lagrangian undetermined multiplier function introduced into the variational problem from the side condition of incompressibility. I was myself not aware of this until recently dr. Høiland at the University of Oslo called my attention to it.

The conditions for the satisfying of (3.3) may now be found from the conditions for the identity

(3.7) $$\frac{dH}{d\varepsilon_{\varepsilon=0}} \equiv 0.$$

Applying here that the limit positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are uneffected when H is varied, we arrive at the following identity:

$$\int_{t_1}^{t_2}\int_{\tau_0}\left[q\frac{\partial^2\boldsymbol{r}}{\partial t^2}+q\nabla\varphi+qc^2\nabla\varphi_c\right]\cdot\frac{\partial\boldsymbol{r}}{\partial\varepsilon_{\varepsilon=0}}d\tau_0 dt\equiv 0.$$

A necessary and sufficient condition for the satisfying of this identity is

(3.8) $$\int_{\tau_0}\left[q\frac{\partial^2\boldsymbol{r}}{\partial t^2}+q\nabla\varphi+qc^2\nabla\varphi_c\right]\cdot\frac{\partial\boldsymbol{r}}{\partial\varepsilon}\bigg|_{\varepsilon=0}d\tau_0\equiv 0.$$

We use the notation $\boldsymbol{S}$ for the vector defined by

$$\boldsymbol{S}=\frac{\partial\boldsymbol{r}(\boldsymbol{r}_0,t;\varepsilon)}{\partial\varepsilon}.$$

Substituting here R, z instead of the variables, R_0, z_0 we get

$$\boldsymbol{S}=\boldsymbol{S}(\boldsymbol{r},t;\varepsilon).$$

$\boldsymbol{S}$ must satisfy the kinematical conditions specified in (1.2), (1.3).

Now, by substitution of R, z in (3.8) instead of R_0, z_0, this identity may be written, having $d\tau_0=d\tau$,

$$\int_{\tau}\left[q\frac{\partial\boldsymbol{v}}{\partial t}+q\boldsymbol{v}\cdot\nabla\boldsymbol{v}+q\nabla\varphi+qc^2\nabla\varphi_c\right]\cdot\boldsymbol{S}_{\varepsilon=0}d\tau\equiv 0.$$

This belongs to the class (1.1) examined in the first section. Consequently, as shown there, we must have

a) $$q\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)+q\nabla\varphi+qc^2\nabla\varphi_c=-\nabla\lambda,$$

(3.9) b) $\lambda=$ const at a free surface,

c) $\triangle\lambda=$ const at surfaces of discontinuity.

For the determination of the scalar λ we have the partial differential equation

(3.10) $$\nabla^2\lambda-\frac{\nabla q\cdot\nabla\lambda}{q}+q\nabla\cdot[\nabla\varphi+c^2\nabla\varphi_c+\boldsymbol{v}\cdot\nabla\boldsymbol{v}]=0$$

and the boundary conditions (3.9) b) and c) together with that valid at the rigid boundary:

$$-\frac{\partial\lambda}{\partial n}=[q\nabla\varphi+qc^2\nabla\varphi_c+q\boldsymbol{v}\cdot\nabla\boldsymbol{v}]\cdot\boldsymbol{n}.$$

The differential equation (3.10) is arrived at by eliminating $\frac{\partial\boldsymbol{v}}{\partial t}$, performing the scalar multiplication $\nabla\cdot$ on each of the terms in (3.9) a). Since now the laminar vector $\nabla\lambda$ is uniqely determined from this equation and the boundary conditions whatever are the constant values of λ at the free surface and of $\triangle\lambda$ at the surfaces of discontinuity, the equations (3.9) must be identical with the meridional equation of motion and the dynamic boundary conditions.

4. The stationary circular vortex considered as an extreme motion.

If boundary conditions are assumed which exclude a supply of work to the fluid from outside, we have the energy equation

$$\int_{\tau_0}\tfrac{1}{2}\boldsymbol{v}^2 q d\tau_0+\int_{\tau_0}\tfrac{1}{2}u^2 q d\tau_0=-\Phi+\text{const.}$$

With the notation K_m for the total kinetic energy of the meridional motion and the former notation K_z for the energy of the zonal motion, the energy equation may also be written

(4.1) $$K_m=-[\Phi+K_z]+\text{const.}$$

We have seen that K_z only depends upon the positions of the fluid particles, so that $\Phi+K_z$ plays the role of a potential for the meridional motion. The problem to find the conditions which must be satisfied if $\Phi+K_z$ shall assume extreme values, is a simple subcase of the more general problem solved in section 3. Thus, they turn out to be

a) $Q\nabla\varphi+QC^2\nabla\varphi_c=-\nabla\lambda$,

(4.2) b) $\lambda=$ const at a free surface,

c) $\triangle\lambda=$ const at surfaces of discontinuity.

The special distribution of density and circulation making $\Phi+K_z$ stationary are denoted by capital letters. The above necessary conditions for extreme values of $\Phi+K_z$ are fulfilled in a stationary circular vortex, pressure then replacing λ in the above equations. Vice verca, an initially circular vortex will be stationary if the distributions of density and circulation satisfy eqs. (4.2).[1] The conditions (4.2) for extremity of

[1]) To see this, we can substitute the laminar vector $-\nabla\lambda$ for $q\nabla\varphi+qc^2\nabla\varphi_c$ in (2.5). Since $\boldsymbol{v}$ is vanishing in a circular vortex, we obtain then initially

$$Q\frac{\partial\boldsymbol{v}}{\partial t}=-\nabla(p-\lambda).$$

Eliminating $\frac{\partial\boldsymbol{v}}{\partial t}$, we obtain

$$\nabla^2(p-\lambda)-\frac{\nabla(p-\lambda)\cdot\nabla Q}{Q}=0.$$

$\Phi + K_z$ are therefore also the conditions for balance in a circular vortex.

The problem to find the positions of the fluid particles furnishing $\Phi + K_z$ with extreme values, may be relatively easily solved if we know the corresponding space distributions $Q(R, z)$ and $C(R, z)$. However, so far no explicit expressions for these distributions have been found, but only certain equations which these space functions must satisfy. To arrive at the explicit solutions is a difficult problem in the general case. In two cases the problem can be solved easily. In the first case the circulation c is assumed to be zero throughout the fluid. Then $\Phi + K_z$ reduces to Φ, and eqs. (4.2) to

$$Q \nabla \varphi = -\nabla \lambda,$$
$\lambda =$ const at a free surface,
$\triangle \lambda =$ const at surfaces of discontinuity.

Hence, by elimination of λ,

$$\nabla Q \times \nabla \varphi = 0.$$

This relation together with the second and third equation above, gives the well known conditions for absolute equilibrium, viz. that the surfaces of equal density, the free surface, and the surfaces of discontinuity must coincide with surfaces of equal potential. By means of this and the equation of continuity it is relatively easy to determine from an initial arbitrary distribution of density and for given boundary conditions the function $Q(R, z)$.

In the other case no effects from the external force are supposed to exist. Then $\Phi + K_z$ reduces to K_z and eqs. (4.2) to

$$QC^2 \nabla \varphi_c = -\nabla \lambda,$$
$\lambda =$ const at a free surface,
$\triangle \lambda =$ const at surfaces of discontinuity.

Hence, by eliminating λ,

$$\nabla QC^2 \times \nabla \varphi_c = 0.$$

This equation together with the second and third equation above, shows that in the balanced circular vortex with no external force, the surfaces of equal values of QC^2, the free surface, and the surfaces of discontinuity must coincide with surfaces $\varphi_c =$ const, i. e. cylindrical surfaces, $R =$ const. The surfaces $qc^2 =$ const being substantial, the problem to find from an initially arbitrary distribution of circulation and density and for given boundary conditions the specially distributions $Q(R, z)$ and $C(R, z)$ which make $\delta K_z = 0$, is solved in the same way as in the former case.

In the general case, when two sets of substantial surfaces, $q(R, z) =$ const and $c(R, z) =$ const are given initially, eqs. (4.2) do not immediately provide us with the necessary means to determine the distributions of density and circulation making $\Phi + K_z$ stationary.

5. The value of $\Phi + K_z$ in the neighbourhood of the extreme values.

Let $\boldsymbol{r}'$ denote the meridional positions of the fluid particles for which $\Phi + K_z$ assumes extreme values, and $d\boldsymbol{r}$ the displacements $\boldsymbol{r} - \boldsymbol{r}'$. In (3.5), (3.6) $\boldsymbol{r}'$ now takes the place of $\boldsymbol{f}$, and t drops out as variable. According to these equations we can write

$$(5.1) \qquad d\boldsymbol{r} = \varepsilon \frac{\partial \boldsymbol{r}}{\partial \varepsilon}_{\varepsilon=0} + \frac{\varepsilon^2}{2} \frac{\partial^2 \boldsymbol{r}}{\partial \varepsilon^2}_{\varepsilon=0} + \cdots$$

Let us denote $[\Phi(\boldsymbol{r}) + K_z(\boldsymbol{r})] - [\Phi(\boldsymbol{r}') + K_z(\boldsymbol{r}')]$ by $\triangle[\Phi + K_z]$. Developing, we obtain

$$(5.2) \quad \triangle[\Phi + K_z] = \int_{\tau_0} [q \nabla \varphi + qc^2 \nabla \varphi_c] \cdot d\boldsymbol{r} d\tau_0$$
$$+ \tfrac{1}{2} \int_{\tau_0} [q d\boldsymbol{r} \cdot \nabla\nabla \varphi \cdot d\boldsymbol{r} + qc^2 d\boldsymbol{r} \cdot \nabla\nabla \varphi_c \cdot d\boldsymbol{r}] \, d\tau_0,$$

neglecting terms of higher order than the second in $d\boldsymbol{r}$. Substituting here for $d\boldsymbol{r}$ from (5.1) and applying the conditions for balance, and well known transformation theorems we arrive at the following expression for $\triangle[\Phi + K_z]$:

With the boundary conditions

$\frac{\partial (p - \lambda)}{\partial n} = 0$ at a rigid surface,

$p - \lambda \quad =$ const at a free surface,

$\triangle(p - \lambda) =$ const at surfaces of discontinuity

this equation has as the only solution,

$p - \lambda \quad =$ const.

So, $\frac{\partial v}{\partial t}$ is initially zero, and this will be the case also for all higher time derivatives.

$$\triangle[\Phi+K_z]=-\tfrac{1}{2}\int_\tau d\boldsymbol{r}\cdot[\nabla Q\nabla\varphi+\nabla QC^2\nabla\varphi_c]\cdot d\boldsymbol{r}d\tau$$
$$(5.3)\qquad +\tfrac{1}{2}\int_{F_f}\varrho_n^2 Q\left[\frac{d\varphi}{dn}+C^2\frac{d\varphi_c}{dn}\right]dF$$
$$-\tfrac{1}{2}\int_{F_d}\varrho_n^2\left[\triangle Q\cdot\frac{d\varphi}{dn}+\triangle QC^2\cdot\frac{d\varphi_c}{dn}\right]dF,$$

ϱ_n denoting the value of $d\boldsymbol{r}$ normal to the boundaries.

6. The criterion of stability for vortex ring perturbations of a balanced circular vortex.

From eq. (5.3) we see that if

a) $d\boldsymbol{r}\cdot[\nabla Q\nabla\varphi+\nabla QC^2\nabla\varphi_c]\cdot d\boldsymbol{r}<0$ for all $d\boldsymbol{r}$,

(6.1) b) $Q\dfrac{d\varphi}{dn}+QC^2\dfrac{d\varphi_c}{dn}\geqq 0$ at a free surface,

c) $\triangle Q\cdot\dfrac{d\varphi}{dn}+\triangle QC^2\cdot\dfrac{d\varphi_c}{dn}\leqq 0$ at surfaces of discontinuity,

then $\triangle[\Phi+K_z]$ will have a positive definite sign. In this case $[\Phi+K_z]_{r=r'}$ will represent a minimum $[\Phi+K_z]_{\min}$ of $\Phi+K_z$. In complete accordance with the well known proof for the stability of a mechanical system charachterized by a minimum of its potential energy we may now equally well prove that a stationary circular vortex is stable for vortex ring perturbations if the vortex is characterized by a minimum of $\Phi+K_z$. In Chapter II some stability criteria for linear flows will be derived which are based on a similar principle. So, it will prove useful to demonstrate shortly the proof. In doing so we shall give it in such a way that it will be valid also in the case in which $[\Phi+K_z]_{\min}$ is only one of several existing minima, and not the smallest one.

