Table of symbols.

The numbers indicate the pages where the symbols are defined.

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B 25
c_p specific heat at const. pressure 5
D 13
D_r, D_θ 7
$\frac{\partial}{\partial t}$ $\frac{d}{dt}$ $\frac{\delta}{\delta t}$ differential symbols 6
G_θ, G_r 7
g gravity acceleration 6
H 13
$h = r - r_0$ 24
n an integer 4
P, $\bar{p}$ pressure 5
P^*, p^* 5
$p = p^{(i)}$ 8
R gas constant 5
R_m, $\bar{R}_m$ 16
r distance from earth's centre 4
r_0 radius of spherical globe 9
T abs. temperature 5
t time 4
$u = u^{(i)}$ 8
V_ψ, v_ψ, V_θ, v_θ angular velocities 5
V_r, v_r vertical velocities 5
$v = v^{(i)}$ 8
$\boldsymbol{V} \cdot \nabla$, $\boldsymbol{v} \cdot \nabla$ differential symbols 6
$w = w^{(i)}$ 8
$\frac{dW}{dt}$ heat supplied per unit mass and unit time 5
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α 12
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δ_a, δ_b, δ_c .... 12—13
$\frac{\delta}{\delta t}$ differential symbol 6
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$\varepsilon = \varepsilon^{(i)}$ 8
η_r, η_ψ, η_p, η_τ, η_θ, η_ε phase angles 7
θ angular distance from North Pole 4
$\varkappa$ 5
λ geographical longitude 5
ν 7
ψ coordinate 4
ϱ_τ, ϱ_ε, ϱ_p 32
σ_r, σ_θ 7
τ_s variation of temperature 5
$\tau = \tau^{(i)}$ 8
Ω angular velocity of earth's rotation 7

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SOME HYDRODYNAMICAL PROPERTIES OF SIMPLE ATMOSPHERIC OSCILLATIONS

WITH APPLICATIONS TO THE SEMIDIURNAL OSCILLATION

BY

P. THRANE

(Manuscript received December 21th, 1949.)

Summary.

The linearized hydrodynamical equations, including the energy equation, are specialized to apply to certain types of simple oscillations which resemble tidal waves but are created by periodic supply and removal of heat. It is shown that the variation in the supplied heat can be computed if the pressure wave is known; furthermore, if the phase angle of the pressure wave at sea level is independent of latitude, and if the changes of state are piezotropic (including adiabatic and isothermal changes), the phase angle must be independent of height. In this connection it is noted that the phase angle of the atmospheric semidiurnal pressure wave is independent of latitude but variable with height.

For a barotropic model atmosphere it is shown that a simple form of oscillation with a 24-hour period, which was considered by *Margules* and others and which satisfies the linearized equations mentioned above, is in disaccord with the exact hydrodynamical equations. Bearing this in mind and taking into account also the boundary condition at the ground, an examination is made of the properties of oscillations having either a 24-hour or a 12-hour period. Owing to the mathematical complexity of the problem this examination is limited to a region near the Pole. The existence of a fundamental difference in the hydrodynamical properties of the two oscillations is indicated, quite apart from possible resonance effects.

The quasistatic equations of Laplace are not applied here. An example is presented to show that the quasistatic method may lead to erroneous results if applied to the 12-hour oscillation.

An attempt is made at applying non-linear perturbation equations containing terms of first and second order to derive certain oscillation properties which are not revealed by the linearized equations.

Through numerical integrations, approximate values of the variation of the supplied heat and of the vertical velocity in the semidiurnal oscillation are obtained, the pressure wave being known from observations. The possibility of explaining this wave as a non-resonant oscillation resulting from thermal processes is discussed. The flux of heat through a model atmosphere, having no other motion than that resulting from the combined diurnal and semidiurnal oscillation, is considered, and a possible explanation of the preponderance of the semidiurnal component is indicated. A brief discussion of the conversion of heat to work and *vice versa* is given.

For convenience certain intermediate computations are reproduced in an appendix.

Introduction.

The aim of this paper is to study the properties of certain simple atmospheric oscillations which may be expressed in the form

$$(1) \qquad f(r, \theta, \psi, t) = A \sin (n\psi + \gamma t + \eta)$$

or as a sum of such expressions, f denoting the variation (i. e. the deviation from a mean value). of any of the significant variables, pressure, t m-perature, velocity, etc. The other symbols are defined as follows:

r distance from the centre of the earth,
θ angular distance from the North Pole,
ψ the angle between a fixed meridian plane and the meridian plane through the point considered,
t time,
n an integer,
γ a constant.

The amplitude A and the phase angle η are functions of r and θ but independent of ψ and t.

Regarding atmospheric oscillations of this kind fundamental investigations were made by *Margules*, following methods developed by *Laplace*. In more recent years important contributions have been rendered by *Lamb*, *Simpson*, *Chapman*, *Taylor*, *Pekeris*, *Weekes* and *Wilkes*, *J. Bjerknes*.

In all mathematical theories of atmospheric oscillations based on the hydrodynamical equations one has to work with simple models of the atmosphere, as for instance the model considered by Taylor (1936)[1]). The atmospheric model considered in the present paper has the following characteristics:

(1) The ellipticity of the figure of the earth is neglected.
(2) The effects of viscosity and friction are omitted.
(3) The fundamental state of motion is assumed to be a zonal current in which each unit mass of air moves with a constant angular velocity along a circle of latitude, while the angular velocity may vary with latitude and elevation.[2])

[1]) A frictionless atmosphere where the temperature distribution was assumed to be independent of latitude and the changes of state in the perturbations to be adiabatic.

[2]) In some of the problems treated here the variation of angular velocity will be taken into account, but mostly we shall assume a constant angular velocity in order to simplify the mathematical formulae. However, an extension of the formulae to the general case would encounter no difficulties of a principal kind.

(4) The oscillations are assumed to be small perturbations superposed on the fundamental current, thus permitting the linearized hydrodynamical equations to be applied.
(5) The gravitational effects of the sun and the moon are not taken into account, the only external effect considered being the heat supplied to the atmosphere by the sun.
(6) In accordance with assumption (5) the oscillations are supposed to undergo non-adiabatic changes of state in certain layers of the atmosphere.
(7) The sun is supposed to be in its equinoctial position and the perturbations to be symmetrical with respect to the equator.

Assumption (2) is supported by the result of *Chapman* (1924). With regard to assumption (4) it may be noted that the equations applied here are not the quasistatic ones, but the terms of vertical acceleration are taken into account.[1])

In connection with assumption (5) it may be observed that the gravitational effect of the sun may play an important part, owing to the resonance effect, if the atmosphere has a period of free oscillations very near to 12 hours. In the frictionless model atmosphere we should then find an oscillation with infinite amplitude. This case cannot be given a satisfactory mathematical treatment without including frictional terms in the hydrodynamical equations. Apart from the case of resonance, which will not be treated here, the tidal waves may be treated separately and superposed on other oscillations.

As regards assumption (6) we shall make the following remarks. Considering a certain oscillation of the form (1) found in the atmosphere, and supposing that the atmospheric model described above can be used with fairly good approximation so far as this oscillation is concerned, we may introduce in the energy equation (v. (1.5) below) the actual observed values of temperature, pressure, velocity and their variations. We thus compute the heat supplied to a unit mass of air per unit time, $\frac{dW}{dt}$, and the variation $\Delta\left(\frac{dW}{dt}\right) = \varepsilon_s$ of this quantity. Since in the atmospheric model considered each parcel

[1]) This method was introduced by *Solberg* (1936).

of air has constant pressure and temperature in the fundamental state of motion, we have $\frac{dW}{dt} = 0$. In oscillations where the air parcels undergo adiabatic changes of state, we find $\varepsilon_s = 0$, but in oscillations generated by the suns radiation we must expect to find ε_s different from zero, at least in certain layers of the atmosphere, and in such cases ε_s will have the form (1). This being the case, it seems convenient to treat ε_s on an equal footing with the other variables (pressure, temperature etc.). We can thus obtain a system of formulae applicable both to adiabatic and non-adiabatic oscillations, including oscillations which are adiabatic in some layers of the atmosphere and non-adiabatic in other layers.

For the purpose of this investigation the variation in the supplied heat may be considered as a given quantity, thus making it unnecessary to refer to the laws of radiation and conduction of heat.

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1. The general hydrodynamical equations applied to a rotating atmosphere.

As we shall be concerned chiefly with waves that travel quickly relative to the surface of the earth, but very slowly relative to the stellar system, it is convenient to introduce, instead of the geographical longitude λ, the coordinate

$$\psi = \lambda + \Omega t$$

where ψ denotes the angle between a meridional plane through a fixed star and the meridional plane through the point considered, and Ω denotes the angular velocity of the earth's rotation.

Denoting by $\frac{d}{dt}$ the time differentiation following the motion of an individual parcel of air we shall write

$$\frac{dr}{dt} = V_r, \quad \frac{d\psi}{dt} = V_\psi, \quad \frac{d\theta}{dt} = V_\theta,$$

The symbols used to describe the state and motion of the atmosphere are given in Table 1. (The last column of this table will be referred to in the next paragraph).

Table 1.

Element	Fundamental state	State of perturbation
Velocity components	V_r	$V_r + v_r$
(as defined above)	V_ψ	$V_\psi + v_\psi$
	V_θ	$V_\theta + v_\theta$
Absolute temperature	T	$T + \tau_s$
Density	Q	$Q + q$
Pressure	P	$P + p$
Auxiliary quantity	$P^* = R \ln P$	$P^* + p^*$
Heat supplied to a unit mass per unit time	$\frac{dW}{dt}$	$\frac{dW}{dt} + \varepsilon_s$

Other symbols used are

R gas constant,

c_p specific heat at constant pressure,

$\varkappa = \frac{c_p}{c_p - R}$,

$\delta = -\frac{\partial T}{\partial r}$ vertical temperature gradient,

Φ potential of exterior force.