Expressing the constant in the energy equation (4.1) by corresponding values of velocity and positions, $\boldsymbol{v}_1$ and $\boldsymbol{r}_1$, the energy equation becomes

$$(6.2)\quad K_m=-[\Phi+K_z]+\int_{\tau_0}\tfrac{1}{2}\boldsymbol{v}_1^2 q d\tau_0+[\Phi+K_z]_{r=r_1}$$

or, by adding and subtracting $[\Phi+K_z]_{\min}$

$$(6.2')\quad K_m=-\triangle[\Phi+K_z]+\int_{\tau_0}\tfrac{1}{2}\boldsymbol{v}_1^2 q d\tau_0+[\Phi+K_z]_{r=r_1}-[\Phi+K_z]_{\min}.$$

Suppose now $\boldsymbol{r}_1$ to represent positions of the fluid particles for which $[\Phi+K_z]_{r=r_1}\geqq[\Phi+K_z]_{\min}$. (If $[\Phi+K_z]_{\min}$ is an absolute minimum of $\Phi+K_z$ this relation is evidently always satisfied). Further, let us suppose that the particles are in the neighbourhood of the considered minimum positions so that $\Phi+K_z$ has to pass at least a value $N>[\Phi+K_z]_{\min}$ before the fluid can approach one of the contingent other lower minima. Since now K_m is positive definite we conclude from (6.2) that

$$\int_{\tau_0}\tfrac{1}{2}\boldsymbol{v}_1^2 q d\tau_0+[\Phi+K_z]_{r=r_1}$$

is the greatest value which $\Phi+K_z$ can assume. Suppose now that the velocities $\boldsymbol{v}_1$ have been chosen so small and the positions $\boldsymbol{r}_1$ so near to the minimum positions that the above sum is below N. Under these conditions the considered minimum is certainly the smallest value which $\Phi+K_z$ can assume. The following relation exists,

$$\Phi+K_z\leqq\int_{\tau_0}\tfrac{1}{2}\boldsymbol{v}_1^2 q d\tau_0+[\Phi+K_z]_{r=r_1}.$$

By subtracting $[\Phi+K_z]_{\min}$ at both sides, we arrive at

$$(6.3)\quad\triangle[\Phi+K_z]\leqq\int_{\tau_0}\tfrac{1}{2}\boldsymbol{v}_1^2 q d\tau_0+[\Phi+K_z]_{r=r_1}-[\Phi+K_z]_{\min}.$$

Substituting here from (5.3), putting $d\boldsymbol{r}$ equal to $\varrho\boldsymbol{l}_I$, where $\boldsymbol{l}_I$ is an unit vector, we obtain

$$(6.4)\left\{\begin{aligned}&-\tfrac{1}{2}\int_\tau\boldsymbol{l}_I\cdot(\nabla Q\nabla\varphi+\nabla QC^2\nabla\varphi_c]\cdot\boldsymbol{l}_I\varrho^2 d\tau\\&\quad+\tfrac{1}{2}\int_{F_f}Q\left[\frac{d\varphi}{dn}+C^2\frac{d\varphi_c}{dn}\right]\varrho_n^2\,dF\\&-\tfrac{1}{2}\int_{F_d}\left[\triangle Q\cdot\frac{d\varphi}{dn}+\triangle QC^2\cdot\frac{d\varphi_c}{dn}\right]\varrho_n^2\,dF\\&\quad\leqq\int_{\tau_0}\tfrac{1}{2}\boldsymbol{v}_1^2 q d\tau_0+[\Phi+K_z]_{r=r_1}-[\Phi+K_z]_{\min}.\end{aligned}\right.$$

Here, the functions before ϱ^2 and ϱ_n^2 are positive, or not all of them are zero, according to the relations (6.1). So, we are able to draw the following conclusion as an expression for the stability in the present case of the stationary circular vortex for vortex ring perturbations:

When

$\boldsymbol{v}_1\to 0$, $\boldsymbol{r}_1\to\boldsymbol{r}'$ (the minimum positions),

then at all times ϱ^2 in the fluid and ϱ_n^2 at a free surface or at surfaces of discontinuity must tend to zero.

Condition (6.4) being an integral condition, exceptions may of course exist for certain parts of the fluid, the free surface, and the surfaces of discontinuity, where the displacements do not tend to zero. However, these parts must diminish to zero simultaneously with $\boldsymbol{v}_1$ and $\boldsymbol{r}_1 - \boldsymbol{r}'$, and so these exceptions cannot change the stable character of the stationary circular vortex characterized by eqs. (6.1).

Having found above that any minimum value of $\Phi + K_z$ under suitable initial conditions will represent the smallest available value of $\Phi + K_z$, it follows from the energy equation that

$$(6.5) \qquad \triangle \underset{\max}{K_m} = [\Phi + K_z]_{r=r_1} - [\Phi + K_z]_{\min}$$

determines the greatest possible increase in the kinetic energy of the meridional flow. In the second place therefore, the stability of the balanced circular vortex implies that the increase in kinetic energy of the meridional flow can be made arbitrarily small at any time by assuming the initial velocities sufficiently small, and the initial positions sufficiently near to the positions in the balanced vortex.

7. The conditions for instability of the stationary circular vortex by axial-symmetric perturbations.

Reversing the unequal signs in (6.1) we obtain

$$(7.1) \quad \begin{cases} \text{a)} \; d\boldsymbol{r} \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] \cdot d\boldsymbol{r} > 0 \text{ for all } d\boldsymbol{r}, \\ \text{b)} \; \frac{d\varphi}{dn} + C^2 \frac{d\varphi_c}{dn} \leqq 0 \text{ at a free surface,} \\ \text{c)} \; \triangle Q \cdot \frac{d\varphi}{dn} + \triangle Q C^2 \cdot \frac{d\varphi_c}{dn} \geqq 0 \text{ at surfaces of discontinuity.} \end{cases}$$

In consequence of these relations, $\triangle [\Phi + K_z]$ is given a negative sign in eq. (5.3) for all displacements $d\boldsymbol{r}$, sufficiently small. Therefore, under the conditions (7.1), $\Phi + K_z$ assumes a maximum value, $[\Phi + K_z]_{\max}$. The greatest possible increase in kinetic energy of the meridional flow is given by $\triangle \underset{\max}{K_m} = [\Phi + K_z]_{r=r_1} - [\Phi = K_z]_{\text{Min}}$, where $[\Phi + K_z]_{\text{Min}}$ now is the value of the lowest minimum. This increase is the greater the nearer are the initial positions to the positions for which $\Phi + K_z$ has the maximum value. In this respect therefore, the conditions are quite different from those in the stable stationary circular vortex. On the other hand, however, the energy equation alone does not exclude the possibility for a stationary circular vortex to behave also in the present case as a stable vortex, since nothing prevents this equation to be fulfilled also for motions taking place arbitrarily near to the equilibrium positions. It is however easy to show that a stationary circular vortex characterized by a maximum of $\Phi + K_z$ cannot be stable in the sence defined in the preceding section. By derivating $\int_{\tau_0} \frac{1}{2}\boldsymbol{v}^2 q d\tau_0$ twice with respect to time we obtain for small motion near the motion in the balanced circular vortex, neglecting terms small of a still higher order:

$$(7.2) \quad \begin{cases} \frac{d^2 K_m}{dt^2} = \int_\tau \boldsymbol{v} \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] \cdot \boldsymbol{v} d\tau \\ \qquad - \int_{F_f} v_n{}^2 \left[\frac{d\varphi}{dn} + C^2 \frac{d\varphi_c}{dn}\right] Q dF \\ + \int_{F_d} v_n{}^2 \left[\triangle Q \cdot \frac{d\varphi}{dn} + \triangle Q C^2 \cdot \frac{d\varphi_c}{dn}\right] \\ \qquad + \int_\tau \left(\frac{d\boldsymbol{v}}{dt}\right)^2 Q d\tau. \end{cases}$$

So, the conditions (7.1) for maximum of $\Phi + K_z$ are also sufficient conditions for positive values of $\frac{d^2 K_m}{dt^2}$. The increase in kinetic energy of the meridional flow will therefore in general exceed any (small) limit. The only exception from this rule will present itself for a singular motion in which the kinetic energy, K_m, decreases to zero simultaneously as the fluid particles approach asymptotically the positions in the balanced vortex.

Having now shown that it is in general impossible to obtain arbitrarily small variations in the kinetic energy of the meridional flow, it follows immediately from the energy equation, written,

$$\triangle K_m = [\Phi + K_z] - [\Phi + K_z]_{r=r_1},$$

that it will in general also be impossible for the fluid particles to remain in the neighbourhood of their positions in the balanced circular vortex. And therefore, both with respect to the variations in kinetic energy of the meridional flow and with respect to the variations in the positions of the fluid particles, the balanced circular vortex will in the present case when it is perturbed by axial-symmetric perturbations behave diametrically different to the stable vortex. Taking this as definition of instability, the conditions (7.1) will represent the necessary and sufficient conditions for instability by vortex ring perturbations of a stationary circular vortex.

8. The stationary circular vortex indifferent for vortex ring perturbations.

If

$$\Phi + K_z = [\Phi + K_z]_{r=r'}$$

for all $\boldsymbol{r}$ in a neighbourhood of the equilibrium positions, $\boldsymbol{r}'$, the balanced vortex reacts indifferent for vortex ring perturbations. The conditions for indifference are found from the relations (6.1) by applying there equal signs instead of the unequal ones. In the indifferent vortex we obtain then from the energy equation that

$$\triangle K_m = 0. \tag{8.1}$$

By indifference therefore, the fluid particles may move finite distances from the positions in the balanced vortex without any change in the kinetic energy of the meridional flow. So, the vortex in this case behaves in one respect as an unstable vortex, and in the other as a stable one.

9. The criteria for kinematically conditioned instability by vortex ring perturbations of a stationary circular vortex.

This case will be defined in the way that the sign of $\triangle[\Phi + K_z]$ may be positive as well as negative depending upon the nature of the displacements $d\boldsymbol{r}$. Such conditions will exist if the fluid contains stable as well as unstable regions. For as a result of such conditions, $\triangle[\Phi + K_z] > 0$ or < 0 according as the displacements are concentrated to a sufficiently high degree in the stable or unstable regions, respectively. In the first case the vortex will behave stably. In the other it will behave unstably, "potential" energy then being transformed to kinetic energy of the meridional flow.

Similar conditions will also exist if

$$\triangle[\Phi + K_z] > 0 \text{ or } < 0$$

depending upon the directions of the displacements. The expression (5.3) for $\triangle[\Phi + K_z]$ is a quadratic expression in ϱ_R, ϱ_z (the components of $d\boldsymbol{r}$ along $\boldsymbol{R}_I$ and $\boldsymbol{z}_I$). The coefficient of the product term $\varrho_R \cdot \varrho_z$ may be brought to vanish by turning the coordinate system a suitable angle α which generally will be a function of R and z. Let η, ζ denote the coordinates along the lines forming the angles α and $\alpha + \frac{\pi}{2}$ with for instance the z-axis, and ϱ_η, ϱ_ζ the corresponding components of $d\boldsymbol{r}$. Eq. (5.3) may then be written

$$\begin{aligned} \triangle[\Phi + K_z] &= \int_\tau [a\,\varrho_\eta^2 + b\varrho_\zeta^2]\, Q d\tau \\ &+ \int_{F_f} \varrho_n^2 \left[\frac{d\varphi}{dn} + C^2 \frac{d\varphi_c}{dn}\right] Q\, dF \\ &- \int_{F_d} \varrho_n^2 \left[\triangle Q \cdot \frac{d\varphi}{dn} + \triangle Q C^2 \cdot \frac{d\varphi_c}{dn}\right] dF. \end{aligned} \tag{9.1}$$

The conditions supposed to exist now require that the quantities a, b must at some localities be of opposite signs. Suppose that a is the one which is negative. Then the balanced circular vortex will behave unstably or stably according as the displacements are directed mainly in the directions η or in the directions ζ. These directions will therefore be denoted the unstable and stable directions, respectively.

In the two cases studied above or in cases where both these conditions are present, the vortex will therefore be unstable for vortex ring perturbations, only if certain kinematical conditions are fulfilled. It will now always be possible to *start* with such a kinematics that initially "potential" energy is transformed to kinetic energy of the meridional flow. To secure the instability however, we must be sure that this kinematics can be maintened for a sufficiently long time, thus preventing a stable kinematics to be established within arbitrarily small time. So, the problem to prove the instability in the present cases reduces to the problem to find a system of perturbations which maintains for a sufficiently long time a kinematics for which the kinetic energy of the meridional flow increases. This will now in particular be the case for a system of perturbations having an unvariable system of streamlines satisfying the kinematical conditions for instability. Such a system must have accelerations given by $d\ddot{\boldsymbol{r}} = -\nu_n^2 d\boldsymbol{r}$ where ν_n^2 is constant. The solution of this equation is given by

$$(9.2) \qquad d\boldsymbol{r} = \boldsymbol{a}_n(R, z) \cos \nu_n (t + \alpha_n)$$

and represents in the stable case a simple oscillation with the frequency ν_n, in the unstable case a flight away from the equilibrium positions with the imaginary frequency ν_n, or the frequency of flight $-i\nu_n$.

10. The equations by axial symmetry for small motion near the motion in the balanced circular vortex.

In section 3 we have derived the meridional equation of motion by axial-symmetry and the corresponding boundary conditions by varying the integral

$$\int_{t_1}^{t_2} \left[\int_{\tau_0} q \tfrac{1}{2} \left(\frac{\partial \boldsymbol{r}}{\partial t} \right)^2 d\tau_0 - \Phi - K_z \right] dt.$$

Here we can write $d\boldsymbol{r}$ instead of $\boldsymbol{r}$, and $\triangle[\Phi + K_z]$ instead of $\Phi + K_z$. In doing so we obtain the variational problem

$$(10.1) \quad \delta \int_{t_1}^{t_2} \left[\int_{\tau_0} q \tfrac{1}{2} \left(\frac{\partial d\boldsymbol{r}}{\partial t} \right)^2 d\tau_0 - \triangle[\Phi + K_z] \right] dt = 0.$$

with the kinematical conditions for $d\boldsymbol{r}$ given in eqs. (1.2), (1.3). Substituting for $\triangle[\Phi + K_z]$ from (5.3) we obtain as sufficient and necessary conditions for the stationarity of the above integral,

$$\text{a)} \quad Q \frac{\partial^2 d\boldsymbol{r}}{\partial t^2} - d\boldsymbol{r} \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] = -\nabla \lambda,$$

(10.2) b) $Q d\boldsymbol{r} \cdot [\nabla \varphi + C^2 \nabla \varphi_c] - \lambda = \text{const}$ at a free surface,

c) $d\boldsymbol{r} \cdot [\triangle Q \cdot \nabla \varphi + \triangle Q C^2 \cdot \nabla \varphi_c] - \triangle \lambda = \text{const}$ at surfaces of discontinuity.

These equations represent for small motions near the motion in the balanced circular vortex: a) the meridional equation of motion b) and c) the dynamic boundary conditions at a free surface and at surfaces of discontinuity.

We shall now postulate the existence of an infinite number of solutions of the form (9.2) by means of which it shall be possible to find by superposition, the quite general solution of (10.2)

$$(10.3) \quad d\boldsymbol{r} = \sum_{n=1}^{\infty} \boldsymbol{a}_n(R, z) \cos \nu_n (t + \alpha_n).$$

Physically this postulate seems reasonable since the equations (10.2) developed by varying an integral, whose integrand is a quadratic function of the impulses $\frac{\partial d\boldsymbol{r}}{\partial t}$ and the displacements $d\boldsymbol{r}$. Thus, the present conditions are the counterpart for a continuum, to the conditions characterizing a system of material points moving near the positions of extremum of its potential energy. And as is well known, the quite general motion of this latter system may be described completely by superposition of a finite number of "eigen"-solutions with a trigonometric time dependency.

Let us suppose that $\boldsymbol{a}_m \cos \nu_m (t + \alpha_m)$ and $\boldsymbol{a}_n \cos \nu_n (t + \alpha_n)$ represent two different "eigen"-solutions of (10.2). Substituting these in (10.2)a) we obtain

$$-Q\nu_m^2 \boldsymbol{a}_m + \boldsymbol{a}_m \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] = -\nabla \lambda_m$$

and

$$-Q\nu_n^2 \boldsymbol{a}_n + \boldsymbol{a}_n \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] = -\nabla \lambda_n.$$

By scalar multiplication of the first equation by $\boldsymbol{a}_n$ and of the other by $\boldsymbol{a}_m$, followed by an integration over the fluid, we obtain by subtraction

$$(10.4) \begin{cases} -(\nu_m^2 - \nu_n^2) \int_\tau Q \boldsymbol{a}_m \cdot \boldsymbol{a}_n d\tau + \int_\tau \boldsymbol{a}_m \cdot [\nabla Q \nabla \varphi \\ \qquad + \nabla Q C^2 \nabla \varphi_c] \cdot \boldsymbol{a}_n d\tau \\ -\int_\tau \boldsymbol{a}_n \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] \cdot \boldsymbol{a}_m d\tau = \\ \qquad -\int_\tau \nabla \lambda_m \cdot \boldsymbol{a}_n d\tau + \int_\tau \nabla \lambda_n \cdot \boldsymbol{a}_m d\tau. \end{cases}$$

From the condition of balance (4.2)a) we obtain by eliminating $\nabla \lambda$

$$(10.5) \qquad \nabla Q \times \nabla \varphi + \nabla Q C^2 \times \nabla \varphi_c = 0.$$

This is the condition also for symmetry of the tensor $\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c$. Owing to this symmetry an interchange of $\boldsymbol{a}_m$ and $\boldsymbol{a}_n$ in for instance the second left-hand side integral in (10.4) will not effect its value. Therefore, the two last left-hand side integrals in (10.4) cancel out. Further, evaluating the right-hand side integrals by means of the theorem of Gauss we obtain

$$-\int_\tau \nabla \lambda_m \cdot \boldsymbol{a}_n d\tau + \int_\sigma \nabla \lambda_n \cdot \boldsymbol{a}_m d\tau =$$
$$= \int_{F_f} [\lambda_m \boldsymbol{a}_n - \lambda_n \boldsymbol{a}_m] \cdot d\boldsymbol{F} - \int_{F_d} [\triangle \lambda_m \boldsymbol{a}_n - \triangle \lambda_n \boldsymbol{a}_m] \cdot d\boldsymbol{F}.$$

Substituting here from the dynamic boundary conditions (10.2) b) and c) these integrals are seen to cancel out, so that we obtain from (10.4)

$$-(\nu^2_m - \nu^2_n) \int_\tau Q \boldsymbol{a}_m \cdot \boldsymbol{a}_n \, d\tau = 0.$$

It follows from this that

$$(10.6) \qquad \int_\tau Q \boldsymbol{a}_m \cdot \boldsymbol{a}_n d\tau = 0, \quad m \neq n.$$

Suppose now that $d\boldsymbol{r}$ represent at time $t = 0$ a quite general system of meridional displacements. According to the postulate (10.3) we may then write

$$(10.7) \qquad d\boldsymbol{r} = \sum_{n=1}^{\infty} c_n \boldsymbol{a}_n,$$

where $c_n = \cos \nu_n \alpha_n$.

The corresponding equation of motion (10.2) at time $t = 0$ may be written

$$-\sum_{n=1}^{\infty} Q \nu_n^2 c_n \boldsymbol{a}_n = d\boldsymbol{r} \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] - \nabla \lambda.$$

Multiplying scalarly by $d\boldsymbol{r}$ followed by an integration over τ, we obtain in consequence of (10.7), (10.6), (10.2) b) and c), and (5.3):

$$(10.8) \qquad -\sum_{n=1}^{\infty} \nu_n^2 \int_\tau Q \, c^2_n \boldsymbol{a}^2_n \, d\tau = -2 \triangle (\Phi + K_z).$$

If now the conditions are those considered in the preceding section, according to which $\triangle [\Phi + K_z]$ could be positive as well as negative depending upon the nature of the displacements $d\boldsymbol{r}$, then in consequence of the last equation "eigen"-solutions must exist with positive as well as negative values of ν_n^2. A stationary, circular vortex with stable as well as unstable regions or directions, is therefore unstable for general vortex ring perturbations.

11. The explicit stability criterion in a particular case under terrestrial conditions.

In section 9 we defined quantities, a, b by means of eq. (9.1). It can be shown that these quantities must satisfy

$$(11.1) \qquad a \cdot b = \nabla Q \times \nabla Q C^2 \cdot \nabla \varphi \times \varphi_c.$$

The criterion of instability which depends upon the directions of the displacements is therefore

$$(11.2) \qquad \nabla Q \times \nabla Q C^2 \cdot \nabla \varphi \times \nabla \varphi_c < 0.$$

Utilizing the condition (10.5) for balance, this criterion can under terrestrial conditions be written approximately

$$(11.3) \qquad \left(\frac{dU}{dz}\right)^2 > -\frac{2\Omega_z - \dfrac{dU}{dy}}{2\Omega_z} \frac{g}{Q} \frac{\partial Q}{\partial z},$$

U denoting the relative zonal velocity.

By replacing $\frac{1}{Q}\frac{\partial Q}{\partial z}$ by $-\frac{1}{\Theta}\frac{\partial \Theta}{\partial z}$ we get the corresponding criterion for the adiabatic atmosphere.

12. On the meridional displacements that, by instability, correspond to extreme values of the acceleration.

In section 10 the existence of a complete system of "eigen"-solutions were postulated. A question which presents itself is whether these simple solutions have an other importance than this pure mathematical one. For the unstable solutions at least this seams to be the case. The reason for this is that the system of displacements which corresponds to solutions with a trigonometric time dependency are the solutions of the following isoperimetrical problem: For a constant value of $\int_\tau Q \frac{1}{2} (d\boldsymbol{r})^2 d\tau$ we shall find the system of displacements giving extreme values, or less restrictively, stationary values to $\triangle [\Phi + K_z]$. The conditions which must then be satisfied are found from the variational problem

$$(12.1) \qquad \delta \left[\nu^2 \int_\tau Q \tfrac{1}{2} (d\boldsymbol{r})^2 d\tau - \triangle (\Phi + K_z)\right] = 0$$

with the side conditions (1.2), (1.3), and turn out to be

a) $-Q\nu^2 d\boldsymbol{r} - d\boldsymbol{r} \cdot [\nabla Q \nabla \varphi + \nabla Q C^2 \nabla \varphi_c] = -\nabla \lambda$,
b) $Q d\boldsymbol{r} \cdot [\nabla \varphi + C^2 \nabla \varphi_c] - \lambda = \text{const}$ at a free surface.
c) $d\boldsymbol{r} \cdot [\triangle Q \cdot \nabla \varphi + \triangle Q C^2 \cdot \nabla \varphi_c] - \triangle \lambda = \text{const}$ at surfaces of discontinuity.

This is the same system of equations which will obtain if we substitute $d\boldsymbol{r} = \boldsymbol{a}_n (R, z) \cos \nu (t + \alpha)$ in the equations (10.2). Particularly, therefore, if there exists under unstable conditions a system of displacements which for a constant value of $\int_\tau Q \frac{1}{2} (d\boldsymbol{r})^2 d\tau$ accelerates the motion in the meridional planes more than all other systems do, this system must generate a motion which represents an "eigen"-solution.

13. A theorem on the problem of re-establishment of a new equilibrium by perturbation of an unstable circular vortex.

If the perturbed balanced vortex is in an unstable state for vortex ring perturbations, there will be a tendency for the fluid particles to move towards the positions of a lower extremum of $\Phi + K_z$. Suppose now that Q, C represent the distributions of density and circulation in the original equilibrium and Q^*, C^* the distributions in one of the other equilibriums which can be established from mere axial-symmetric displacements of the fluid particles. In these new equilibrium states the quantities a, b defined from (9.1) assume values which we denote by a^*, b^*. These quantities satisfy (11.1), so that besides

$$a \cdot b = \nabla Q \times \nabla QC^2 \cdot \nabla \varphi \times \nabla \varphi_c$$

we have in the new equilibrium

$$a^* \cdot b^* = \nabla Q^* \times \nabla Q^* C^{*2} \cdot \nabla \varphi \times \nabla \varphi_c .$$

The iso-surfaces $q = \text{const}$, $c = \text{const}$ being individual surfaces, it follows immediately from reasons of continuity that the direction of the vector product $\nabla q \times \nabla qc^2$ cannot change during the displacements of the fluid particles. So, the direction of $\nabla Q^* \times \nabla Q^* C^{*2}$ will be equal to the direction of $\nabla Q \times \nabla QC^2$, and accordingly since $\nabla \varphi \times \nabla \varphi_c$ is a vector unvariable with time, the sign of $a^* \cdot b^*$ must be equal to the sign of $a \cdot b$. This result may be thus stated: By mere displacements of the fluid particles it is not possible to arrive from an unstable state characterized by $a \cdot b < 0$ to a stable state in which $\Phi + K_z$ has a minimum value, but only to new unstable equilibriums.

If a statically stable atmosphere is unstable according to the criterion (11.3), the unstable directions will, as shown by *H. Solberg* (2), be approximately along the isentropic surfaces, provided these are quasihorizontal. By the breaking up of this atmosphere, the above theorem implies that there will be a tendency to establish new unstable conditions, with approximately vertical unstable directions.

CHAPTER II.

ON THE STABILITY OF THE CIRCULAR VORTEX FOR NOT AXIAL-SYMMETRIC PERTURBATIONS.

14. General concepts and definitions.

In this chapter we shall deal with a motion of an incompressible, homogeneous, and inviscid fluid supposed to be enclosed within fixed boundaries symmetrical with respect to an axis z. Supposing further that there is no effects from external forces the hydrodynamic equations are

$$\frac{D\boldsymbol{v}}{dt} = -\nabla p \tag{14.1}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{14.2}$$

having introduced nondimensional quantitaties and put density epual to 1. We shall in the following not be interested in finding exact solutions of these equations, but shall derive by means of some integral theorems sufficient criteria for the stability of certain simplified fluid motions.

With R, ψ, z as cylindrical coordinates, and v_R, u, v_z as the corresponding velocity components, the velocity may be written $\boldsymbol{v} = v_R \boldsymbol{R}_I + u\boldsymbol{i} + v_z \boldsymbol{z}_I$ or, with the notation $\boldsymbol{v}_m$ for $v_R \boldsymbol{R}_I + v_z \boldsymbol{z}_I$: $\boldsymbol{v} = \boldsymbol{v}_m + u\boldsymbol{i}$.[1]) We define average values of the velocity components according to the following equations:

$$\text{a)} \quad \bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u \, d\psi$$

$$\text{(14.3)} \quad \text{b)} \quad \bar{v}_R = \frac{1}{2\pi}\int_0^{2\pi} v_R \, d\psi$$

$$\text{c)} \quad \bar{v}_z = \frac{1}{2\pi}\int_0^{2\pi} v_z \, d\psi .$$

For convenience we adopt the notation $\bar{\boldsymbol{v}}_m$ for $\bar{v}_R \boldsymbol{R}_I + \bar{v}_z \boldsymbol{z}_I$. The irregular velocities $\boldsymbol{v}'$ are defined by

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1) It should be remembered that $\boldsymbol{v}$ in this chapter denotes the general velocity and not the meridional one as in the preceding chapter.

(14.4) $$\boldsymbol{v} = \bar{u}\boldsymbol{i} + \bar{\boldsymbol{v}}_m + \boldsymbol{v}'.$$

$\bar{u}$, $\bar{v}_R$, and $\bar{v}_z$ are independent of the ψ-coordinate so that an arbitrary velocity field of a fluid enclosed within the chosen boundaries may be considered as being composed of a pure rotating velocity field, $\bar{u}\boldsymbol{i}$, with z as the axis of rotation, a meridional axial-symmetric field, $\bar{\boldsymbol{v}}_m$, and one irregular field $\boldsymbol{v}'$. The additive velocity components must satisfy

(14.5) $$\int_0^{2\pi} u'd\psi = \int_0^{2\pi} v_R'd\psi = \int_0^{2\pi} v_z'd\psi = 0.$$

The integral principles which form the base of the developments in this chapter are those of conservation of total kinetic energy and total angular momentum, both certainly being true under the assumptions made for the considered fluid. With τ denoting the total volume of the fluid, these principles are

$$\int_\tau \tfrac{1}{2}\,\boldsymbol{v}^2\,d\tau = \text{const},$$

and

$$\int_\tau u\,R\,d\tau = \text{const}.$$

Substituting $\boldsymbol{v} = \bar{u}\boldsymbol{i} + \bar{\boldsymbol{v}}_m + \boldsymbol{v}'$ in the former and $u = \bar{u} + u'$ in the latter of these equations we obtain, using (14.5)

(14.6) $$\int_\tau \tfrac{1}{2}\boldsymbol{v}'^2 d\tau + \int_\tau \tfrac{1}{2}\,\bar{\boldsymbol{v}}_m^{\,2}\,d\tau = -\int_\tau \tfrac{1}{2}\bar{u}^2\,d\tau + \text{const},$$

(14.7) $$\text{const} = \int_\tau \bar{u}\,R\,d\tau.$$

In the form (14.6) the energy equation allows for the interpretation that the mean "zonal" flow is the only source for the kinetic energy of the irregular flow and the axial-symmetric mean meridional flow.