With the above notations the general equations of motion, the equation of continuity and the energy equation may be written

$$\frac{dV_r}{dt} - rV_\theta^2 - r\sin^2\theta V_\psi^2 + \frac{1}{Q}\frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} = 0, \tag{1.1}$$

$$r^2\sin^2\theta\frac{dV_\psi}{dt} + 2r\sin^2\theta\, V_r V_\psi + 2r^2\sin\theta\cos\theta V_\psi V_\theta + \frac{1}{Q}\frac{\partial P}{\partial \psi} + \frac{\partial \Phi}{\partial \psi} = 0, \tag{1.2}$$

$$r^2\frac{dV_\theta}{dt} + 2rV_rV_\theta - r^2\sin\theta\cos\theta\, V_\psi^2 + \frac{1}{Q}\frac{\partial P}{\partial \theta} + \frac{\partial \Phi}{\partial \theta} = 0, \tag{1.3}$$

$$\frac{1}{Q}\frac{dQ}{dt} + \frac{2}{r}V_r + \frac{\partial V_r}{\partial r} + \frac{\partial V_\psi}{\partial \psi} + \frac{\partial V_\theta}{\partial \theta} + \operatorname{cotg}\theta \cdot V_\theta = 0, \tag{1.4}$$

$$c_p\frac{dT}{dt} - \frac{RT}{P}\frac{dP}{dt} - \frac{dW}{dt} = 0. \tag{1.5}$$

Using the equation of state

$$P = QRT$$

the density may be eliminated from Eqs. (1.1—5) by putting

$$(1.6)\quad \begin{aligned} &\frac{1}{Q}\frac{\partial P}{\partial r} = T\frac{\partial P^*}{\partial r}, \quad \frac{1}{Q}\frac{\partial P}{\partial \psi} = T\frac{\partial P^*}{\partial \psi}, \\ &\frac{1}{Q}\frac{\partial P}{\partial \theta} = T\frac{\partial P^*}{\partial \theta}, \quad \frac{1}{Q}\frac{dQ}{dt} = \frac{1}{R}\frac{dP^*}{dt} - \frac{1}{T}\frac{dT}{dt}. \end{aligned}$$

The equations (1.1—5) will be applied to an atmosphere surrounding a rotating globe. To be exact one should take into account the ellipticity of the figure of the earth. How this can be achieved has been shown by *H. Solberg* (1936) for the case of a homogeneous ocean. The mathematical computations involved are, however, rather difficult and we shall therefore in the present paper consider the rotating globe as a sphere. In making this simplification we can still take account of the essential features of the effect of the earth's rotation upon the fundamental state of motion. This is done by a suitable choice of the potential Φ, which must be different from the real potential of gravitation.

Let us choose for Φ the following expression

$$(1.7)\qquad \Phi = \tfrac{1}{2}\, r^2\, \bar{\Omega}^2 \sin^2\theta - \frac{kM_e}{r},$$

where k is the "gravitation constant", M_e the mass of the earth and $\bar{\Omega} = \bar{\Omega}(r)$ is a function which is equal to Ω near the earth's surface and tends rapidly towards zero at great distance from the earth in such a manner that the last term in Eq. (1.7) preponderates for large values of r.

If the equations (1.1—5) had to be applied at all distances from the earth it would be necessary to make certain assumptions as to the value of $\bar{\Omega}(r)$ and $\frac{d\bar{\Omega}}{dr} \neq 0$ for all values of r but if we limit our investigation to the troposphere and the lower stratosphere we may put $\bar{\Omega} = \Omega = \text{const.}$

Let us apply Eqs. (1.1—3) to the case when the atmosphere is at rest, i.e.

$$V_r = 0,\ \ V_\theta = 0\ \ V_\psi = \Omega.$$

Introducing the value (1.7) for Φ, putting $g = \frac{kM_e}{r^2}$ and $\bar{\Omega} = \Omega$ and having regard to (1.6) we then obtain

$$(1.8)\qquad \frac{\partial P^*}{\partial \psi} = 0, \quad \frac{\partial P^*}{\partial \theta} = 0, \quad \frac{\partial P^*}{\partial r} = -\frac{g}{T},$$

i.e. the isobaric surfaces are parallell to the earth's surface. In choosing Φ as shown above we thus obtain dynamical conditions quite similar to those found on the real globe. It should be noted, however, that g as defined above is independent of latitude whereas the real acceleration of gravity shows a slight variation with latitude.[1])

2. The equations of perturbation in the case of a zonal current.

In the general state of perturbation the significant variables may be written in the form shown in the last column of Table 1, where the variations $v_r, v_\psi \ldots \varepsilon_s$ are assumed to be so small that products of such variations may be neglected.

It will be convenient in many cases to introduce the differential operator

$$(2.1)\qquad \boldsymbol{V}\cdot\nabla = V_r\frac{\partial}{\partial r} + V_\psi\frac{\partial}{\partial \psi} + V_\theta\frac{\partial}{\partial \theta}\ ^{2)}$$

and the corresponding operator $\boldsymbol{v}\cdot\nabla$.

Further we shall henceforth distinguish three different time differentiations defined in the following manner

(a) $\frac{\partial}{\partial t}$ denotes a differentiation where ψ, r, θ are kept constant.

(2.2) (b) $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla$

(c) $\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + (\boldsymbol{V} + \boldsymbol{v})\cdot\nabla$

In the zonal current we have by definition

$$(2.3)\qquad V_r = 0,\ V_\theta = 0,\ V_\psi = V_\psi(r, \theta),$$

[1]) The above method of disregarding the ellipticity of the earth is well known, but it is not usually made clear that the method consists in a suitable choice of Φ which differs from the real gravitation potential.

[2]) V. Bjerknes (1929) p. 41.

whence it follows that, if F denotes any one of the variables listed in the first column of Table 1, we then have

$$(2.4)\qquad \frac{\partial F}{\partial \psi} = 0, \quad \frac{\partial F}{\partial t} = 0, \quad \frac{dF}{dt} = 0.$$

It is seen from the energy equation that we have $\frac{dW}{dt} = 0$.

In accordance with the general method developed by *V. Bjerknes* (1929) we proceed as follows: After substituting the expressions (1.6) in the equations (1.1—5) we compute the first variation of each term in these equations. In the expressions thus obtained we finally introduce the conditions (2.3) and (2.4). In order to simplify the form of the equations, the following symbols are introduced:

$$D_r = \frac{1}{R}\frac{\partial P^*}{\partial r} - \frac{1}{T}\frac{\partial T}{\partial r}, \quad D_\theta = \frac{1}{R}\frac{\partial P^*}{\partial \theta} - \frac{1}{T}\frac{\partial T}{\partial \theta},$$

$$(2.5)\quad G_r = c_p\frac{\partial T}{\partial r} - T\frac{\partial P^*}{\partial r}, \quad G_\theta = c_p\frac{\partial T}{\partial \theta} - T\frac{\partial P^*}{\partial \theta},$$

$$\sigma_r = \frac{\partial V_\psi}{\partial r} + \frac{2}{r}V_\psi, \qquad \sigma_\theta = \frac{\partial V_\psi}{\partial \theta} + 2\,\mathrm{cotg}\,\theta\, V_\psi,$$

and the equations of perturbation may then be written as follows.[1])

$$(2.6)\quad \text{I} \qquad \frac{dv_r}{dt} - 2V_\psi r\sin^2\theta\, v_\psi + T\frac{\partial p^*}{\partial r} + \frac{\partial P^*}{\partial r}\tau_s = 0,$$

$$\text{II} \qquad \frac{dv_\psi}{dt} + \sigma_\theta v_\theta + \sigma_r v_r + \frac{T}{r^2\sin^2\theta}\frac{\partial p^*}{\partial \psi} = 0,$$

$$\text{III} \qquad \frac{dv_\theta}{dt} - 2\sin\theta\cos\theta\, V_\psi v_\psi + \frac{T}{r^2}\frac{\partial p^*}{\partial \theta} + \frac{1}{r^2}\frac{\partial P^*}{\partial \theta}\tau_s = 0,$$

$$\text{IV} \qquad \frac{1}{R}\frac{dp^*}{dt} - \frac{1}{T}\frac{d\tau_s}{dt} + \left(D_r + \frac{2}{r}\right)v_r + (D_\theta + \mathrm{cotg}\,\theta)\, v_\theta + \frac{\partial v_r}{\partial r} + \frac{\partial v_\psi}{\partial \psi} + \frac{\partial v_\theta}{\partial \theta} = 0,$$

$$\text{V} \qquad c_p\frac{d\tau_s}{dt} - T\frac{dp^*}{dt} + G_r v_r + G_\theta v_\theta - \varepsilon_s = 0.$$

Since all products of variations have been neglected, the equations are linear. The second-order terms are listed in Table 2, as some of them will be referred to in later paragraphs.

Table 2.

Terms of the second order which have been disregarded in deducing the equations (2.6). Each horizontal row has the number of the equation to which it belongs.

$$\text{I} \qquad \boldsymbol{v}\cdot\nabla v_r - rv_\theta^2 - r\sin^2\theta\, v_\psi^2 + \tau_s\frac{\partial p^*}{\partial r}.$$

$$\text{II} \qquad \boldsymbol{v}\cdot\nabla v_\psi + \frac{2}{r}v_r v_\psi + 2\,\mathrm{cotg}\,\theta\, v_\theta v_\psi + \frac{\tau_s}{r^2\sin^2\theta}\frac{\partial p^*}{\partial \psi}.$$

$$\text{III} \qquad \boldsymbol{v}\cdot\nabla v_\theta + \frac{2}{r}v_r v_\theta - \sin\theta\cos\theta\, v_\psi^2 + \frac{\tau_s}{r^2}\frac{\partial p^*}{\partial \theta}.$$

$$\text{IV} \qquad \frac{1}{R}\boldsymbol{v}\cdot\nabla p^* - \frac{1}{T}\boldsymbol{v}\cdot\nabla\tau_s + \frac{\tau_s}{T^2}\frac{d\tau_s}{dt} + \frac{1}{T^2}\frac{\partial T}{\partial r}\tau_s v_r + \frac{1}{T^2}\frac{\partial T}{\partial \theta}\tau_s v_\theta.$$

$$\text{V} \qquad c_p\boldsymbol{v}\cdot\nabla\tau_s - T\boldsymbol{v}\cdot\nabla p^* - \tau_s\frac{dp^*}{dt} - \frac{\partial P^*}{\partial r}\tau_s v_r - \frac{\partial P^*}{\partial \theta}\tau_s v_\theta.$$

The coefficients of Eqs. (2.6) are functions of r and θ but independent of t and ψ. The equations are valid for any perturbation in the zonal current, but we shall here be concerned with a special simple form of perturbation. Writing for brevity

$$(2.7)\qquad \nu = n\psi + \gamma t$$

where n is an integer or zero and γ a constant, we consider the following functions

$$(2.8)\quad \begin{aligned} p^* &= A_p\sin(\nu + \eta_p), & v_r &= A_r\cos(\nu + \eta_r), \\ \tau_s &= A_\tau\sin(\nu + \eta_\tau), & v_\theta &= A_\theta\cos(\nu + \eta_\theta), \\ v_\psi &= A_\psi\sin(\nu + \eta_\psi), & \varepsilon_s &= A_\varepsilon\cos(\nu + \eta_\varepsilon), \end{aligned}$$

[1]) A few details of the computations are given in the appendix.

where the amplitudes $A_p, A_\tau \dots A_\varepsilon$ and the phase angles $\eta_p, \eta_\tau \dots \eta_\varepsilon$ are functions of r and θ only. A perturbation of the form (2.8) will be called a "simple cscillation".

In order to show how the functions (2.8) can be made to satisfy the system (2.6) we write each of them as a sum of two functions in the following manner:

$$
(2.9)\qquad
\begin{aligned}
p^* &= p^{(1)} \sin \nu + p^{(2)} \sin\left(\nu + \frac{\pi}{2}\right) & v_r &= w^{(1)} \cos \nu + w^{(2)} \cos\left(\nu + \frac{\pi}{2}\right)\\
\tau_s &= \tau^{(1)} \sin \nu + \tau^{(2)} \sin\left(\nu + \frac{\pi}{2}\right) & v_\theta &= u^{(1)} \cos \nu + u^{(2)} \cos\left(\nu + \frac{\pi}{2}\right)\\
v_\psi &= v^{(1)} \sin \nu + v^{(2)} \sin\left(\nu + \frac{\pi}{2}\right) & \varepsilon_s &= \varepsilon^{(1)} \cos \nu + \varepsilon^{(2)} \cos\left(\nu + \frac{\pi}{2}\right)
\end{aligned}
$$

thus introducing 12 auxiliary variables

$$p^{(1)}, \tau^{(1)}, \dots \varepsilon^{(1)} \qquad p^{(2)}, \tau^{(2)}, \dots \varepsilon^{(2)}$$

which are functiors of r and θ but independent of ψ and t. It is immediately apparent that we have

$$
(2.10)\qquad
\begin{aligned}
p^{(1)} &= A_p \cos \eta_p & p^{(2)} &= A_p \sin \eta_p\\
\tau^{(1)} &= A_\tau \cos \eta_\tau & \tau^{(2)} &= A_\tau \sin \eta_p\\
&\vdots & &\vdots\\
\varepsilon^{(1)} &= A_\varepsilon \cos \eta_\varepsilon & \varepsilon^{(2)} &= A_\varepsilon \sin \eta_\varepsilon
\end{aligned}
$$

$$
(2.11)\qquad \operatorname{tg} \eta_p = \frac{p^{(2)}}{p^{(1)}} \qquad A_p = \sqrt{p^{(1)2} + p^{(2)2}}
$$

and similar expressions for the other phase angles and amplitudes.

Introducing the expressions (2.9) in Eqs. (2.6) we may write the first of these equations in the following form

$$
(2.12)\qquad S^{(1)} \sin \nu + S^{(2)} \sin\left(\nu + \frac{\pi}{2}\right) = 0
$$

where $S^{(1)}$ is an expression containing only the variables $p^{(1)}, \tau^{(1)} \dots \varepsilon^{(1)}$ and their derivates, and $S^{(2)}$ is obtained by substituting in $S^{(1)}$, $p^{(2)}, \tau^{(2)} \dots \varepsilon^{(2)}$ for $p^{(1)}, \tau^{(1)} \dots \varepsilon^{(1)}$. As (2.12) must hold for all values of ν we have

$$S^{(1)} = 0,\ S^{(2)} = 0.$$

Applying this procedure to each of the equations (2.6) we obtain 5 equations for the six variables

$$p^{(i)}\ \tau^{(i)} \dots \varepsilon^{(i)}\ (i = 1 \text{ or } 2).$$

In order to have simple symbols that are convenient for computations we shall usually drop the index (i) in the subsequent formulae.[1])

Introducing for brevity the symbol

$$
(2.13)\qquad \gamma_n = \gamma + n V_\psi,
$$

the five equations for $p^{(i)}\ \tau^{(i)} \dots \varepsilon^{(i)}$ may be written as follows

$$
(2.14)\qquad
\begin{aligned}
&\text{I} & -\gamma_n w - 2r V_\psi \sin^2\theta \cdot v + T\frac{\partial p}{\partial r} + \frac{\partial P^*}{\partial r}\tau &= 0,\\
&\text{II} & \gamma_n v + \sigma_r w + \sigma_\theta u + \frac{nT}{r^2 \sin^2\theta} p &= 0,\\
&\text{III} & -\gamma_n u - 2V_\psi \sin\theta\cos\theta \cdot v + \frac{T}{r^2}\frac{\partial p}{\partial\theta} + \frac{1}{r^2}\frac{\partial P^*}{\partial\theta}\tau &= 0,\\
&\text{IV} & \frac{\gamma_n}{R}p - \frac{\gamma_n}{T}\tau + \left(D_r + \frac{2}{r}\right)w + (D_\theta + \operatorname{cotg}\theta)\,u + \frac{\partial w}{\partial r} + \frac{\partial u}{\partial\theta} + nv &= 0,\\
&\text{V} & \gamma_n c_p \tau - \gamma_n T p + G_r w + G_\theta u - \varepsilon &= 0.
\end{aligned}
$$

Here r and θ are the only independent variables.

If $p^{(1)}, \tau^{(1)} \dots \varepsilon^{(1)}$ and $p^{(2)}, \tau^{(2)} \dots \varepsilon^{(2)}$ be two different sets of functions, each of them satisfying Eqs. (2.14), then Eq. (2.11) and the corresponding formulae for the other variables determine amplitudes and phase angles so as to make the functions (2.8) satisfy Eqs. (2.6).

[1]) It is important to bear in mind that p stands for the auxiliary quantity $p^{(i)}$, whereas the pressure itself has been denoted by $\overline{p}$ (Table 1). The connection is given by $p^* = p^{(1)} \sin \nu + p^{(2)} \sin\left(\nu + \frac{\pi}{2}\right)$ and $p^* = R\,\frac{p}{P}$

3. Some fundamental properties of the simple oscillation.

Eqs. (2.14) together with the boundary condition, which is to be examined presently, determine the properties of the oscillation (2.8), i. e. the relation between pressure, temperature, velocity and the heat supplied. So far we have made no assumption as to which of the variables have to be considered as known or unknown quantities.

In the atmosphere the pressure variations can be observed with greater accuracy than the other variables. It is therefore important to note that if, in a simple oscillation, the pressure is known, the other variables, $\boldsymbol{v}$, τ_s and ε_s, are then completely determined and can be computed. This will appear from a relation which we shall now establish between the auxiliary quantities p and ε which represent the variation of the pressure and the supplied heat.

Solving the equations (2.14 I, II, III and V) with respect to u, v, w, τ we find

$$(3.1)\qquad \begin{aligned} w &= ap + b\frac{\partial p}{\partial r} + c\frac{\partial p}{\partial \theta} + d\cdot\varepsilon, \\ u &= a_1 p + b_1\frac{\partial p}{\partial r} + c_1\frac{\partial p}{\partial \theta} + d_1\cdot\varepsilon, \end{aligned}$$

and similar expressions for τ and v; a, b, ... a_1, b_1 ... being known functions of r and θ. Computing $\frac{\partial w}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ from (3.1) and introducing the expressions thus obtained in Eq. (2.14, IV), we arrive at an equation of the following form:

$$(3.2)\qquad \begin{aligned} &A_{11}\frac{\partial^2 p}{\partial r^2} + 2A_{12}\frac{\partial^2 p}{\partial r\,\partial\theta} + A_{22}\frac{\partial^2 p}{\partial\theta^2} + A_1\frac{\partial p}{\partial r} \\ &+ A_2\frac{\partial p}{\partial\theta} + A_0 p + B_1\frac{\partial\varepsilon}{\partial r} + B_2\frac{\partial\varepsilon}{\partial\theta} + B_0\varepsilon = 0. \end{aligned}$$

The coefficients A_{11} .... B_0 can be computed by means of the coefficients of Eqs. (2.14) and can thus be considered as given functions of r and θ. Eq. (3.2) is the relation sought between p and ε. We shall now combine it with the boundary condition at the earth's surface ($r = r_0$). From $v_r = 0$ it follows that $w^{(i)} = 0$. Referring to Eqs. (3.1) we can thus write the boundary condition in the form

$$(3.3)\qquad \left\{ap + b\frac{\partial p}{\partial r} + c\frac{\partial p}{\partial\theta} + d\cdot\varepsilon\right\}_{r=r_0} = 0.$$

Suppose now that the pressure oscillation p^* in Eqs. (2.8) is known from observations at all heights from the surface upwards to a certain altitude. $p^{(1)}$ and $p^{(2)}$ are then given by Eqs. (2.10). Knowing $p^{(i)}$ we can compute $(\varepsilon^{(i)})_{r=r_0}$ from Eq. (3.3) and simultaneously (3.2) expresses $\varepsilon^{(i)}$ through a linear partial differential equation of the first order. According to the theory of such equations there exists a single-valued integral $\varepsilon^{(i)}(r, \theta)$ which for $r = r_0$ assumes a prescribed value. The latter is in this case the value determined by Eq. (3.3).