Now let c denote the velocity circulation along an arbitrary circle symmetrical with respect to the z-axis. Hence, if R is the radius of this circle,

$$c = \int_0^{2\pi} u\,R\,d\psi.$$

So, according to the definition of $\bar{u}$.

$$\bar{u} = \frac{c}{2\pi R}.$$

Substituting this into eqs. (14.6), (14.7) we obtain

(14.8) $$\int_\tau \tfrac{1}{2}\boldsymbol{v}'^2\,d\tau + \int_\tau \tfrac{1}{2}\bar{\boldsymbol{v}}_m^{\,2}\,d\tau = -\int_\tau \frac{c^2\,d\tau}{8\pi^2R^2} + \text{const},$$

(14.9) $$\int_\tau c\,d\tau = \text{const}.$$

Suppose c_0 to denote the velocity circulation when $t = 0$. The constant in the last equation may then be written $\int_\tau c_0\,d\tau$, and the equation itself

(14.9') $$\int_\tau (c - c_0)\,d\tau = 0.$$

Let us suppose that the velocity distribution initially is known and given by

$$\boldsymbol{v}_{t=0} = \boldsymbol{v}_0\,(R, \psi, z)\,,$$

and let $d\boldsymbol{r}$ denote the displacements up to time t of the fluid particles

$$d\boldsymbol{r} = \boldsymbol{r}\,(\boldsymbol{r}_0\,, t) - \boldsymbol{r}_0$$

where $\boldsymbol{r}_0$ is defined by $\boldsymbol{r}_0 = \boldsymbol{r}_{t=0}$. We consider now, at time t, a circle symmetrical with respect to the z-axis. The particles forming this circle will at time $t = 0$ form a certain closed curve which we denote by L. In consequence of the theorem of conservation of velocity circulation along physical curves, the velocity circulation c along the circle at time t must be equal to the velocity circulation along L at time $t = 0$. This may be written

(14.10) $$c = \oint_L \boldsymbol{v}_0 \cdot \delta\boldsymbol{r}.$$

L is completely determined when the displacements $d\boldsymbol{r}$ and the position of the chosen circle are given. We may therefore write

$$c = c\,(d\boldsymbol{r}; \boldsymbol{v}_0; R, z).$$

Substituting this into the right-hand integrals of eqs. (14.8), (14.9), the parameters R, z drop out under the integration over τ. These integrals are therefore, when the velocity field is given initially, only depending upon the displacements $d\boldsymbol{r}$.

In the following the main problem is to use eqs. (14.8), (14.9) to find conditions for a not arbitrarily small increase in the kinetic energy expressed by the terms on the left-hand side of eq. (14.8). The notation "arbitrarily small" will be explained in the later developments. In the case of complete axial symmetry this problem has been considered already in Chapter I. By axial symmetry $\boldsymbol{v}' = 0$, $\bar{\boldsymbol{v}}_m = \boldsymbol{v}_m$, so that in this case eq. (14.8) reduces to $\int_\tau \tfrac{1}{2}\boldsymbol{v}^2_m\,d\tau = -\int_\tau \frac{c^2 d\tau}{8\pi^2R^2} + \text{const}$, which is a special case of the general equation (4.1). In the following we shall only examine cases where we can at the outset exclude the existence of axial-symmetric meridional velocities, giving thus the conditions

$$(14.11) \qquad \bar{v}_R = \bar{v}_z = 0.$$

Under this condition the energy equation (14.8) reduces to

$$(14.12) \quad \int_{\tau} \tfrac{1}{2}\, \boldsymbol{v}'^2 \, d\tau = -\int_{\tau} \frac{c^2 d\tau}{8\pi^2 R^2} + \text{const.}$$

We now proceed to develop explicit expressions for c in the depency upon the displacements and the initial velocity field. By substituting $\boldsymbol{v}_0 = \bar{u}_0 \boldsymbol{i} + \boldsymbol{v}_0'$ in (**14.10**) we obtain

$$(14.13) \qquad c = \oint_L \bar{u}_0 \boldsymbol{i} \cdot \delta \boldsymbol{r} + \oint_L \boldsymbol{v}_0' \cdot \delta \boldsymbol{r}.$$

The velocity field being known for the initial positions of the fluid particles, we can from the principle of conservation of velocity circulation along physical curves, or if we will, from the principle of "moving with the fluid" of the vortex tubes, determine uniquely the distribution of vorticity for any other positions of the fluid particles. The well known differential expression for this is

$$\frac{\partial \nabla \times \boldsymbol{v}}{\partial t} = -\boldsymbol{v} \cdot \nabla \nabla \times \boldsymbol{v} + \nabla \times \boldsymbol{v} \cdot \nabla \boldsymbol{v}.$$

Let us suppose that $\nabla \times \boldsymbol{v}^*$ denotes the vorticities which would obtain owing to a "moving with the fluid" separately of the vortex tubes of the initial mean flow, $\bar{u}_0 \boldsymbol{i}$, so that

$$(14.14) \qquad \nabla \times \boldsymbol{v}^*_{t=0} = \nabla \times \bar{u}_0 \boldsymbol{i}$$

and

$$(14.15) \quad \frac{\partial \nabla \times \boldsymbol{v}^*}{\partial t} = -\boldsymbol{v} \cdot \nabla \nabla \times \boldsymbol{v}^* + \nabla \times \boldsymbol{v}^* \cdot \nabla \boldsymbol{v}.$$

Especially, we must have

$$(14.15') \quad \int_0^{2\pi} \boldsymbol{v}^* (R, \psi, z, t) \cdot \boldsymbol{i} R d\psi = \oint_L \boldsymbol{v}_0^* \cdot \delta \boldsymbol{r}$$

From (14.15) we obtain

$$(14.16) \qquad \frac{\partial \boldsymbol{v}^*}{\partial t} = \boldsymbol{v} \times \nabla \times \boldsymbol{v}^* + \nabla \lambda$$

where $\nabla \lambda$ is some ascendental vector the closer determination of which we are not interested in here. Derivating this equation locally with respect to time and substituting from (14.15) we obtain

$$(14.17) \quad \frac{\partial^2 \boldsymbol{v}^*}{\partial t^2} = \frac{\partial \boldsymbol{v}}{\partial t} \times \nabla \times \boldsymbol{v}^* + \boldsymbol{v} \times [-\boldsymbol{v} \cdot \nabla \nabla \times \boldsymbol{v}^* + \nabla \times \boldsymbol{v}^* \cdot \nabla \boldsymbol{v}] + \nabla \frac{\partial \lambda}{\partial t}.$$

According to (14.14), $\boldsymbol{v}^*_{t=0}$ and $\bar{u}_0 \boldsymbol{i}$ contingently only differ with respect to a field with a cyclic constant, so that

$$(14.18) \qquad \oint \boldsymbol{v}^*_{t=0} \cdot \delta \boldsymbol{r} = \oint \bar{u}_0 \boldsymbol{i} \cdot \delta \boldsymbol{r} + K.$$

Using (14.15') we thus get

$$\oint_L \bar{u}_0 \boldsymbol{i} \cdot d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{v}^* \cdot \boldsymbol{i} R d\psi - K.$$

Substituting this in (14.13), and expanding $\boldsymbol{v}^*$ in powers of t, we get

$$(14.19) \; c = -K + \int_0^{2\pi} \boldsymbol{v}^*_{t=0} \cdot \boldsymbol{i} R d\psi + t \int_0^{2\pi} \frac{\partial \boldsymbol{v}^*}{\partial t_{t=0}} \cdot \boldsymbol{i} R \, d\psi + \frac{t^2}{2} \int_0^{2\pi} \frac{\partial^2 \boldsymbol{v}^*}{\partial t^2_{t=0}} \cdot \boldsymbol{i} R d\psi + \ldots + \oint_L \boldsymbol{v}_0' \cdot \delta \boldsymbol{r}.$$

At time $t = 0$ the closed curve L coincides with the circle with radius R. This gives in connection with (14.18),

$$(14.20) \quad \int_0^{2\pi} \boldsymbol{v}^*_{t=0} \cdot \boldsymbol{i} R d\psi = \int_0^{2\pi} \bar{u}_0 R d\psi + K = c_0 + K.$$

We have

$$(14.21) \qquad d\boldsymbol{r} = \boldsymbol{v}_0 t + \tfrac{1}{2} \frac{D\boldsymbol{v}}{dt}_{t=0} t^2 + \ldots.$$

Substituting in (14.19) from (14.20), (14.16), (14.17), (14.21), and using (14.14) and the conditions (14.11), we obtain

$$(14.22) \quad c = c_0 + \tfrac{1}{2} \int_0^{2\pi} [-d\boldsymbol{r}_m \cdot \nabla \nabla \times \bar{u}_0 \boldsymbol{i} + \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \nabla d\boldsymbol{r}_m] \cdot \boldsymbol{i} \times d\boldsymbol{r}_m R d\psi + \int_0^{2\pi} h(0) \, R d\psi + \oint_L \boldsymbol{v}_0' \cdot \delta \boldsymbol{r},$$

$d\boldsymbol{r}$ having been replaced by the meridional displacements $d\boldsymbol{r}_m$, since it will be only these components which must be taken into account in the first right-hand-side integral. $h(0)$ denotes terms of still higher order in $d\boldsymbol{r}$.

Quite generally we must have

$$(14.23) \qquad \oint_L \boldsymbol{v}_0' \cdot \delta \boldsymbol{r} \to 0 \;\text{ when } \nabla \times \boldsymbol{v}_0' \to 0.$$

This is a simple consequence of Stokes theorem. Suppose for instance that R_1 is the radius of a circle at the boundary, symmetrical with respect to the z-axis. Let f be some surface enclosed within this circle and the closed curve L. Then from Stokes theorem

$$\oint_L \boldsymbol{v}'_0 \cdot \delta \boldsymbol{r} - \int_0^{2\pi} u'_0 R_1 \, d\psi = \int_f \nabla \times \boldsymbol{v}'_0 \cdot \delta \boldsymbol{f}.$$

Since now, according to (14.5), $\int_0^{2\pi} u_0' R_1 d\psi = 0$, we obtain (14.23) from the last equation. For sufficiently small $\nabla \times \boldsymbol{v}'_0$ eq. (14.22) therefore reduces to

$$(14.24) \quad c = c_0 + \tfrac{1}{2} \int_0^{2\pi} [- d\boldsymbol{r}_m \cdot \nabla \nabla \times \bar{u}_0 \boldsymbol{i} + \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \nabla d\boldsymbol{r}_m] \cdot \boldsymbol{i} \times d\boldsymbol{r}_m R d\psi + \int_0^{2\pi} h(0) \, R d\psi \, .$$

This formula determines the variations in c which are caused by the transport of the vortex tubes of the initial mean flow.

The infinite flow between straight, parallel boundaries may be considered as a limiting case of the flow hitherto examined. For, if R_M and R_m denote respectively the longest and shortest distances from the z-axis to the walls we obtain the infinite flow in a straight channel by letting R_M and R_m tend to infinity, $R_M - R_m$ simultaneously remaining finite. We define directions x, y, z in the straight flow from $-R\,d\psi \rightarrow dx$, $dR \rightarrow dy$, $z = z$. The velocity is written $\boldsymbol{v} = u\boldsymbol{i} + v_y \boldsymbol{y}_I + v_z \boldsymbol{z}_I$. Let a denote any of the variables or expressions derived from the variables of our hydrodynamic equations. Then averages are defined according to

$$(14.25) \qquad \bar{a} = - \lim_{R \rightarrow \infty} \cdot \frac{\int_0^{2\pi} a R \, d\psi}{2 \pi R} \, .$$

We write $a = \bar{a} + a'$. As previously, we obtain $\overline{a'} = 0$. With the assumption (14.11), $\bar{v}_y = \bar{v}_z = 0$, the equation of energy becomes

$$(14.26) \quad \int_\tau \tfrac{1}{2} \boldsymbol{v}'^2 \, d\tau = - \int_\tau \tfrac{1}{2} \bar{u}^2 d\tau + \text{const},$$

where τ stands for the infinite volume of the fluid. Instead of the condition (14.7) for constant angular momentum we obtain now a condition for constant total momentum

$$(14.27) \qquad \int_\tau \bar{u} \, d\tau = \text{const.}$$

or

$$(14.28) \qquad \int_\tau (\bar{u} - \bar{u}_0) \, d\tau = 0.$$

From (14.25) we obtain $\bar{u} = - \lim_{R \rightarrow \infty} \frac{c}{2\pi R}$. Substituting here for c from (14.22) and using the symbol of averages, we arrive at

$$(14.29) \quad \bar{u} = \bar{u}_0 + \tfrac{1}{2} \overline{[- d\boldsymbol{r}_m \cdot \nabla \nabla \times \bar{u}_0 + \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \nabla d\boldsymbol{r}_m] \cdot \boldsymbol{i} \times d\boldsymbol{r}_m} + \overline{h(0)} - \lim_{R \rightarrow \infty} \frac{\oint_L \boldsymbol{v}'_0 \cdot \delta \boldsymbol{r}}{2 \pi R} \, ,$$

having used the notation $d\boldsymbol{r}_m$ for the displacements in the y, z-planes: $d\boldsymbol{r}_m = \varrho_y \boldsymbol{y}_I + \varrho_z \boldsymbol{z}_I$. Again for sufficiently small $\nabla \times \boldsymbol{v}'_0$, the last term in the equation above may be neglected in comparison with the others, so that in this case

$$(14.30) \quad \bar{u} = \bar{u}_0 + \tfrac{1}{2} \overline{[- d\boldsymbol{r}_m \cdot \nabla \nabla \times \bar{u}_0 \boldsymbol{i} + \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \nabla d\boldsymbol{r}_m] \cdot \boldsymbol{i} \times d\boldsymbol{r}_m} + \overline{h(0)}.$$

15. Definition of stability and instability.

We have hitherto considered a motion which apart from the chosen boundaries and the limitation expressed in (14.11), has been allowed to be a quite general one. In the following we shall make the special limitation that in the initial state the irregular velocities together with their space derivatives are small in comparison with the corresponding quantities in the mean flow, so that the initial motion is simply a perturbation of a circular vortex. The assumption that also the space derivatives are small is essential for the arguments in the following. For it implies particularly that the vorticities $\nabla \times \boldsymbol{v}'_0$ are small compared with those in the mean flow, so that we can use the simplified expressions (14.24), (14.30) for respectively c and $\bar{u}$. It should be noted that the assumption that the additive velocities are small not of necessity implies that their vorticities are small, since by suitable space variations of the additive velocities their vorticities can become arbitrarily great.

A circular vortex will be said to be stable if the meridional displacements up to any time, however great, will remain arbitrarily small, at least on an average, if the additive vorticities,

$\nabla \times \boldsymbol{v}'_0$, are assumed arbitrarily small. We may express this as follows:

General criterion for stability:

$$(15.1) \qquad \int_\tau (d\boldsymbol{r}_m)^2 \, d\tau \to 0 \text{ when } \nabla \times \boldsymbol{v}'_0 \to 0.$$

A simple consequence of the stability of a circular vortex will be that the increase $\int_\tau \frac{1}{2} \boldsymbol{v}'^2 \, d\tau - \int_\tau \frac{1}{2} \boldsymbol{v}_0'^2 \, d\tau$ in the kinetic energy of the irregular flow up to any time will become arbitrarily small with $\nabla \times \boldsymbol{v}'_0$:

$$(15.2) \qquad \int_\tau \tfrac{1}{2} \boldsymbol{v}'^2 \, d\tau - \int_\tau \tfrac{1}{2} \boldsymbol{v}_0'^2 \, d\tau \to 0 \text{ when } \nabla \times \boldsymbol{v}'_0 \to 0.$$

This is seen from the energy equation (14.8) if one takes account of the continious dependency of c upon the meridional displacements. However, it is not always allowed to conclude in the reverse way that relation (15.1) will be a necessary consequence of (15.2). In mean flows where finite displacements in meridional direction are possible, simultaneously as $\int_\tau \frac{1}{2} \boldsymbol{v}'^2 \, d\tau - \int_\tau \frac{1}{2} \boldsymbol{v}_0'^2 d\tau$ can be made arbitrarily small with $\nabla \times \boldsymbol{v}'_0$, the circular vortex will be said to be indifferent for the considered perturbations. If neither of the relations (15.1), (15.2) can be satisfied, the circular vortex will be defined as unstable. The following criterion for instability is sufficient and necessary:

Criterion of instability:

An increase in the kinetic energy of the irregular flow shall exist which cannot be made arbitrarily small with $\nabla \times \boldsymbol{v}'_0$. From the energy equation it follows then that the meridional displacements cannot remain arbitrarily small.

16. Two-dimensional motion in planes perpendicular to the z-axis.

Suppose at time $t = 0$ that

$$(16.1) \qquad v_z = \frac{\partial v_R}{\partial z} = \frac{\partial u}{\partial z} = 0,$$

and that the boundaries are concentric cylinders symmetrical with respect to the z-axis. It follows then from reasons of symmetry, already, that (16.1) will be satisfied at all times, implying that the motion will be completely described if we know it in an arbitrary plane, $z = \text{const}$. As a consequence of the condition of incompressibility and (16.1) we obtain $\bar{v}_R = \frac{1}{2\pi R}\int_0^{2\pi} v_R R \, d\psi = 0$. Therefore, the axialsymmetric mean meridional motion disappears from the equations, and we can use the energy equation in the form (14.12) and the expression for c as given by (14.22). Now, let us consider a fluid mass within a sheet of unit thickness perpendicular to the z-axis, and let F denote the total area between the cylinders. This fluid is uncapable of receiving work or angular momentum through its boundaries, so that the energy equation and the condition for constant angular momentum may be written

$$\int_z^{z+1}\int_F \tfrac{1}{2} \boldsymbol{v}'^2 \, dF \, dz = -\int_z^{z+1}\int_F \frac{c^2 \, dF \, dz}{8\pi^2 R^2} + \text{const},$$

and

$$\int_z^{z+1}\int_F (c - c_0) \, dF \, dz = 0.$$

Since now $\boldsymbol{v}$ is assumed to be independent of z, we obtain from these equations

$$(16.2) \qquad \int_F \tfrac{1}{2} \boldsymbol{v}'^2 \, dF = -\int_F \frac{c^2 dF}{8\pi^2 R^2} + \text{const},$$

$$(16.3) \qquad \int_F (c - c_0) \, dF = 0.$$

In the motions studied now the expression for c in (14.22), using the notation ϱ_R for $d\boldsymbol{r} \cdot \boldsymbol{R}_I$, reduces to

$$(16.4) \qquad c = c_0 + \tfrac{1}{2}\int_0^{2\pi} \varrho^2_R \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R d\psi + \int_0^{2\pi} h(0) \, R d\psi + \oint_L \boldsymbol{v}'_0 \cdot \delta \boldsymbol{r},$$

since $\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \nabla d\boldsymbol{r}_m = \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I \frac{\partial d\boldsymbol{r}_m}{\partial z}$ which becomes zero according to (16.1). This simplification in the expression for c is a consequence of the fact that the vorticity is individually conserved in a two-dimensional motion: $\frac{D\nabla \times \boldsymbol{v}}{dt} = 0$.

17. The Rayleigh-Taylor criterion of stability.

Substituting from (16.4) in (16.3) we obtain

$$(17.1)\quad 0 = \pi \int_F \varrho_R{}^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R dF + 2\pi \int_F h(0)\, R dF + \int_F \oint_L \boldsymbol{v}_0' \cdot \delta \boldsymbol{r}\, dF.$$

Let us suppose that the circular vortex with velocities $\bar{u}_0 \boldsymbol{i}$ has either steadily increasing or steadily decreasing vorticity between the boundaries, so that

$$(17.2)\quad \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} \text{ is of one sign throughout the fluid.}$$

Then the first integral in (17.1) has a definite sign. whatever are the displacements ϱ_R. Its numerical value will tend to zero if, and only if $\int_F \varrho^2{}_R dF \rightarrow 0$. The second integral is small in comparison with the first one for sufficiently small ϱ_R. The last integral represents the effect upon $\int_F (c - c_0)\, dF$ from the "moving with the fluid' of the vortex tubes belonging to the additive field, $\boldsymbol{v}'_0$, and becomes small with $\nabla \times \boldsymbol{v}_0'$, as shown in (14.23). Due to these facts we obtain from (17.1) that under condition (17.2),

$$(17.3)\quad \int_F \varrho_R{}^2 dF \rightarrow 0 \text{ when } \nabla \times \boldsymbol{v}_0' \rightarrow 0.$$

Therefore, a circular vortex in which the condition (17.2) is fulfilled, must be stable for two-dimensional perturbations. This was at first indicated by *Rayleigh* (3). The proof as given above, is essentially *Taylor's* (4) proof for this stability, differing however from his in stating $\nabla \times \boldsymbol{v}_0' \rightarrow 0$ as a necessary base for the proof. If the vorticities $\nabla \times \boldsymbol{v}_0'$ are finite nothing can be proved with respect to the stability from mere considerations of the conditions for constant angular momentum. When turbulence is formed in laminar flows, it may well be that the vorticities $\nabla \times \boldsymbol{v}_0'$ are of the same order as or even greater than the vorticities of the smoothed flow, even if $\boldsymbol{v}'_0$ is small compared with $\bar{u}_0 \boldsymbol{i}$. I hope to return to the question as to what will happen if $\nabla \times \boldsymbol{v}_0'$ is finite, in a later paper.

It was found above that under the condidition stated in (17.2) the two first right-hand-side integrals in (16.4) contribute to a change of a definite sign in c, at least for small ϱ_R. We shall show that this result is correct however large are the radial displacements. Let the stippled curve in fig. 1 represent the closed curve L

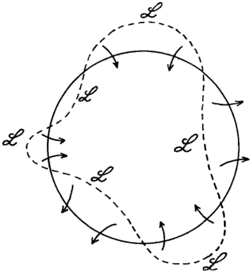

Fig. 1.

which under certain displacements transforms to the circle drawn in the same figure. Let further A_1 denote the total area moved into this circle and A_2 that moved out of it. The motion being assumed two-dimensional, the condition of incompressibility requires that

$$(17.4)\quad A_1 = A_2.$$

From Stokes theorem we obtain

$$(17.5)\quad \oint_L \bar{u}_0 \boldsymbol{i} \cdot \delta \boldsymbol{r} - \int_0^{2\pi} \bar{u}_0 R d\psi = \int_{A_2} \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I dA - \int_{A_1} \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I dA.$$

This equation in connection with (17.2), (17.4) shows that the changes in c caused by the transport of the vortex tubes of the initial mean flow, will be of the same sign everywhere in the fluid, however large are the radial displacements of the fluid particles.

18. An other form of Taylors proof.

From the condition for constant angular momentum

$$\text{const} = \int_F c\,dF = 2\pi \int_{R_1}^{R_2} cR\,dR$$

we obtain, when integrating by part:

(18.1) $$\text{const} = 2\pi[\tfrac{1}{2}R_2^2\,c(R_2) - \tfrac{1}{2}R_1^2\,c(R_1)] - 2\pi\int_{R_1}^{R_2}\frac{1}{2}\frac{dc}{dR}R^2\,dR.$$

R_1, R_2 denote the radii of the inner and outer cylinder, respectively. We have

$$\frac{dc}{dR} = \frac{d}{dR}\int_0^{2\pi} uR\,d\psi = \int_0^{2\pi} R\left(\frac{\partial u}{\partial R} + \frac{u}{R}\right)d\psi$$

or, since $\nabla\times\boldsymbol{v}\cdot\boldsymbol{z}_I = \frac{\partial u}{\partial R} + \frac{u}{R} - \frac{\partial v_R}{R\partial\psi}$ and

$$\int_0^{2\pi}\frac{\partial v_R}{\partial\psi}\,d\psi = 0,$$

$$\frac{dc}{dR} = \int_0^{2\pi}\nabla\times\boldsymbol{v}\cdot\boldsymbol{z}_I\,R\,d\psi.$$

Substituting from this into (18.1), and noting that $c(R_1)$, $c(R_2)$ are independent of time since they represent velocity circulations along physical curves, we arrive at

$$\text{const} = \int_F \zeta R^2\,dF,$$

having written ζ for $\nabla\times\boldsymbol{v}\cdot\boldsymbol{z}_I$. For the plane motion now considered, ζ is individually conserved and may therefore be considered as a function of the Lagrangian particle variables, only. If these variables are substituted in the last integral we obtain

(18.2) $$\text{const} = \int_{F_0}\zeta(\boldsymbol{r}_0)\,R^2\,dF_0 = G(\boldsymbol{r}).$$

This integral being formally equal to the total potential energy of an incompressible fluid, $\Phi(\boldsymbol{r}) = \int_{\tau_0} q(\boldsymbol{r}_0)\,\varphi\,d\tau_0$, we may from a comparison with the results derived in Chapter I determine the space distribution of vorticity for extreme values of $G(\boldsymbol{r})$. Thus, if Z denotes this particular distribution of vorticity, then

$$Z\nabla R^2 = \text{laminar} = -\nabla\lambda$$

is the condition for stationary values of $G(\boldsymbol{r})$. The vorticity thus being a function only of R, the corresponding motion is in concentric circles. The two-dimensional motion, independent of z, in concentric circles will be denoted a cylindric flow. If the vorticity in the cylindric flow is steadily increasing or steadily decreasing with R, the extremum is a minimum or a maximum, respectively, corresponding to Φ assuming a maximum or minimum value according as the density increases or decreases steadily with φ. Suppose now that a cylindric flow characterized by either a maximum or minimum of $G(\boldsymbol{r})$, is disturbed by superimposed small vorticities. The isolines for vorticity will then be but slightly different from circles $R = \text{const}$. Accordingly, small displacements are sufficient to bring the fluid particles to the positions for which $G(\boldsymbol{r})$ assumes an extreme value. The constant in (18.2) is therefore slightly different from a maximum or minimum value of G, and we obtain

(18.3) $$G(\boldsymbol{r}) = G_{\text{extr.}} + \varepsilon,$$

where the constant ε now is a small quantity. This equation can only be satisfied if the fluid particles remain, at least on an average, in the neighbourhood of the positions for which $G(\boldsymbol{r})$ obtains its maximum or minimum value. This expresses the stability of cylindric flows characterized by either steadily increasing or decreasing vorticity between the walls.[1])

It may be useful to consider shortly the analogous conditions for a material point moving in a x, y-plane and satisfying the condition

$$f(x, y) = \text{const.}$$

Let us suppose that $f(x, y)$ assumes an extreme value in the point x_1, y_1. We may write the above condition

$$f(y, y) = f(x_1, y_1) + \varepsilon,$$

ε being a constant. Suppose x_1, y_1 to be a

[1]) This proof for stability was derived without knowing Taylors proof. On the whole, it seems as if Taylors proof has called but slightly attention in spite of its, after my opinion, great theoretical interest

maximum or minimum point for f. Then the curves $f(x, y) = \text{const}$ must be closed curves in the neighbourhood of x_1, y_1. If therefore at a given instant the material point has a position near to x_1, y_1 so that the quantity ε is small, it must at all times move at a closed curve in the neighbourhood of the point x_1, y_1, fig. 2 a.

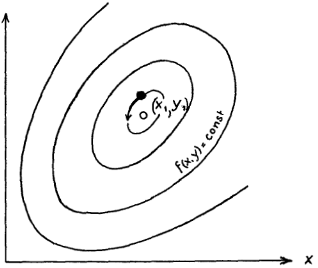

Fig. 2 a.

If, however, the extremum is a saddelpoint, fig. 2 a, or if ε is not a small quantity,

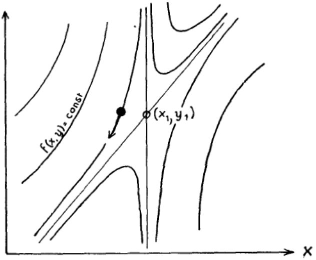

Fig. 2 b.

fig. 2 c, then the material point may change its position considerably without violating the condition $f(x, y) = \text{const}$. In the corresponding cases for the cylindric flow the vorticity is partly increasing and partly decreasing between the walls, giving a "saddelpoint"-extremum to G, or the isolines for vorticity in the disturbed state are far from having the shape of concentric circles, giving thus a finite value to the quantity ε in (18.3).