Knowing $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ we find A_ε and η_ε, and ε_s in Eq. (2.8) is thereby determined. From (3.1) and the corresponding formulae we can finally compute temperature and velocity.

The result obtained can be extended to oscillations of a more general kind. Let us consider a more or less irregular perturbation, where the variables are periodical functions of ψ and t. They can then generally be expanded in double Fourier series, e. g.

$$(3.4)\qquad p^* = \sum_{n=0}^{n=\infty}\sum_{\mu=0}^{\mu=\infty}\left\{p^{(1)}{}_{n,\mu}\sin(n\psi + \mu\gamma t) + p^{(2)}{}_{n,\mu}\sin\left(n\psi + \mu\gamma t + \frac{\pi}{2}\right)\right\},$$

$$(3.5)\qquad \varepsilon_s = \sum_{n=0}^{n=\infty}\sum_{\mu=0}^{\mu=\infty}\left\{\varepsilon^{(1)}{}_{n,\mu}\cos(n\psi + \mu\gamma t) + \varepsilon^{(2)}{}_{n,\mu}\cos\left(n\psi + \mu\gamma t + \frac{\pi}{2}\right)\right\},$$

$p^{(i)}{}_{n,\mu}$, $\varepsilon^{(i)}{}_{n,\mu}$ .. being functions of r and θ. Introducing the above expressions, together with the similar expansions of $\boldsymbol{v}$ and τ_s, in Eq.s. (2.6) the left-hand side of each of these equations also assumes the form of a double Fourier series. The latter being equal to zero for all values of ψ and t, each of its coefficients must vanish and this leads to equations of the form (2.14) with the variables $p^{(i)}{}_{n,\mu}$ ... $\varepsilon^{(i)}{}_{n,\mu}$. In this way we come back to the simple oscillation examined above. To each term $p_{n,\mu}\sin(n\psi + \mu\gamma t)$ of Eq. (3.4) corresponds a term $\varepsilon_{n,\mu}\cos(n\psi + \mu\gamma t)$, and thus ε_s of Eq. (3.5) is uniquely determined when p^* is known.

The possibility of computing ε_s when p^* is given has a certain practical interest since ε_s is not directly observable. A theoretically more interesting but also more difficult problem is to determine p^* when ε_s is considered as a given quantity. This problem is related to questions concerning the cause of some of the simple oscillations found in the atmosphere, since ε_s depends directly upon radiation from the sun.

It is immediately apparent that the latter problem is essentially different from the former, for Eq. (3.2) is of first order in ε but of second order in p. The boundary condition at the ground combined with Eq. (3.2) determines ε when p is given, but in order to determine p when ε is given further conditions must be specified. Such conditions can be found by examining the properties of the simple oscillations at the Pole, and this will be discussed in a later paragraph.

We shall now prove the following proposition:

If the phase angle η_p of the pressure oscillation is independent of latitude for $r = r_0$ and if the air particles are subject to adiabatic changes of state, then η_p is also independent of height in a region extending from the ground upwards to a certain altitude.

Suppose that $p^{(1)}$ and $p^{(2)}$ can be expanded in Taylor series

$$\text{(3.6)} \quad \begin{aligned} &\text{(a)}\ p^{(1)} = p^{(1)}_0 + \left(\frac{\partial p^{(1)}}{\partial r}\right)_0 (r - r_0) + \tfrac{1}{2}\left(\frac{\partial^2 p^{(1)}}{\partial r^2}\right)_0 (r - r_0)^2 + \dots \\ &\text{(b)}\ p^{(2)} = p^{(2)}_0 + \left(\frac{\partial p^{(2)}}{\partial r}\right)_0 (r - r_0) + \tfrac{1}{2}\left(\frac{\partial^2 p^{(2)}}{\partial r^2}\right)_0 (r - r_0)^2 + \dots \end{aligned}$$

the subscript nought denoting the values of the variables at the ground ($r = r_0$) and the coefficients of the expansions thus being functions of θ alone, then the above proposition may be formulated as follows:

If $\left(\frac{\partial \eta_p}{\partial \theta}\right)_0 = 0$ and $\varepsilon_s = 0$, then $\frac{\partial \eta_p}{\partial r} = 0$ in the region where Eqs. (3.6) are valid.

Taking account of Eq. (2.11) this can also be expressed as follows

If $p^{(2)}_0 = kp^{(1)}_0$ (k constant) and $\varepsilon_s = 0$, then

$$\text{(3.7)} \qquad p^{(2)} = kp^{(1)}$$

for all values of r in the region considered.

To prove this it is obviously sufficient to show that

$$\text{(3.8)} \qquad \left(\frac{\partial^m p^{(2)}}{\partial r^m}\right)_0 = k\left(\frac{\partial^m p^{(1)}}{\partial r^m}\right)_0,$$

where m denotes any positive integer.

From the relation $p^{(2)}_0 = kp^{(1)}_0$ it follows that $\frac{\partial p^{(2)}_0}{\partial \theta} = k\frac{\partial p^{(1)}_0}{\partial \theta}$. Recalling that Eq. (3.3) stands for two equations with indices $i = 1$, $i = 2$, and putting $\varepsilon^{(i)} = 0$ and $r = r_0$ it is easily seen that

$$\left(\frac{\partial p^{(2)}}{\partial r}\right)_0 = k\left(\frac{\partial p^{(1)}}{\partial r}\right)_0$$

and therefore also

$$\left(\frac{\partial^2 p^{(2)}}{\partial r\, \partial \theta}\right)_0 = k\left(\frac{\partial^2 p^{(1)}}{\partial r\, \partial \theta}\right)_0.$$

Next, writing Eq. (3.2) with $i = 1$ and with $i = 2$, putting $\varepsilon^{(i)} = 0$, $r = r_0$ and comparing these two equations, it is seen that

$$\left(\frac{\partial^2 p^{(2)}}{\partial r^2}\right)_0 = k\left(\frac{\partial^2 p^{(1)}}{\partial r^2}\right)_0,$$

and therefore

$$\left(\frac{\partial^3 p^{(2)}}{\partial r^2 \partial \theta}\right)_0 = k\left(\frac{\partial^3 p^{(1)}}{\partial r^2 \partial \theta}\right)_0.$$

Differentiating Eqs. (3.2), with $i = 1$ and $i = 2$, with respect to r and then putting $\varepsilon^{(i)} = 0$ and $r = r_0$ we find similarly

$$\left(\frac{\partial^3 p^{(2)}}{\partial r^3}\right)_0 = k\left(\frac{\partial^3 p^{(1)}}{\partial r^3}\right)_0.$$

Proceeding in this manner we arrive, by successive differentiations, at Eq. (3.8) for all values of m, and hence Eq. (3.7) follows, and our proposition is proved.

The property thus found in simple adiabatic oscillations can easily be shown to hold in a more general case. The condition $\varepsilon_s = 0$ can be considered as an equation which holds in addition to the five equations (2.14). Let us replace the equation $\varepsilon_s = 0$ by an equation of the form

$$\text{(3.9)} \qquad L\left(u, v, w, \tau, \varepsilon, p, \frac{\partial p}{\partial r}, \frac{\partial p}{\partial \theta}\right) = 0$$

(representing two equations with indices $i = 1$ and $i = 2$ respectively), L being a linear, homogeneous function of these 8 variables. This equation is assumed to be independent of Eqs. (2.14) and not contradictory to them. We consider Eq. (3.9) together with Eqs. (2.14, I, II, III, V), (omitting Eq. IV for the present). These 5 equations can generally be solved with respect to u, v, w, τ, ε, viz.

$$(3.10)\qquad \begin{aligned} w &= \bar{a}p + \bar{b}\frac{\partial p}{\partial r} + c\frac{\partial p}{\partial \theta}, \\ u &= \bar{a}_1 p + \bar{b}_1\frac{\partial p}{\partial r} + \bar{c}_1\frac{\partial p}{\partial \theta}, \end{aligned}$$

and the corresponding linear expressions for τ, v and ε. Computing $\frac{\partial u}{\partial \theta}$ and $\frac{\partial w}{\partial r}$ from (3.10) and introducing the expressions obtained in Eq. (2.14) IV we arrive at an equation of the form

$$(3.11)\qquad a_{11}\frac{\partial^2 p}{\partial r^2} + 2a_{12}\frac{\partial^2 p}{\partial r\,\partial \theta} + a_{22}\frac{\partial^2 p}{\partial \theta^2} + a_1\frac{\partial p}{\partial r} + a_2\frac{\partial p}{\partial \theta} + a_0 p = 0.\ ^{1)}$$

According to Eq. (3.10) the boundary condition is

$$(3.12)\qquad \left\{\bar{a}p + \bar{b}\frac{\partial p}{\partial r} + \bar{c}\frac{\partial p}{\partial \theta}\right\}_{r=r_0} = 0$$

From Eqs. (3.11) and (3.12) we arrive, by the method used above, at the same result concerning the phase angle of pressure in the case where Eq. (3.9) is valid, as in the simple case where (3.9) reduces to $\varepsilon = 0$.