It is interesting to note that the above

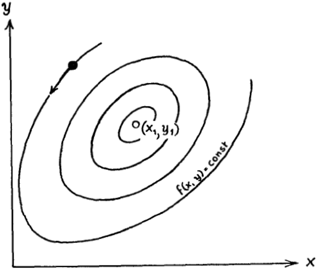

Fig. 2 c.

developments show that a motion in two dimensions between concentric cylinders which is not a pure cylindric flow can generally not become so at any later stage. For in this case the constant value of G will generally be different from a stationary value. If therefore the motion should become a pure cylindric flow, G would have to change its value, and thus to violate the condition of constant angular momentum. For the same reason it is generally not possible to have a motion in which all particles oscillate at the same rate and with the same phase around the circles, $R = \text{const}$, along which they are moving in the mean.

19. The general criteria for energy transformations.

Inserting for the constant in the energy equation (14.12) the value obtained from initial values of $\boldsymbol{v}'$ and c, we obtain

$$\int_\tau \tfrac{1}{2}\boldsymbol{v}'^2 d\tau - \int_\tau \tfrac{1}{2}\boldsymbol{v}_0'^2\, d\tau = -\int_t \frac{c^2 d\tau}{8\pi^2R^2} + \int_t \frac{c_0^2 d\tau}{8\pi^2R^2}.$$

By substituting here $c = c_0 + (c - c_0)$ it follows

$$\int_\tau \tfrac{1}{2}\,\boldsymbol{v}'^2\,d\tau - \int_\tau \tfrac{1}{2}\boldsymbol{v}'_0{}^2 d\tau =$$

$$-\int_\tau \frac{c_0(c-c_0)d\tau}{4\pi^2R^2} - \int_\tau \frac{(c-c_0)^2 d\tau}{8\pi^2 R^2}.$$

We have

$$\frac{c_0}{4\pi^2R^2} = \frac{\overline{\omega}_0}{2\pi},$$

$\overline{\omega}_0$ denoting the average angular speed along the circles $R = \text{const}$ at time $t = 0$. Thus

$$(19.1)\quad \int_\tau \tfrac{1}{2}\boldsymbol{v}'^2\,d\tau - \int_\tau \tfrac{1}{2}\,\boldsymbol{v}_0{}^2\,d\tau =$$

$$-\frac{1}{2\pi}\int_\tau \overline{\omega}_0\,(c-c_0)\,d\tau - \int_t \frac{(c-c_0)^2\,d\tau}{8\,\pi^2R^2}.$$

We divide τ into two parts, τ_1, τ_2, in which $c - c_0$ has opposite signs. Let τ_1 be the volume where $c - c_0$ is negative and τ_2 the volume where $c - c_0$ is positive. Integrating in the first right-hand-side integral of (19.1) at first over τ_1 and then over τ_2, and putting suitable average values of $\overline{\omega}_0$, respectively $\overline{\omega}_0\,(R_1, z_1)$ and $\overline{\omega}_0\,(R_2, z_2)$ outside the integral signs, we obtain

$$\int_\tau \tfrac{1}{2}\boldsymbol{v}'^2\,d\tau - \int_\tau \tfrac{1}{2}\,\boldsymbol{v}_0{}'^2 d\tau = -\frac{\overline{\omega}_0\,(R_1, z_1)}{2\pi}\int_{\tau_1}(c-c_0)d\tau$$

$$-\frac{\overline{\omega}_0\,(R_2, z_2)}{2\pi}\int_{\tau_2}(c-c_0)\,d\tau - \int_\tau \frac{(c-c_0)^2\,d\tau}{8\pi^2R^2}.$$

It follows from the condition (14.8) for constant angular momentum that

$$\int_{\tau_1}(c-c_0)\,d\tau = -\int_{\tau_2}(c-c_0)\,d\tau.$$

Substituting from this in the last equation we get

$$(19.2)\quad \int_\tau \tfrac{1}{2}\,\boldsymbol{v}'^2\,d\tau - \int_t \tfrac{1}{2}\,\boldsymbol{v}_0{}'^2\,d\tau =$$

$$\frac{1}{2\pi}[\overline{\omega}_0\,(R_1,z_1) - \overline{\omega}_0\,(R_2,z_2)]\int_{\tau_2}(c-c_0)d\tau - \int_\tau \frac{(c-c_0)^2 d\tau}{8\,\pi^2R^2}.$$

The analogous equation for the infinite flow in a straight channel is obtained from eqs. (14.26), (14.27) by a similar reasoning

$$(19.2')\quad \int_\tau \tfrac{1}{2}\,\boldsymbol{v}'^2\,d\tau - \int_\tau \tfrac{1}{2}\,\boldsymbol{v}_0{}'^2\,d\tau =$$

$$[\bar{u}_0\,(y_1,z_1) - \bar{u}_0\,(y_2,z_2)]\int_{\tau_2}(\bar{u}-\bar{u}_0)d\tau - \int_\tau \tfrac{1}{2}\,(\bar{u}-\bar{u}_0)^2\,d\tau.$$

From these equations we can state the following rules:

If all values of $\overline{\omega}_0$, respectively $\bar{u}_0$ in τ_2 (the region where the mean velocity increases with time) are *greater* than all values of $\overline{\omega}_0$, respectively $\bar{u}_0$ in τ_1 (the region where the mean velocity decreases with time), the kinetic energy of the irregular flow is destroyed and transformed to kinetic energy of the mean flow.

For in this case $\overline{\omega}_0\,(R_1, z_1) - \overline{\omega}_0\,(R_2, z_2)$ and $\int_{\tau_2}(c-c_0)\,d\tau$, respectively $\bar{u}_0\,(y_1, z_1) - \bar{u}_0\,(y_2, z_2)$ and $\int_{\tau_2}(\bar{u}-\bar{u}_0)\,d\tau$ must have opposite signs, so that both right-hand-side terms in (19.2) become negative.

On the other hand:

If all values of $\overline{\omega}_0$, respectively $\bar{u}_0$, in the region where the velocity increases with time are *smaller* than all values of $\overline{\omega}_0$, respectively $\bar{u}_0$, in the region where it decreases, then kinetic energy of the mean flow is transformed to kinetic energy of the irregular flow, at least for sufficiently small displacements of the fluid particles.

This rule is based on the facts that the first right-hand-side term in (19.2), respectively (19.2′) now is positive, and that the second integrals are small in comparison with the first ones for small displacements.

20. Motion in planes perpendicular to the z-axis. — The case where the vorticity of the mean flow decreases as well as increases with the distance from the centrum of the motion.

Suppose that F_1 denotes that part of the area between the cylinders where the vorticity $\nabla \times \bar{u}_0\boldsymbol{i}\cdot\boldsymbol{z}_I$ is steadily decreasing with R_1, and F_2 that part where it is steadily increasing with R. For sufficiently small ϱ_R and $\nabla\times\boldsymbol{v}_0'$ we obtain according to (16.4)

$$(20.1)\begin{cases} \text{a)} \int_{F_1} (c - c_0)\, dF = \pi \int_{F_1} \varrho_R^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R dF < 0 & \text{for all } \varrho_R \\ \text{b)} \int_{F_2} (c - c_0)\, dF = \pi \int_{F_2} \varrho_R^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R dF > 0 & \text{for all } \varrho_R, \end{cases}$$

and from the condition (16.3) for constant angular momentum we obtain

$$(20.1)\ \text{c)} \int_F (c - c_0)\, dF = \pi \int_{F_1} \varrho_R^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R dF + \pi \int_{F_2} \varrho_R^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R dF = 0.$$

The two integrals to the right in the last equation now having opposite signs, we can by suitable choice of ϱ_R make $\int_F (c - c_0)\, dF$ arbitrarily small without necessarily having $\int_F \varrho_R^2 dF \to 0$.

Therefore, in the present case it does not follow from the condition of constant angular momentum that $\int_F \varrho_R^2\, dF \to 0$ with $\nabla \times \boldsymbol{v}_0'$. However, utilizing the conditions for energy transformations stated in the previous section it is possible also in the present case to examine the stability more closely. Replacing τ by F in eq. (19.2) and remembering that the last integral may for sufficiently small ϱ_R be neglected in comparison with the others, we obtain

$$\int_F \tfrac{1}{2} \boldsymbol{v}'^2\, dF - \int_F \tfrac{1}{2} \boldsymbol{v}_0'^2\, dF = \frac{1}{2\pi} [\bar{\omega}_0(R_1) - \bar{\omega}_0(R_2)] \int_{F_2} (c - c_0)\, dF.$$

For sufficiently small ϱ_R and $\nabla \times \boldsymbol{v}_0'$ we can substitute for $c - c_0$ from (20.1) b), thus obtaining

$$(20.2)\quad \int_F \tfrac{1}{2} \boldsymbol{v}'^2\, dF - \int_F \tfrac{1}{2} \boldsymbol{v}_0'^2\, dF = [\bar{\omega}_0(R_1) - \bar{\omega}_0(R_2)] \int_{F_2} \tfrac{1}{2} \varrho_R^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R dF.$$

The sufficient criterion (19.3) for transformation of kinetic energy from the irregular to the mean flow assumes now the following explicit form that

(20.3) all values of $\bar{\omega}_0$ in that part of the fluid where $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR}$ is positive be greater than all values of $\bar{\omega}_0$ where $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR}$ is negative.

Under this condition the right-hand side in (20.2) must be definite negative, implying that

$$\int_\tau \tfrac{1}{2} \boldsymbol{v}'^2\, d\tau - \int_\tau \tfrac{1}{2} \boldsymbol{v}_0'^2\, d\tau < 0.$$

Then, since $\int_\tau \tfrac{1}{2} \boldsymbol{v}_0'^2\, d\tau \to 0$ with $\nabla \times \boldsymbol{v}'_0$, we must necessarily also have

$$\int_\tau \tfrac{1}{2} \boldsymbol{v}'^2\, d\tau - \int_\tau \tfrac{1}{2} \boldsymbol{v}_0'^2\, d\tau \to 0 \text{ with } \nabla \times \boldsymbol{v}_0'.$$

Therefore, the right-hand side in (20.2) must tend to zero with $\nabla \times \boldsymbol{v}'_0$ which is possible only if

$$\int_{F_2} \varrho_R^2\, dF \to 0 \text{ with } \nabla \times \boldsymbol{v}'_0.$$

Applying this result in (20.1)c), and noting that $\int_{F_1} \varrho_R^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} R dF$ also has a definite sign, we obtain

$$(20.4)\quad \int_F \varrho_R^2\, dF \to 0 \text{ with } \nabla \times \boldsymbol{v}_0'.$$

This expresses the stability of the cylindric flow now considered.

If we take account also of the effect from the transport of the additive vorticities, we obtain an additional term to the right in (20.2) given by

$$(20.5)\quad \frac{1}{2\pi} [\bar{\omega}_0(R_1) - \bar{\omega}_0(R_2)] \int_{F_2} \Big[\oint_L \boldsymbol{v}_0' \cdot \delta \boldsymbol{r} \Big]\, dF.$$

Since this term becomes arbitrarily small with $\nabla \times \boldsymbol{v}_0'$, it is easily understood that the above criterion (20.3) for stability will hold even if we take account of the effect from the transport of the additive vorticities, $\nabla \times \boldsymbol{v}_0'$. If, however, the vorticities $\nabla \times \boldsymbol{v}_0'$ are finite, then finite energy transformations can develop from the term (20.5). That is why finite vorticities probably is a factor of importance for the formation of turbulence.

To examine the conditions for instability we need not take account of the term (20.5), since this becomes arbitrarily small with the vorticities $\nabla \times \boldsymbol{v}_0'$. Eq. (20.2) gives then:

(20.6) If all values of $\bar{\omega}_0$ in the region with negative $\frac{d\nabla \times \bar{u} \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR}$ are greater than all values in the region with positive $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR}$, then as a result of the transport of the vorticites of the initial mean flow, kinetic energy of the mean flow is transformed to kinetic energy of the irregular flow.

To arrive at a criterion of instability of the kind defined in (15.3) it remains however to prove that the increase in kinetic energy of the irregular flow arrived at above does not become arbitrarily small with $\nabla \times \boldsymbol{v}_0'$. As we shall see in section 24 many types of perturbations must exist which will be stable in spite of condition (20.6). Therefore, the only conclusion which we can draw from (20.2) is that under condition (20.6) a system of not arbitrarily small *virtual* displacements will lead to a not arbitrarily small increase in the kinetic energy of the irregular flow. Whether or not such displacements will exist in an *actual* motion cannot be decided by means of energy considerations alone. In spite of our disability to give a strict proof for instability under the present conditions, we will nevertheless refer to (20.6) as the condition for "instability".

21. Explicit criteria of stability for the two-dimensional linear flow.

In the following sections we are going to deal with some explicit cases of the two-dimensional flow considered above, starting with the infinite flow in a straight channel.[1] It was shown at the end of section 14 that this flow could be considered as a limiting case of a flow within a channel symmetrical with respect to the z-axis. Let us now consider a particular case of this infinite flow, assuming

$$v_z = \frac{\partial \boldsymbol{v}}{\partial z} = 0. \tag{21.1}$$

Having $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} \rightarrow -\frac{d^2\bar{u}_0}{dy^2}$ when $R \rightarrow \infty$ and $\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \nabla d\boldsymbol{r} = \nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I \frac{\partial d\boldsymbol{r}}{\partial z}$ which is zero according to (21.1), eq. (14.29) reduces to

$$\bar{u} - \bar{u}_0 = \tfrac{1}{2}\,\overline{\varrho_y{}^2}\,\frac{d^2\bar{u}_0}{dy^2} + h(\bar{0}) - \lim_{R\to\infty} \frac{\oint_L \boldsymbol{v}'_0 \cdot \delta\boldsymbol{r}}{2\pi R}. \tag{21.2}$$

[1]) The notation "linear flow" is used here for the corresponding mean flow.

For sufficiently small ϱ_y and additive vorticities $\nabla \times \boldsymbol{v}_0'$ it can be written

$$\bar{u} - \bar{u}_0 = \tfrac{1}{2}\,\overline{\varrho_y{}^2}\,\frac{d^2\bar{u}_0}{dy^2}. \tag{21.2'}$$

The criterion of stability (17.2), (17,3) now assumes the form:

(21.3) If $\frac{d^2\bar{u}_0}{dy^2}$ is of one sign throughout the fluid, then at all times

$$\frac{1}{F}\int_F \varrho_y{}^2\, dF \rightarrow 0 \text{ with } \nabla \times \boldsymbol{v}'_0.$$

This criterion is simply an expression of the fact that, assuming small additive vorticities at time $t = 0$, finite displacements can not occur without violating the condition (14.28) of constant total momentum.

The energy equation to deal with develops from (19.2) by writing F instead of τ, and thus becomes

$$\int_F \tfrac{1}{2}\boldsymbol{v}'^2\, dF - \int_F \tfrac{1}{2}\boldsymbol{v}'_0{}^2\, dF = [\bar{u}_0(y_1) - \bar{u}_0(y_2)]\int_{F_s} (\bar{u} - \bar{u}_0)\, dF - \int_F \tfrac{1}{2}(\bar{u} - \bar{u}_0)^2 dF. \tag{21.4}$$

Hence, by substituting for $\bar{u} - \bar{u}_0$ from (21.2'), and disregarding the last integral which becomes negligible for sufficiently small ϱ_y, we obtain

$$\int_F \tfrac{1}{2}\boldsymbol{v}'^2\, dF - \int_F \tfrac{1}{2}\boldsymbol{v}_0'^2\, dF = [\bar{u}_0(y_1) - \bar{u}_0(y_2)]\int_{F_s} \tfrac{1}{2}\,\varrho_y{}^2\,\frac{d^2\bar{u}_0}{dy^2}\, dF. \tag{21.5}$$

By a reasoning similar to that in the preceding section we arrive at the following criterion of stability:

(21.6) If all $\bar{u}_0$ in the region where $\frac{d^2\bar{u}_0}{dy^2} > 0$ are greater than all $\bar{u}_0$ where $\frac{d^2\bar{u}_0}{dy^2} < 0$, then at all times

$$\frac{1}{F}\int_F \varrho_y{}^2\, dF \rightarrow 0 \text{ with } \nabla \times \boldsymbol{v}'_0$$

On the other hand:

(21.7) If all values of $\bar{u}_0$ in the region where $\frac{d^2\bar{u}_0}{dy^2} > 0$ are smaller than all $\bar{u}_0$ where $\frac{d^2\bar{u}_0}{dy^2} < 0$, the corresponding flow is "unstable".

Fig. 3 and fig. 4 show examples of linear flows which are stable according to criterion (21.3). The full drawn lines are velocity profiles

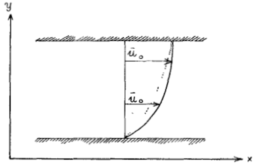

Fig. 3.

for $\bar{u}_0$ and the stippled lines indicate the velocity profiles for the mean flow at a later in-

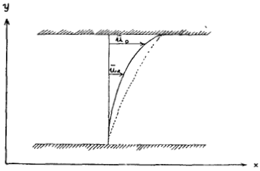

Fig. 4.

stant. Fig. 5 shows a linear flow which is stable according to criterion (21.6) and fig. 6 one which is "unstable" according to criterion (21.7).

It should be noted that a linear flow with a velocity profile shown in fig. 6 will have as a limiting case a flow with gliding along a discontinuity line, if the middle zone of maxi-

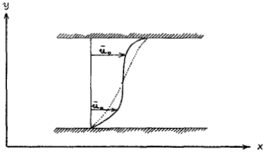

Fig. 5.

mum shear is narrowed more and more, fig. 7. This limiting flow could also be examined directly by the methods used above. We must

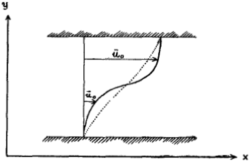

Fig. 6.

then take account of the effect upon the mean flow from the transport of the gliding vorticity.

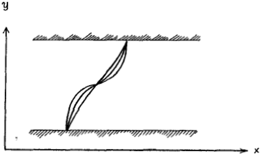

Fig. 7.

As a result we would then find a transformation of kinetic energy from the mean to the irregular flow.

Rayleigh (3) has found that linear flows with broken velocity profiles are stable if they are of a kind analogous to that shown in fig. 5, and unstable if they are of the kind shown in fig. 6. Thus, these results are in agreement with those found above. It is worth noticing that the simple method used in this paper enables us to give an exact proof for the stability of linear flows with velocity profiles shown in fig. 5, whereas the analytical method of proof by finding the general solution of linearized equations of motion by superposition of elementary solutions is so difficult that until now only elementary solutions have been found, and for the broken velocity profiles only.

22. The cylindric flow.

For this flow we have

$$\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I = \frac{\bar{u}_0}{R} + \frac{d\bar{u}_0}{dR}$$

so that now the stability criterion (17.2) assumes the form, that

$$\frac{d^2\bar{u}_0}{dR^2} + \frac{d}{dR}\left(\frac{\bar{u}_0}{R}\right) \tag{22.1}$$

be of one sign throughout the fluid.

Since the expression for vorticity contains R, the necessary condition for instability: that $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR}$ changes sign between the boundaries, may now be satisfied even if $\frac{d^2\bar{u}_0}{dR^2}$ is of one sign throughout the fluid. Suppose for instance that $\bar{u}_0$ is given by

$$\bar{u}_0 = -\frac{a}{2} R^3 + a R_2 R^2$$

where a is a positive constant and R_2 the radius of the outer cylinder. It follows then,

$$\frac{d^2\bar{u}_0}{dR^2} = -a\,(3R - 2R_2),$$

$$\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR} = -a\,(4R - 3R_2),$$

$$\frac{d\bar{\omega}_0}{dR} = -a\,(R - R_2).$$

Consider the case with $R_1 = \frac{2}{3} R_2$, i. e.

$$R_2 \geqq R \geqq \tfrac{2}{3} R_2,$$

R_1 denoting the radius of the inner cylinder. Then $\frac{d^2\bar{u}_0}{dR^2}$, save for the zero value at the inner wall, is negative throughout the fluid, whereas $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR}$ changes sign for $R = 3/4\, R_2$. Since $\bar{\omega}_0$ according to the above formula is steadily increasing with R and $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{z}_I}{dR}$ is negative in the outer part of the fluid, the flow studied in this example is "unstable" according to criterion (20.6).

23. Motion on concentric spheres.

Let ψ, φ, a represent spherical coordinates which are related to the cylindrical coordinates used hitherto through: $R = a \cos\varphi$, $z = a \sin\varphi$, and let us suppose that

$$v_a = \frac{\partial \boldsymbol{v}}{\partial a} = 0. \tag{23.1}$$

If the fluid is enclosed within the fixed boundaries given by the surfaces

$$a = a_1,\ a = a_2,\ \varphi = \varphi_1,\ \varphi = \varphi_2,$$

and if F is the area of one arbitrary of the spherical stream surfaces a = const, then

$$\int_F \tfrac{1}{2} \boldsymbol{v}'^2\, dF = -\int_F \frac{c^2 dF}{8\pi^2 R^2} + \text{const} \tag{23.2}$$

and

$$\int_F (c - c_0)\, dF = 0 \tag{23.3}$$

develop from the energy equation and from the condition for constant angular momentum as did previously the eqs. (16.2), (16.3) for the plane two-dimensional flow. On account of (23.1) the expression (14.22) for c reduces to

$$c = c_0 - \int_0^{2\pi} \tfrac{1}{2} \varrho_\varphi^2 \frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{a\, d\varphi} R\, d\psi + \int_0^{2\pi} h(0)\, R\, d\psi + \oint_L \boldsymbol{v}_0' \cdot \delta \boldsymbol{r}. \tag{23.4}$$

The criterion (17.2), (17.3) assumes the form:

(23.5) If $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi}$ is of one sign for all φ between the boundaries, then $\int_F \varrho_\varphi^2\, dF \to 0$ with $\nabla \times \boldsymbol{v}'_0$.

A simple example of a flow which is stable in this sence is given by a fluid rotating at the constant angular speed of the earth, $\bar{\omega}_0 = \Omega$. In this flow,

$$\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi} = 2\,\Omega \cos\varphi$$

which is of one sign at all latitudes φ. This is the physical base for the stability of the so-called *Rossby* — waves (5); they are stable because finite displacements ϱ_φ, by small vorticities in the field of perturbations, would be inconsistent with the condition of constant angular momentum.

It appears from (23.4) that the mean zonal velocities will decrease, at least for small displacements ϱ_φ, as a result of the north- and southwards transports of the vortex tubes of the initial mean flow. That this will be the case for all ϱ_φ, however large, can be seen from a reasoning similar to that given at the end of section 17.

We have seen above that the "moving with the fluid" of the vorticities of the mean flow will create a relative mean easterly flow. Thus to secure the constancy of the total angular momentum a compensating mean westerly flow must be created from the motion of the additive vorticities, $\nabla \times \boldsymbol{v}'_0$. This implies, as may be seen by applying the theorem of Stokes, that the fluid particles with cyclonic additive vorticites must move in the mean to the north and those with anticyclonic vorticity to the south, fig. 8.

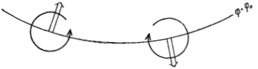

Fig. 8.

If $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi}$ changes sign with latitude, the energy equation (23.2) written in the form

$$\int_F \tfrac{1}{2} \boldsymbol{v}'^2 \, dF - \int_F \tfrac{1}{2} \boldsymbol{v}_0'^2 \, dF =$$
$$- \frac{1}{2\pi} \int_F \omega_0 (c - c_0) \, dF - \int_F \frac{(c - c_0)^2 \, dF}{8 \pi^2 R^2},$$

in connection with the expression (23.4) for $c - c_0$ gives the criteria for stability and "instability" in just the same way as for the analogous cases studied in the preceding sections. Thus as sufficient criterion for stability:

(23.6) If all angular speeds $\bar{\omega}_0$ in the region of positive $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi}$ are smaller than all angular speds $\bar{\omega}_0$ in the region of negative $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi}$, then independently of time,
$$\int_F \varrho^2_\varphi \, dF \to 0 \text{ with } \nabla \times \boldsymbol{v}'_0.$$

On the other hand we obtain as criterion for "instability":

(23.7) If all $\bar{\omega}_0$ in the region where $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi} > 0$ are greater than all $\bar{\omega}_0$ in the region where $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi} < 0$, then the vortex is "unstable".

Let U denote the mean relative velocity when $t = 0$. Then

$$\omega_0 = \Omega + \frac{U}{a \cos \varphi} \tag{23.8}$$

and

$$\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I = 2 \, \Omega \sin \varphi - \frac{dU}{a d\varphi} + \frac{U}{a} \cdot \operatorname{tg} \varphi.$$

In the diagram in fig. 9 we have drawn to the right a curve giving the variation of $2 \, \Omega \sin \varphi$ with latitude. To the left are drawn two profiles, I, II, for the relative mean velocity U. The corresponding variation of absolute vorticity is given by the curves I, II to the right in the diagram. Both profiles show a maximum of westerly wind at middle latitudes. In consequence of this a relative cyclonic vorticity occurs to the north of the latitude with maximum velocity, so that a northerly region exists where the absolute vorticity of the initial mean flow either obtains a less marked increase with latitude (profile I) or even decreases with latitude (profile II). In the first case the atmosphere is stable according to criterion (23.5). In the other case the stability can be examined by means of the criteria (23.6), (23.7). The stippled curve to the left in the diagram represents the variation of $\frac{U}{\cos \varphi}$ with latitude corresponding to profile II. There exists for this profile (indicated by arrows in the diagram) one northerly region with negative $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi}$ and one southerly region with positive $\frac{d\nabla \times \bar{u}_0 \boldsymbol{i} \cdot \boldsymbol{a}_I}{d\varphi}$. All $\frac{U}{\cos \varphi}$ being greater in the southerly region than in the northerly we obtain, applying (23.8) and criterion (23.7), that the circular vortex with the velocity profile II, is "unstable" for horizontal perturbations.

It is interesting to note that it is because of the tendency from the term $2 \, \Omega \sin \varphi$ to make the vertical component of vorticity in the initial mean flow steadily increase with latitude,

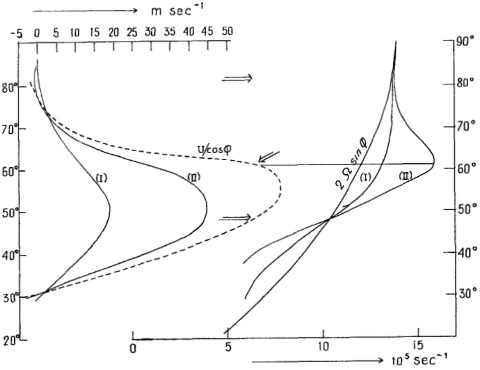

Fig. 9.

that the magnitude of the maximum of the westerly velocities must exceed a certain critical value in order to have "unstable" conditions.

24. On the stabilizing influence from the kinematics of the perturbations.

It was pointed out in the preceding sections that the transport of the vortex tubes of the additive velocity field could have no destabilizing influenzes if the additive vorticities were assumed as small. As we shall see in the following, this fact does not prevent the initial additive velocities from having stabilizing influences.