Eq. (3.9) includes the case of piezotropy, which, in the fundamental state, can be expressed by the formula:

$$(3.13)\qquad \frac{dT}{dt} = f(P^*)\frac{dP^*}{dt}.$$

To apply this to the state of perturbation we perform the variation on both sides (v. Appendix) and obtain the following expression:

$$(3.14)\qquad \begin{aligned} &\frac{d\tau_s}{dt} + v_r\frac{\partial T}{\partial r} + v_\theta\frac{\partial T}{\partial \theta} \\ &= f(P^*)\left(\frac{dp^*}{dt} + v_r\frac{\partial P^*}{\partial r} + v_\theta\frac{\partial P^*}{\partial \theta}\right). \end{aligned}$$

Introducing here the expressions (2.9) we obtain

$$(3.15)\qquad \begin{aligned} &\gamma_n\tau + w\frac{\partial T}{\partial r} + u\frac{\partial T}{\partial \theta} \\ &= f(P^*)\left(\gamma_n p + w\frac{\partial P^*}{\partial r} + u\frac{\partial P^*}{\partial \theta}\right), \end{aligned}$$

which is a special case of (3.9).[2]

[1]) In the case when Eq. (3.9) reduces to $\varepsilon = 0$ (3.11) is identical with (3 2) but in the general case (3 2) and (3.11) are different from each other and both are valid simultaneously.

[2]) In Eq. (3.15) is included the case of isothermal changes of state, $f(P^*) \equiv 0$.

It should be noted that Eq. (3.9) is more general than (3.15). For instance the simple case $\tau = 0$, which is included in (3.9), is generally a non-piezotropic case, as it is not possible from Eqs. (2.14) with $\tau = 0$ to deduce an equation of the form (3.15).

We shall next consider the case of horizontal motion in the simple oscillation, i.e. $v_r = 0$, $w^{(i)} = 0$. Although this case is included in (3.9) it is different from the other cases in that Eq. (3.12) gives no independent condition but follows from (3.9). Therefore the above reasonings do not hold in this case. For simplicity we shall limit our examination of the horizontal motion to the atmosphere at the equator. Putting $w = 0$, $\frac{\partial w}{\partial r} = 0$ in Eqs. (2.14), solving I, II, III and V with respect to u, v, τ and ε, and introducing the expressions thus obtained in IV, we arrive at an equation of the form (3.11) but without the term $a_{11}\frac{\partial^2 p}{\partial r^2}$. At the equator we have, on account of the symmetry, $\frac{\partial p}{\partial \theta} = 0$, so that (3.11) reduces to

$$(3.16)\qquad a_{22}\frac{\partial^2 p}{\partial \theta^2} + a_1\frac{\partial p}{\partial r} + a_0 p = 0.$$

The symmetry also involves $\frac{\partial \eta_p}{\partial \theta} = 0$ i.e. $p^{(2)} = f(r)\,p^{(1)}$ at the equator. Introducing this expression in Eq. (3.16), which is valid both for $p^{(1)}$ and $p^{(2)}$, it is seen that $f(r)$ must reduce to a constant. This result can be expressed in the following manner:

In a simple oscillation which is symmetrical with respect to the equator and where the motion is everywhere horizontal, the phase angle of the pressure oscillation at the equator must be independent of height.

The results obtained concerning the phase angle η_p have a bearing upon the theory of the semidiurnal oscillation in the atmosphere. Observations show that in this oscillation $\left(\frac{\partial \eta_p}{\partial \theta}\right)_0 = 0$,[1]) but simultaneously η_p varies with height. It seems therefore very likely that the semidiurnal oscillation must be non-horizontal and have non-piezotropic changes of state. However, a strict

[1]) Simpson (1918) p. 8.

proof cannot be given here, as the simplifying conditions on which our formulae are based are but approximately fulfilled in the atmosphere. In particular it should be borne in mind that all frictional terms have been omitted from our equations. In this connection, however, it may be pointed out that, if the friction at the ground were the cause of the variation of η_p with height, one should expect to find that the numerical value of $\frac{d\eta_p}{dr}$ would have a maximum at the ground and decrease with height, but this is not the case, $\frac{d\eta_p}{dr}$ being practically constant with height.

In the investigations of *Margules* horizontal motion and isothermal changes of state were assumed, while *Taylor* (1936) and *Pekeris* (1937) assumed adiabatic changes of state. As far as the semidiurnal oscillation is concerned it seems preferable to drop these assumptions.

If this oscillation is non-adiabatic in certain layers then the quantity ε_s, when expressed as a sum of simple harmonic functions, must have a semidiurnal as well as a diurnal term in these layers. An example of this may be seen when considering a region at the ground at the equator, where $v_r = 0$, $v_\theta = 0$ and Eq. (2.6, V) reduces to

$$\varepsilon_s = c_p \frac{d\tau_s}{dt} - T \frac{dp^*}{dt}.$$

Introducing here the observed values of τ_s and p^* it will appear that the first term on the right-hand side preponderates. Since observations show clearly that the temperature has a semidiurnal term this must be the case also with ε_s, at least near the ground. With increasing elevation the term $G_r v_r$ of Eq. (2.6, V) gains importance, and then no simple relation exists between ε_s and τ_s.

4. Equations of perturbation for a barotropic atmosphere. Order of magnitude of the variables at the Pole.

In the foregoing paragraphs we have studied the simple oscillation solely by means of the hydrodynamical equations and the boundary condition at the ground. We shall now examine other conditions which have to be fulfilled in order that the solutions shall have physical interpretations.

An examination of the coefficients of Eq. (3.2) shows that $\theta = 0$ represents a singularity. It is therefore necessary to examine in detail the properties of the simple oscillation at the Pole.

We have so far considered perturbations in a baroclinic zonal current. In order to simplify computations we shall henceforth consider oscillations of an atmosphere originally at rest, i. e. we assume

$$V_\varphi = \Omega.$$

Hence it follows that

$$\frac{\partial P^*}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial P^*}{\partial r} = -\frac{g}{T},$$

$$D_\theta = 0, \quad G_\theta = 0, \quad \sigma_\theta = 2\Omega \operatorname{cotg} \theta, \tag{4.1}$$

$$D_r = \frac{\delta}{T} - \frac{g}{RT}, \quad G_r = g - c_p \delta, \quad \sigma_r = \frac{2}{r}\,\Omega.$$

The atmosphere is thus barotropic in the fundamental state. Introducing the symbols

$$\delta_a = \frac{g}{c_p} \qquad \delta_b = \frac{g}{R} - \delta \tag{4.2}$$

and the auxiliary quantity α defined by

$$2\Omega\alpha = \gamma + n\Omega \;^{1)} \tag{4.3}$$

and taking account of Eqs. (4.1) we may write Eqs. (2.14) as follows

$$\text{(4.4)} \quad \text{I} \qquad -2\Omega\alpha w - 2\Omega r \sin^2\theta \cdot v + T\frac{\partial p}{\partial r} - \frac{g}{T}\tau = 0,$$

$$\text{II} \qquad 2\Omega\alpha v + 2\Omega \operatorname{cotg}\theta \cdot u + \frac{2\Omega}{r} w + \frac{nT}{r^2 \sin^2\theta} p = 0,$$

$$\text{III} \qquad -2\Omega\alpha u - 2\Omega \sin\theta\cos\theta \cdot v + \frac{T}{r^2}\frac{\partial p}{\partial \theta} = 0,$$

$$\text{IV} \qquad \frac{2\Omega\alpha}{R} p - \frac{2\Omega\alpha}{T}\tau + \left(\frac{2}{r} - \frac{\delta_b}{T}\right) w + \frac{\partial w}{\partial r} + nv + \frac{\partial u}{\partial \theta} + \operatorname{cotg}\theta \cdot u = 0,$$

$$\text{V} \qquad 2\Omega\alpha c_p \tau - 2\Omega\alpha T p + c_p(\delta_a - \delta) w - \varepsilon = 0.$$

[1]) It is easily seen that the period of the oscillation (2.8) in a coordinate system fixed to the earth is $\frac{12}{\alpha}$ hours.

Putting for brevity

$$(4.5)\quad \begin{aligned} H &= \frac{g\,(\delta_a - \delta)}{4\Omega^2 T}, \quad D = H\,(a^2 - \cos^2\theta) + a^2(1 - a^2) \\ \delta_c &= \frac{2}{r} - \frac{\delta_b}{T} + \frac{\delta_a - \delta}{T} = \frac{2}{r} - \frac{g}{\varkappa R T}, \end{aligned}$$

and eliminating v and τ from the system (4.4) we obtain the following equations

$$(4.6)\qquad w = ap + b\frac{\partial p}{\partial r} + c\frac{\partial p}{\partial \theta} + d\cdot\varepsilon,$$

$$(4.7)\qquad u = a_1 p + b_1\frac{\partial p}{\partial r} + c_1\frac{\partial p}{\partial \theta} + d_1\cdot\varepsilon,$$

$$(4.8)\qquad \frac{\partial w}{\partial r} + \frac{\partial u}{\partial \theta} + \delta_c w + \frac{\cos^2\theta - na}{\sin\theta\cos\theta}u + \frac{2\Omega a}{\varkappa R}p + \frac{nT}{2\Omega r^2\sin\theta\cos\theta}\cdot\frac{\partial p}{\partial \theta} - \frac{\varepsilon}{c_p T} = 0.$$

For the coefficients we obtain, after a simple but somewhat lengthy calculation, the following values

$$a = \frac{-aT}{2\Omega D}\left[\frac{na}{r} - \frac{\delta_a}{T}(a^2 - \cos^2\theta)\right],$$

$$b = \frac{-aT}{2\Omega D}(a^2 - \cos^2\theta), \quad c = \frac{-aT}{2\Omega r D}\sin\theta\cos\theta,$$

$$(4.9)\qquad d = \frac{\delta_a}{4\Omega^2 T D}(a^2 - \cos^2\theta),$$

$$a_1 = \frac{1}{2\Omega r^2 D}[ra\delta_a\sin\theta\cos\theta + (H - a^2)nT\,\mathrm{cotg}\,\theta],$$

$$b_1 = c, \quad c_1 = \frac{aT}{2\Omega r^2 D}(H - a^2 + \sin^2\theta),$$

$$d_1 = \frac{\delta_a}{4\Omega^2 r T D}\sin\theta\cos\theta.$$ [1])

Computing $\frac{\partial w}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ from (4.6—7) and introducing the expressions for u, w, $\frac{\partial w}{\partial r}$, $\frac{\partial u}{\partial \theta}$ in Eq. (4.8) we arrive at an equation of the form (3.2). It is easily seen that we must have

$$A_{11} = b \quad A_{22} = c_1 \quad A_{12} = \tfrac{1}{2}(c + b_1) = c.$$

Further we obtain

$$A^2_{12} - A_{11}A_{22} = c^2 - bc_1 = \left(\frac{aT}{2\Omega r}\right)^2\frac{1}{D},$$

whence it follows that Eq. (3.2) is of the hyperbolic or elliptic type according as

$$D = \frac{g\,(\delta_a - \delta)}{4\Omega^2 T}(a^2 - \cos^2\theta) + a^2(1 - a^2)$$

is positive or negative. This is a generalization of the result obtained by *Solberg* (1936) for simple oscillations with constant phase angles in a homogeneous ocean. In the case of indifferent static equilibrium ($\delta_a = \delta$) we have the hyperbolic case when $a^2 < 1$ i. e. the period of oscillation $\frac{12}{a}$ is greater than 12 siderial hours. If, however, $\delta_a > \delta$ and $a^2 < 1$, D may be positive in low latitudes and negative in high latitudes. This is the case for instance with the diurnal oscillation ($a = \frac{1}{2}$).