The equation of motion in the ψ-direction for the two-dimensional flow considered in section 16 can be written

$$\frac{\partial u}{\partial t} = -\frac{\partial (p + \frac{1}{2}\boldsymbol{v}^2)}{R\,\partial\psi} - v_R \nabla \times \boldsymbol{v} \cdot \boldsymbol{z}_I$$

having substituted $\frac{\partial \frac{1}{2}\boldsymbol{v}^2}{R\partial\psi} + v_R \nabla \times \boldsymbol{v} \cdot \boldsymbol{z}_I$ for the convective accelerations $\boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{i}$. Hence by averaging, noting that $\bar{v}_R = 0$,

$$(24.1) \quad \frac{\partial \bar{u}}{\partial t} = -\frac{1}{2\pi R}\int_0^{2\pi} v_R \nabla \times \boldsymbol{v} \cdot \boldsymbol{z}_I \, R d\psi = -\frac{1}{2\pi R}\int_0^{2\pi} v'_R \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I \, R d\psi.$$

Derivating the energy equation (16.2) with respect to time and using (24.1) we obtain

$$\frac{d}{dt}\int_F \tfrac{1}{2}\boldsymbol{v}'^2 \, dF = \int_F \bar{u} v'_R \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I dF.$$

Substituting here $\nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I = \frac{u'}{R} + \frac{\partial u'}{\partial R} - \frac{\partial v'_R}{R\partial\psi}$, this equation becomes

$$(24.2) \quad \frac{d}{dt}\int_F \tfrac{1}{2}\boldsymbol{v}'^2 \, dF = -\int_F \frac{d\bar{\omega}}{dR} u' v'_R R dF.$$

The corresponding equation for the two-dimensional infinite flow in a straight channel is

$$(24.3) \quad \frac{d}{dt}\int_F \tfrac{1}{2}\boldsymbol{v}'^2 \, dF = -\int_F \frac{d\bar{u}}{dy} u' v'_y \, dF.$$

In the following we suppose $\frac{d\bar{u}}{dy}$ to have a definite, say positive sign. Then, as is readily seen, the kinetic energy of the irregular flow will steadily increase or steadily decrease according as $u'v'_y$, at least on an average in the fluid, is negative or positive. $u'v'_y$ will be negative if the streamlines for $\boldsymbol{v}'$ are declined, at least on an average, in the direction of negative x, and be positive if they are declined in the opposite direction. In order to know which declination the streamlines for $\boldsymbol{v}'$ will have it suffices to know how the isolines $\nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I = \text{const}$ are declined, these being, at least on an average declined in the same way as are the streamlines.

The velocity of propagation in the direction of x of the isolines for additive vorticity being denoted by d, we have

$$d = \frac{-\dfrac{\partial \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I}{\partial t}}{\dfrac{\partial \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I}{\partial x}}.$$

From the vorticity equation we obtain, neglecting second order terms, that

$$\frac{\partial \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I}{\partial t} = -\bar{u}_0 \frac{\partial \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I}{\partial x} + v'_y \frac{d^2 \bar{u}_0}{dy^2}.$$

Substituting this in the formula for d, it follows

$$\text{(24.4)} \qquad d = \bar{u}_0 - \frac{v_y' \dfrac{d^2 \bar{u}_0}{dy^2}}{\dfrac{\partial \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I}{\partial x}}$$

Suppose now that the additive velocities, when $t = 0$, are determined from a streamfunction $\psi' = f(y) \sin \frac{2\pi x}{L}$, so that initially

$$\text{(24.5)} \qquad u' = \frac{df}{dy} \sin \frac{2\pi x}{L}, \; v_y' = -f \frac{2\pi}{L} \cos \frac{2\pi x}{L}.$$

Then the streamlines have initially the neutral form with no changes in phase with the y-coordinate, so that

$$\frac{d}{dt} \int_F \tfrac{1}{2} \boldsymbol{v}'^2 \, dF = 0, \; t = 0$$

as might be seen by substituting from (24.5) in (24.3) and effecting the integration. Whether or not the kinetic energy of the irregular flow shall later on decrease or increase will therefore depend upon how the isolines for the additive vorticities (and thereby the streamlines) become declined during the following motion. This can be examined by applying formula (24.4) for the velocity of propagation of these isolines. In doing so we suppose $f(y)$ to have the same sign for all y between the walls apart from the boundaries themselves where it is zero in consequence of the boundary conditions. This implies that there are no rectilinear streamlines in the direction of x apart from those at the boundaries. Taking the velocity circulation around the curve enclosing, say the region with positive values of the streamfunction ψ', we obtain, applying the theorem of Stokes to this circulation integral, that $\nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I$ must have, at least on an average, the same sign as ψ'. Having

$$\text{(24.6)} \qquad \nabla \times \boldsymbol{v}' \cdot \boldsymbol{z}_I = \left(\frac{4\pi^2}{L^2} f - \frac{d^2 f}{dy^2} \right) \sin \frac{2\pi x}{L},$$

$\frac{4\pi^2}{L^2} f - \frac{d^2 f}{dy^2}$ has therefore on an average between the boundaries the same sign as f. Let us consider the simple case that $\frac{4\pi^2}{L^2} f - \frac{d^2 f}{dy^2}$ has the same sign as f in all levels. Then, by substituting from (24.5), (24.6) in (24.4) it follows that

$$\text{(24.7)} \qquad d = \bar{u}_0 + h(y) \frac{d^2 \bar{u}_0}{dy^2}, \; t = 0$$

where $h(y)$ is given by

$$\text{(24.8)} \qquad h(y) = \frac{f}{\dfrac{4\pi^2}{L^2} f - \dfrac{d^2 f}{dy^2}} \geq 0 \text{ for all } y \text{ between the walls.}$$

We investigate at first a linear flow in which the velocity profile is of the stable type shown in fig. 5. It follows then from (24.7), (24.8) that the velocity d will be greater above the inflection point than below it, so that the isolines for additive vorticity, and thereby the streamlines will initially have the tendency to become declined towards increasing x. This implies, according to what was pointed out above, that the kinetic energy of the irregular flow starts decreasing at the next instants. Therefore, the stability proof given in section 21 is confirmed by the above qualitative arguments of combined kinematical and dynamical nature.

As the next example take the linear flow with the "unstable" profile shown in fig. 6. It follows now that the second term to the right in (24.7) will be negative above the inflection point and positive below it, so that this term will tend to compensate the effect from the mean velocity, $\bar{u}_0$, to give the streamlines the "stable" declination. For sufficiently small wave-lengths, however, the compensation can certainly not be complete. For, considering (24.8) we obtain

$$h(y) \to 0 \text{ when } L \to 0,$$

whence, in connection with (24.7),

$$d \to \bar{u}_0 \text{ when } L \to 0.$$

Therefore, for perturbations with sufficiently small wave-lengths, the lines of equal additive vorticity, and thereby the streamlines, will become declined in the direction of positive x, leading thus to conditions under which the kinetic energy of the irregular flow decays as for a stable flow.

If we divide eq. (24.3) by $\int_F \frac{1}{2} \boldsymbol{v}'^2 dF$, we obtain as a measure, ν of the stability:

$$\nu = -\frac{\int_F \frac{d\bar{u}_0}{dy} u'^2 \frac{v_y'}{u'} dF}{\int_F \frac{1}{2} \boldsymbol{v}'^2 dF}.$$

From reasons of continuity we can conclude that when $L \to \infty$, then on an average in $F, \frac{v'_y}{u'} \to 0$ and $u'^2 \to \boldsymbol{v}'^2$. Utilizing this in the above expression for ν, we obtain

$$\text{(24.9)} \qquad \nu \to 0 \text{ when } L \to \infty.$$

This result shows, in connection with that derived above with respect to the stability for perturbations with sufficiently small wave-lengths that if the flow reacts unstably for certain perturbations there must exist at least one intermediate wave-length with a maximum of instability.

The results derived above are all in agreement with those found by *Rayleigh* (6) for broken velocity profiles from a discussion of solutions of the linearized equations of motion. It remains, however, to be shown that unstable solutions really exist in the present case. This means that it remains to be shown that for sufficiently great wave-lengths there will be a tendency for the lines of equal additive vorticity to become declined towards decreasing x. For that purpose we try to solve the problem of finding the particular wave-length, L_c, for which d is constant: $d = \text{const} = c$. This wave-length will represent, according to our qualitative reasoning, the critical wave-length separating *the stable short waves from the unstable long waves.* In this case we may write (24.7)

$$\text{(24.10)} \quad \frac{d^2 f}{dy^2} - \left[\frac{4\pi^2}{L_c^2} + \frac{\frac{d^2(\bar{u}_0 - c)}{dy^2}}{\bar{u}_0 - c}\right] f = 0.$$

With f as a solution of this equation, the streamfunction

$$\psi' = f(y) \sin \frac{2\pi}{L_c}(x - ct)$$

represents a permanent wave-solution. It is seen that in order to avoid a singularity in the above differential equation, c cannot be equal to a value of $\bar{u}_0$ in any layer different from the layer where we have the inflection point. And further c must, as shown by *Rayleigh* (7), equal $\bar{u}_0$ in some layer between the walls. Then, the only remaining possibility is to assume $c = \bar{u}_0$ in the inflection point. With $y = 0$ in this layer (24.10) may be written

$$\frac{d^2 f}{dy^2} - \left[\frac{4\pi^2}{L_c^2} + \frac{6by + \ldots}{ay + by^3 + \ldots}\right] f = 0,$$

having developed $\bar{u}_0 - c$ in a Taylor series, $\bar{u}_0 - c = ay + by^3 + \ldots$, or it may be written

$$\text{(24.11)} \quad \frac{d^2 f}{dy^2} - \left[\frac{4\pi^2}{L_c^2} + \frac{6b + \ldots}{a + by^2 + \ldots}\right] f = 0.$$

Here the constants a, b are of opposite signs according to our assumption regarding the character of the velocity profile. The wave-length L_c is now to be found from the "eigen"-value $\frac{4\pi^2}{L_c^2}$ consistent with the boundary conditions.

We shall consider a simple case of this equation by assuming $\bar{u}_0 - c = \sin \frac{2\pi y}{H}$ and the boundaries at distances $H/2$ and $-H/2$ from the level $y = 0$. For this velocity profile, eq. (24.11) assumes the form

$$\text{(24.12)} \quad \frac{d^2 f}{dy^2} - 4\pi^2 \left(\frac{1}{L_c^2} - \frac{1}{H^2}\right) f = 0.$$

The solution of this equation is

$$f = k_1 \sin\left[2\pi y \sqrt{\frac{1}{H^2} - \frac{1}{L_c^2}}\right] + k_2 \cos\left[2\pi y \sqrt{\frac{1}{H^2} - \frac{1}{L_c}}\right.$$

This, in connection with the boundary conditions

$$f=0 \text{ for } y=\pm\frac{H}{2}$$

gives

$$L_c=\frac{H}{\sqrt{1-\frac{n^2}{4}}}$$

where n is integer. Using the condition that L must be real, we obtain for the critical wave-length

$$L_c=\frac{2H}{\sqrt{3}}. \qquad (24.13)$$

In the preceding section it was pointed out by means of energy considerations that a circular vortex with a velocity profile as shown by the curve II in fig. 9 is unstable. Applying the considerations in this section to motions in concentric spheres we obtain that the circular vortex with this velocity profile will be: (*a*) stable for wave-lengths below a certain critical wave-length, and (*b*) possess a maximum of instability for at least one wave-length. It will now be a question of considerable interest, in view of the application to the atmosphere, to determine these wave-lengths when unstable conditions exist in the atmosphere. I hope to return to this in a later paper.

25. Three-dimensional perturbations of the linear flow.

Hitherto we have derived certain criteria of stability for motions in two dimensions only. This limitation led to a simplification in the general expression (14.24) for c which is lost if three-dimensional perturbations are considered. The nature of this simplification is revealed from a study of the vorticity equation. This equation, in the two-dimensional motion, reduces to an expression for the individual conservation of vorticity: $\frac{D\nabla\times\boldsymbol{v}\cdot\boldsymbol{z}_I}{dt}=0$ in the cylindric and linear flow, and $\frac{D\nabla\times\boldsymbol{v}\cdot\boldsymbol{a}_I}{dt}=0$ for the motion on concentric spheres. In the general three-dimensional motion this equation is

$$\frac{D\nabla\times\boldsymbol{v}}{dt}=\nabla\times\boldsymbol{v}\cdot\nabla\boldsymbol{v}$$

so that the number of changes possible in the vorticity field is increased immensely. We shall now study the influence from three-dimensional perturbations upon an infinite, straight flow with a linear distribution of velocity:

$$\bar{u}_0=\frac{d\bar{u}_0}{dy}y.$$

By two-dimensional perturbations in the x, y-planes this flow is certainly not unstable. For since $\frac{d\bar{u}_0}{dy}$ is a space constant and the vorticity $\nabla\times\boldsymbol{v}\cdot\boldsymbol{z}_I$ is individually conserved, no changes in the vorticity field can arise from the transport of the vorticities $\frac{d\bar{u}_0}{dy}$. So, the changes in the vorticity field, and thereby in the velocities, are due to the transport of the initial additive vorticities only and thus become arbitrarily small with $\nabla\times\boldsymbol{v}_0'$. Having now $\frac{d^2\bar{u}_0}{dy^2}=0$, the expression (14.29), assuming small displacements ϱ_y, and small additive vorticities initially, reduces to

$$\bar{u}=\bar{u}_0-\tfrac{1}{2}\frac{d\bar{u}_0}{dy}\varrho_y\frac{\partial\bar{\varrho}_z}{\partial z}. \qquad (25.1)$$

Suppose the system of small virtual displacements to be given by

$$\begin{aligned}\varrho_x&=\varepsilon\lambda_1\sin\lambda_1x\cos\lambda_2y\cos\lambda_3z\\ \varrho_y&=\varepsilon\lambda_2\cos\lambda_1x\sin\lambda_2y\cos\lambda_3z\\ \varrho_z&=-\varepsilon\frac{\lambda_1^2+\lambda_2^2}{\lambda_3}\cos\lambda_1x\cos\lambda_2y\sin\lambda_3z\end{aligned} \qquad (25.2)$$

where λ_2 and λ_3 are given by

$$\lambda_2=\frac{\pi}{H},\ \lambda_3=\frac{\pi}{B},$$

H and B denoting the widths of the channel in the y- and z-direction, respectively. This system of displacements satisfies the condition of incompressibility and the boundary conditions. Substituting from (25.2) in (25.1) we obtain

$$\bar{u}-\bar{u}_0=k\frac{d\bar{u}_0}{dy}\overline{\cos^2\lambda_1x}\cos\frac{\pi z}{B}\sin\frac{2\pi y}{H}. \qquad (25.3)$$

where k a positive constant.

The integral of this expression in the y-direction between $y=0$ and $y=H$ vanishes for all z, so that also $\int_\tau(\bar{u}-\bar{u}_0)\,d\tau$ will vanish. The condition of constant total momentum, $\int_\tau(\bar{u}-\bar{u}_0)d\tau=0$, is therefore satisfied for the displacements