The other coefficients of Eq. (3.2) are rather complicated expressions. However, it will suffice for our purpose to determine their order of magnitude near the Pole, as shown in the next paragraph.

In order to avoid misconceptions it will be convenient in the following to distinguish between «mathematical» and «physical» solutions of the equations (2.6), (2.14), (4.4). Any set of 6 functions

$$(4.10)\qquad p^*, \tau_s \ldots\ldots \varepsilon_s \text{ or } p, \tau \ldots\ldots \varepsilon$$

satisfying (2.6) or (2.14), (4.4) respectively will be called a mathematical solution. Assumptions concerning known and unknown quantities are not necessarily involved. The six functions of Eq. (2.6) will be called a physical solution if they have the properties necessary to represent the physical quantities in question. Considering Eqs. (2.9) it is seen that these properties can be expressed as follows:

$p, \tau, v, u, w, \varepsilon$ must be

(4.11) (a) single-valued and

(b) finite for all values of θ and $r \gtrless r_0$.

(c) The functions $p^*, \tau_s \ldots\ldots \varepsilon_s$ obtained from (2.9) by introducing $p^{(i)}\,\tau^{(i)}\ldots\varepsilon^{(i)}$ on the right-hand side, must be approximate solutions of the complete hydrodynamical equations for all values of ψ, t, θ and $r \gtrless r_0$. [1])

In addition to the above condition we must also have

$$(4.11\text{ d})\qquad w^{(i)} = 0 \text{ for } r = r_0.$$

[1]) $a_1\,b_1\ldots c_1, d_1$ are not identical with the symbols of (3.1) where the conditions (4.1) had not yet been introduced

[1]) The latter property is usually taken for granted, but the next paragraph will show the necessity of a close examination on this point.

We shall henceforth confine ourselves to the case $n \neq 0$. If condition (a) shall hold we must have

(4.12) $A_r = 0,\ A_p = 0,\ A_\tau = 0,\ A_\varepsilon = 0$ for $\theta = 0$.[1])

From Eq. (4.12) and (2.11) it follows that

(4.13) $w = 0,\ p = 0,\ \tau = 0,\ \varepsilon = 0$ for $\theta = 0$.

It will be assumed that the six variables in the physical solution (4.10) are continuous and have continuous derivates and can be developed in a Taylor series in a certain region around the Pole. This assumption seems natural, since the variables in question represent pressure, temperature, velocity and supplied heat, and the model atmosphere is supposed to contain no discontinuity.[2])

It will be convenient to introduce a symbol for «order of magnitude in θ when $\theta \to 0$». Suppose that

(4.14) $f(\theta) = f_m \theta^m + f_{m+1}\theta^{m+1} + f_{m+2}\theta^{m+2} + \ldots$

where m is a positive or negative integer or zero and $f_m \neq 0$. We then write

(4.15) $$f(\theta) = O(\theta^m)$$

to express that $f(\theta)$ is of the same order of magnitude as θ^m when $\theta \to 0$. In the special case $m = 0$ we write $f(\theta) = O(1)$. Further we shall write

(4.16) $$f(\theta) \gtrsim O(\theta^m)$$ [3])

to express that $f(\theta)$ may possibly be of a higher order than θ^m e. g. θ^{m+1}.[3])

In previous paragraphs «order of magnitude» has been used in a different sense. p was said to be of the first order, p^2 of the second order etc. In some cases it will therefore be convenient to write

$$f(\theta) = O_\mu(\theta^m)$$

when, for a given value of θ, f is of the same order as p^μ.[4])

From Eqs. (4.13) it follows that

(4.17) $$p,\ \tau,\ w,\ \varepsilon \gtrsim O(\theta)$$

and from Eqs. (4.4, I) and (2.9) it may be inferred that

(4.18) $$2\Omega r \sin^2\theta \cdot v \gtrsim O(\theta),\quad v \gtrsim O\left(\frac{1}{\theta}\right),$$

$$v_\varphi \gtrsim O\left(\frac{1}{\theta}\right).$$

It is possible to find a solution of Eqs. (2.6) with $v_\varphi = O\left(\frac{1}{\theta}\right)$ satisfying the conditions (4.11 a, b). Such a solution would give a linear velocity $r \sin\theta\, v_\varphi$, finite for $\theta = 0$, but the condition (4.11 c) would not hold. This is seen in the following manner: The term $\frac{\partial v_\varphi}{\partial \psi}$ in Eq. (2.6 IV) would have to be counterbalanced by some other term of the same order $O\left(\frac{1}{\theta}\right)$. The only possibility is the term $\operatorname{cotg}\theta \cdot v_\theta$ so that we must have:

$$\operatorname{cotg}\theta \cdot v_\theta = O\left(\frac{1}{\theta}\right),\quad v_\theta = O(1).$$

If a solution (2.8) of Eqs. (2.6) with

$$v_\varphi = O\left(\frac{1}{\theta}\right)\ v_\theta = O(1),\ p^* \gtrsim O(\theta)\ v_r \gtrsim O(\theta)$$

corresponds to an approximate solution of Eqs. (1.1—5) we must have for the order of magnitude in θ of the variables of Eq. (1.2), having regard to the properties of the zonal current:

$$V_\varphi = O\left(\frac{1}{\theta}\right),\ V_\theta = O(1),\ V_r \gtrsim O(\theta),\ \frac{\partial P}{\partial \psi} \gtrsim O(\theta).$$

But this is not possible since all the terms of Eq. (1.2) would then vanish for $\theta = 0$ except the term $2r^2 \sin\theta \cos\theta V_\varphi V_\theta = O(1)$, so that (1.2) would not be satisfied. Hence it follows that the hypothesis $v_\varphi = O\left(\frac{1}{\theta}\right)$ must be abandoned.

From the above considerations it follows that we must always have $v < O\left(\frac{1}{\theta}\right)$ and consequently $v \gtrsim O(1)$.

Combining this with the conditions (4.17) we infer from Eq. (4.4, IV)

$$\frac{\partial u}{\partial \theta} + \operatorname{cotg}\theta \cdot u \gtreqless O(1),$$

whence it follows that $u \gtreqless O(\theta)$ and from (4.4, III):

$$\frac{\partial p}{\partial \theta} \gtreqless O(\theta).$$

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[1]) If we had for instance $A_p \neq 0$ for $\theta = 0$ we should at a given instant get an infinite number of values of p^* at the Pole depending on the choice of ψ.

[2]) Mathematical solutions may include functions which are discontinuous at the Pole.

[3]) Example: $\sin m\theta \gtreqless O(\theta)$ when m is a positive integer

[4]) Example: If $v_r = O_1(\theta^2)$, $v_\theta = O_1(\theta)$ then $v_r v_\theta = O_2(\theta^3)$.

The final result is then:

$$(4.19)\qquad \left.\begin{matrix} w \\ u \\ \tau \\ \varepsilon \end{matrix}\right\} \gtreqless O(\theta) \qquad \begin{matrix} v \gtreqless O(1) \\ p \gtreqless O(\theta^2) \end{matrix}$$ [1]

In order to see if this is compatible with the condition (4.11 c) we return to Eqs. (2.6) and Table 2. Let φ_1 denote any term in these equations and let φ_2 denote any term in the corresponding row of Table 2. It is readily verified that

$$\frac{\varphi_2}{\varphi_1} \gtreqless O(1)$$

when the conditions (4.19) are fulfilled. Therefore the condition $\left|\frac{\varphi_2}{\varphi_1}\right| << 1$ holds for a small perturbation also when $\theta \to 0$.

Thus Eqs. (4.19) are necessary and sufficient for the validity of the conditions (4.11 a b c) at the Pole.

From the assumption that the variables $p, \tau \ldots \varepsilon$ shall be continuous and have continuous differential coefficients at the Pole we can further deduce a property of the series (4.14) where f is taken to represent any one of the significant variables.

The points of a plane through the earth's axis can be defined by means of ψ and θ in two ways, viz.

(a) $\quad 0 \gtreqless \theta \gtreqless \pi,\ \psi = \psi_0$ and $\psi = \psi_0 + \pi$,

(b) $\quad -\pi \gtreqless \theta \gtreqless \pi,\ \psi = \psi_0$.

Considering for instance the pressure oscillation

$$p^*(r, \theta, \psi_0, t)$$
$$= p^{(1)} \sin(n\psi + \gamma t) + p^{(2)} \sin\left(n\psi + \gamma t + \frac{\pi}{2}\right)$$

we have in the case (a)

$$p^*(r, \theta, \psi_0 + \pi, t) = (-1)^n p^*(r, \theta, \psi_0, t)$$

which corresponds to

$$p^{(1)}(r, \theta) = (-1)^n p^{(1)}(r, -\theta),$$
$$p^{(2)}(r, \theta) = (-1)^n p^{(2)}(r, -\theta)$$

in the case (b). Differentiating with respect to θ and putting $\theta = 0$ we obtain

$$p(r, 0) = (-1)^n p(r, 0),$$
$$p'(0) = (-1)^{n+1} p'(0),\ p''(0) = (-1)^{n+2} p''(0)$$

[1] In the diurnal oscillation Margules used a pressure of the order $O(\theta)$ which is not permissible. (V. Exner (1925) Dynamische Meteorologie p. 406).

etc. If all derivates are to be continuous for $\theta = 0$ it is seen that

$$(4.20)\qquad \begin{aligned} p &= p_2\theta^2 + p_4\theta^4 + p_6\theta^6 + \ldots \text{ when } n \text{ is even,} \\ p &= p_3\theta^3 + p_5\theta^5 + p_7\theta^7 + \ldots \text{ when } n \text{ is odd.} \end{aligned}$$

The same considerations apply to w, τ, v and ε. From Eq. (4.4 III) it follows that

$$(4.21)\qquad \begin{aligned} u &= u_1\theta + u_3\theta^3 + u_5\theta^5 + \ldots \text{ when } n \text{ is even,} \\ u &= u_2\theta^2 + u_4\theta^4 + u_6\theta^6 + \ldots \text{ when } n \text{ is odd.} \end{aligned}$$

(Some of the first terms in the series (4.20—21) may vanish).

5. Properties of Eq. (3.2) near the Pole.

Previously we have considered certain relations which permit ε_s to be evaluated when p^* is known. We shall now consider other relations between p^* and ε_s obtained by combining Eq. (3.2) with the conditions at the Pole as developed in para. 4.

Since it follows from Eq. (4.5) that $D = O(\theta^2)$ when $\alpha^2 = 1$ and $D = O(1)$ when $\alpha^2 \neq 1$, we shall treat these two cases separately.

A. The case $\alpha^2 = 1$.

Expanding the coefficients in Eqs. (4.6—8) in Taylor series we may write

$$(5.1)\qquad \begin{aligned} w = &\left(\frac{\alpha_{-2}}{\theta^2} + \alpha_0 + \alpha_2\theta^2 + \ldots\right) p \\ &+ (\beta_0 + \beta_2\theta^2 + \ldots)\frac{\partial p}{\partial r} \\ &+ \left(\frac{\gamma_{-1}}{\theta} + \gamma_1\theta + \ldots\right)\frac{\partial p}{\partial \theta} + (\delta_0 + \delta_2\theta^2 + \ldots)\,\varepsilon, \end{aligned}$$

$$(5.2)\qquad \begin{aligned} u = &\left(\frac{\alpha_{-3}}{\theta^3} + \frac{\alpha_{-1}}{\theta} + \alpha_1\theta + \ldots\right) p \\ &+ \left(\frac{\beta_{-1}}{\theta} + \beta_1\theta + \ldots\right)\frac{\partial p}{\partial r} \\ &+ \left(\frac{\gamma_{-2}}{\theta^2} + \gamma_0 + \ldots\right)\frac{\partial p}{\partial \theta} + \left(\frac{\delta_{-1}}{\theta} + \delta_1\theta + \ldots\right)\varepsilon, \end{aligned}$$

$$(5.3)\qquad \begin{aligned} &\frac{\partial w}{\partial r} + \frac{\partial u}{\partial \theta} + \delta_c w + \left(\frac{1 - na}{\theta} + \ldots\right) u + \frac{2\Omega a}{\varkappa R}\, p \\ &+ \frac{nT}{2\Omega r^2}\left(\frac{1}{\theta} + \ldots\right)\frac{\partial p}{\partial \theta} - \frac{\varepsilon}{c_p T} = 0. \end{aligned}$$

Deducing $\frac{\partial w}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ from Eqs. (5.1—2) and introducing in Eq. (5.3) the expressions thus obtained, we may write Eq. (3.2) as follows

$$(5.4)\quad \left(\frac{a_{-2}}{\theta^2}+a_0+\dots\right)\frac{\partial^2 p}{\partial\theta^2}+2\left(\frac{b_{-1}}{\theta}+b_1\theta+\dots\right)\frac{\partial^2 p}{\partial r\partial\theta}$$
$$+(c_0+c_2\theta^2+\dots)\frac{\partial^2 p}{\partial r^2}+\left(\frac{d_{-2}}{\theta^2}+d_0+\dots\right)\frac{\partial p}{\partial r}$$
$$+\left(\frac{f_{-3}}{\theta^3}+\frac{f_{-1}}{\theta}+\dots\right)\frac{\partial p}{\partial\theta}+\left(\frac{g_{-4}}{\theta^4}+\frac{g_{-2}}{\theta^2}+\dots\right)p$$
$$+(h_0+h_2\theta^2+\dots)\frac{\partial\varepsilon}{\partial r}+\left(\frac{j_{-1}}{\theta}+j_1\theta+\dots\right)\frac{\partial\varepsilon}{\partial\theta}$$
$$+\left(\frac{k_{-2}}{\theta^2}+k_0+\dots\right)\varepsilon=0.$$

The coefficients of (5.1—4) $a_{-2}, \beta_0 \dots a_{-2}, b_{-1}\dots$ are functions of r and can be computed by means of the formulae (4.6—9).

Supposing that p and ε can be expanded in Taylor series in a certain region $0 \lesseqgtr \theta \lesseqgtr \theta_0$ we may write, having regard to Eqs. (4.19—21)

$$(5.5)\quad \begin{aligned} p &= p_m\theta^m + p_{m+2}\theta^{m+2}+\dots \\ \varepsilon &= \varepsilon_\mu\theta^\mu + \varepsilon_{\mu+2}\theta^{\mu+2}+\dots \end{aligned}$$

We shall assume that $p_m \neq 0$. In the case of adiabatic oscillations all the coefficients $\varepsilon_\mu, \varepsilon_{\mu+2}\dots$ vanish. In the non-adiabatic case we may assume $\varepsilon_\mu \neq 0$ with the possible exception of special values of r (e.g. $\varepsilon_\mu = 0$ for $r = r_0$). Further we note that $m \gtrless 2$ and $\mu \gtrless 1$ are even or odd numbers according as n is even or odd.

Inserting these series for p and ε in Eq. (5.4) we obtain on the left-hand side a power series that may be written as follows

$$(5.6)\quad R_m p_m\theta^{m-4}+B_{m-2}\theta^{m-2}+B_m\theta^m+\dots$$
$$+S_\mu\varepsilon_\mu\theta^{\mu-2}+C_\mu\theta^\mu+C_{\mu+2}\theta^{\mu+2}+\dots=0,$$

where $R_m, B_{m-2}\dots S_\mu, C_\mu, C_{\mu+2}\dots$ are functions of r which can be derived from the coefficients of Eqs. (5.4—5). We have

$$(5.7)\quad R_m = m(m-1)a_{-2}+mf_{-3}+g_{-4}$$
$$=\frac{T(H-1)}{2\Omega r^2 H}(m-na-2)(ma+n)\ ^{1)}$$
where $a = \pm 1$.

$$(5.8)\quad S_\mu = k_{-2}+\mu j_{-1} = \frac{\delta_a}{4\Omega^2 rTH}(\mu - na).$$
$$(a=\pm 1)\ ^{1)}$$

It is immediately apparent from Eq. (5.6) that

(5.9)
(a) If $m-2<\mu$ we must have $R_m = 0$,
(b) If $m-2=\mu$: $R_m p_m + S_\mu\varepsilon_\mu = 0$,
(c) If $m-2>\mu$: $S_\mu\varepsilon_\mu = 0$.

[1]) Details of the computations are given in the appendix.

B. The case $a^2 \neq 1$, $a^2 \neq H$.

We may here write equations quite similar to (5.1—4) but the coefficients will have other values which we shall denote by

$$\bar a_{-2}\ \bar a_0\ \bar\beta_0 \dots \bar a_{-2}\ \bar a_0\ \bar b_{-1}\dots$$

Since we have in this case $D \neq 0$ when $\theta = 0$, it is seen that the following coefficients

$$(5.10)\quad \begin{aligned} &\bar a_{-2}\ \bar\gamma_{-1}\ \bar a_{-3}\ \bar\beta_{-1}\ \bar\gamma_{-2}\ \bar\delta_{-1} && \text{in (5.1—2)}, \\ &\bar a_{-2}\ \bar b_{-1}\ \bar d_{-2}\ \bar g_{-4}\ \bar j_{-1}\ \bar k_{-2}\ \bar f_{-3} && \text{in (5.4)} \end{aligned}$$

all vanish.

Introducing the series (5.5) in the equation that corresponds to (5.4) and putting $\frac{d\varepsilon_\mu}{dr} = \varepsilon'_\mu$ etc. we obtain here a power series analogous to (5.7), viz.

$$(5.11)\quad R_m p_m\theta^{m-2}+\bar B_m\theta^m+\bar B_{m+2}\theta^{m+2}+\dots$$
$$+(\bar S_\mu\varepsilon_\mu+\bar h_0\varepsilon'_\mu)\theta^\mu+\bar C_{\mu+2}\theta^{\mu+2}+\dots=0$$

where

$$(5.12)\quad R_m = \frac{aT}{2\Omega r^2(a^2-1)}(m^2-n^2).\ ^{1)}$$

Corresponding to (5.9) we may state the following conditions

(5.13)
(a) If $m-2<\mu$ we must have $\bar R_m = 0$,
(b) If $m-2=\mu$: $\bar R_m p_m+\bar S_\mu\varepsilon_\mu+\bar h_0\varepsilon'_\mu=0$,
(c) If $m-2>\mu$: $\bar S_\mu\varepsilon_\mu+\bar h_0\varepsilon'_\mu=0$.

In the case of adiabatic oscillations ($\varepsilon \equiv 0$) the second line of Eq. (5.11) vanishes and we must have

$$(5.14)\quad \bar R_m = 0,\ \bar B_m = 0,\ B_{m+2}=0,\dots$$

We note that this condition cannot be fulfilled in the case $\gamma = 0$, $n = 1$ since $m \gtrless 2$ and therefore $\bar R_m \neq 0$ according to (5.12). On the assumptions specified above it follows then that an adiabatic oscillation of the form $A\sin(\psi+\eta)$ is not possible.

6. An application to the theory of the diurnal and the semidiurnal oscillation of the atmosphere.

The formulae of the preceding paragraph determine certain relations between the variables p and ε near the Pole. These relations are a

[1]) Details of the computations are given in the appendix

consequence of the hydrodynamical equations (4.4) and the conditions (4.11 a b c). It remains however to take the boundary condition (4.11 d) into account. In doing this we shall limit our considerations to cases that bear a close resemblance to the diurnal and the semidiurnal oscillation.

Since these waves retain a fixed position relative to the sun, they have a very slow motion relative to our coordinate system. For simplicity we shall first consider waves having no motion relative to the coordinate system. We then have

$$\gamma = 0, \ a = \frac{n}{2}.$$

A. The oscillation $\gamma = 0$, $n = 1$, $a = \frac{1}{2}$.

According to the conditions (4.19) we must have $m \gtrless 2$ and it is seen from (5.12) that $\bar{R}_m \neq 0$ so that the condition (5.13 a) does not apply. If ε is considered as a given quantity (determined by radiation and the physical properties of the atmosphere) the special condition (5.13 c) will generally not be valid. We must have therefore $m - 2 = \mu$. p_m is then determined by (5.13 b) and p_{m+2}, $p_{m+4} \ldots$ can be determined successively, all of them being expressed as functions of $\varepsilon_\mu, \varepsilon_{\mu+2} \ldots$ and their derivates.

Considering next the boundary condition we put in Eq. (5.1) $r = r_0$, $w = 0$ and introduce the series (5.5). Since we consider a case where $a^2 \neq 1$ we must have regard to (5.10). The power series thus obtained on the right-hand side of (5.1) being zero it is seen that

(6.1) $$\varepsilon_\mu = \varepsilon_{m-2} = 0 \ \text{for} \ r = r_0,$$

(6.2) $$\left\{ \left(a_0 + m\bar{\gamma}_1 \right) p_m + \bar{\beta}_0 p'_m + \bar{\delta}_0 \varepsilon_m \right\}_{r=r_0} = 0,$$

and similar expressions can be deduced from higher terms of the series. Introducing in these expressions the values of p_m, $p_{m+2} \ldots$ determined from Eq. (5.13 b) etc., as mentioned above, we obtain successively an infinite number of conditions which must hold for the coefficients ε_μ, $\varepsilon_{\mu+2} \ldots$ in order that the hydrodynamical equations and the boundary condition may be satisfied simultaneously.

Thus the simple oscillation (2.8) with $\gamma = 0$, $n = 1$ will be possible only if $\varepsilon_s = A_\varepsilon \cos(\psi + \eta_\varepsilon)$ fulfills certain special conditions.[1])

[1]) Is has been shown in the foregoing paragraph that such an oscillation cannot be adiabatic.

B. The oscillation $\gamma = 0$, $n = 2$ $(a = 1)$.

According to Eq. (5.7—8) we have here

(6.3) $$R_m = \frac{T(H-1)}{2\Omega r^2 H}(m-4)(m+2),$$

(6.4) $$S_\mu = \frac{\delta_a}{4\Omega^2 r T H}(\mu - 2).$$

It can be shown that, if $m \neq 4$, the above reasonings concerning the oscillation $(a = \frac{1}{2}, n = 1)$ are valid also in the case $(a = 1, n = 2)$. Thus, when $m \neq 4$, the latter oscillation is possible only if $\varepsilon_s = A_\varepsilon \cos(2\psi + \eta_\varepsilon)$ fulfills special conditions. Further it is easily seen that the combination $m = 4$, $\mu \neq 2$ is not compatible with the boundary condition. (v. Appendix).

The remaining case

(6.5) $$m = 4, \ \mu = 2,$$

is of special interest, for, Eq. (5.9 b) being then satisfied for all values of p_4 and ε_2, it becomes possible to satisfy this equation and the boundary condition simultaneously. Introducing the series

(6.6) $$p = p_4\theta^4 + p_6\theta^6 + \ldots.$$

(6.7) $$\varepsilon = \varepsilon_2\theta^2 + \varepsilon_4\theta^4 + \ldots.$$

into Eq. (5.1) we get

$$w = \left\{ (a_{-2} + 4\gamma_{-1}) p_4 + \delta_0 \varepsilon_2 \right\} \theta^2 + \ldots$$

Having recourse to Eq. (4.6) we easily find a_{-2}, γ_{-1} and δ_0 (v. Appendix) and

(6.8) $$w = \frac{-T}{2\Omega H}\left(\frac{6}{r} p_4 - \frac{\delta_a}{2\Omega T^2} \varepsilon_2 \right) \theta^2 + \ldots.$$

From the boundary condition it follows then.

(6.9) $$\left\{ \frac{6}{r} p_4 - \frac{\delta_a}{2\Omega T^2} \varepsilon_2 \right\}_{r=r_0} = 0$$

which determines $(p_4)_{r=r_0}$, ε_2 being given.

So far we have considered only the first terms of the series (6.6—8). We shall now consider higher terms as well in order to show that when ε is given, p and the other significant variables can be determined, not only for $r = r_0$, but also in a certain region above the ground.

Introducing the series (6.6—7) in Eq. (5.4) we obtain on the left-hand side a power series where every term must vanish. Denoting by f_i, φ_i, F_i, Φ_i $(i = 2, 4, 6 \ldots)$ linear functions we may write the coefficient of θ^2 in the form

(6.10) $$f_2(p_6, p_4, p'_4) + \varphi_2(\varepsilon_4, \varepsilon_2, \varepsilon'_2) = 0,$$

and the coefficient of θ^4:

(6.11) $$f_4(p_8, p_6, p_4, p'_6, p'_4, p''_4) + \varphi_4 = 0$$

etc.

Introducing Eqs. (6.6—7) in (5.1) and putting $r = r_0$, $w = 0$, we obtain another series equal to zero. The coefficient of θ^4 may be written

$$(6.12) \qquad \left\{ F_4 \left(p_6, p_4, p'_4\right) + \Phi_4 \right\}_{r=r_0} = 0,$$

and the coefficient of θ^6:

$$(6.13) \qquad \left\{ F_6 \left(p_8, p_6, p_4, p'_6, p'_4\right) + \Phi_6 \right\}_{r=r_0} = 0,$$

etc., φ_i and Φ_i depending solely upon the given quantities $\varepsilon_2, \varepsilon_4, \varepsilon_6 \ldots.$

Repeating the above procedure we obtain successively and infinite number of linear equations, each of them being based upon the foregoing equations according to simple rules which will become apparent by the consideration of Table 3.

Table 3.

	a)	b)	c)
I	p_4 p_6 p_4'	p_6' p_4''	p_6'' p_4'''
II $(k=k_0)$	p_4 p_6 p_4'		
III	p_4 p_6 p_8 p_4' p_6' p_4''	p_8' p_6'' p_4'''	
IV $(k=k_0)$	p_4 p_6 p_8 p_4' p_6'		
V	p_4 p_6 p_8 p_{10} p_4' p_6' p_8' p_4'' p_6''		
VI $(k=k_0)$	p_4 p_6 p_8 p_{10} p_4' p_6' p_8'		

In Table 3 each filled-in box represents a linear equation. Thus the first box represents Eq. (6.10), the p-coefficients occurring in the equation being listed in the box. Differentiating this equation with respect to r we obtain a new equation represented by box I (b) and containing the variables listed in I (a + b). Differentiating again we obtain an equation represented by box I (c) and containing the variables listed in I (a + b + c).

The box III (a) represents Eq. (6.11). Treating this in the same manner as I (a) we obtain III (b), etc. After Eq. (6.11) follows $f_6 + \varphi_6 = 0$ which is represented by box V (a) in the table.

Eqs. (6.12—13) and the following equation $\left\{ F_8 + \Phi_8 \right\}_{r=r_0} = 0$ are represented by the boxes II, IV, VI. These equations are valid only for $r = r_0$ and cannot be differentiated.

It is easily seen that Table 3 can be extended ad infinitum by repetition of the procedure described above. For $r = r_0$ all equations of the table are valid simultaneously. $(p_4)_{r=r_0}$ is already determined from Eq. (6.9). The table shows that:

(A) Eqs. I (a) and II determine $(p_6, p'_4)_{r=r_0}$.

(B) Eqs. I (b), III (a) and IV determine $(p_8, p'_6, p''_4)_{r=r_0}$.

(C) Eqs. I (c), III (b), V (a) and VI determine $(p_{10}, p_8', p_6'', p_4''')_{r=r_0}$,

etc.

Extending the table step by step we can thus determine

$$(p_4, p_4', p_4'' \ldots)_{r=r_0}$$
$$(p_6, p_6', p_6'' \ldots)_{r=r_0}$$

etc.

Hence it follows that

$$p_4 = (p_4)_0 + (p_4')_0 (r - r_0) + \ldots.$$

and similarly $p_6, p_8 \ldots$ can be determined in a certain region, from the ground upwards, in the vicinity of the Pole, which means that p is determined in this region. As we have seen u, v, w and τ can be found when p and ε are known.

The results obtained here concerning the oscillations ($n = 1$, $a = \frac{1}{2}$) and ($n = 2$, $a = 1$) have a bearing upon the theory of the semidiurnal and the diurnal oscillation in the atmosphere. Denoting by ψ_s the angle between the meridional plane through the sun and the meridional plane through the point considered and by Ω_s the angular velocity of the earth's rotation relative to the sun and putting $\Omega - \Omega_s = \omega$ we have

$$\psi = \lambda + \Omega t, \quad \psi_s = \lambda + \Omega_s t,$$
$$n\psi + \gamma t = n\psi_s + (\gamma + n\omega)\, t.$$

For waves that retain a fixed position relative to the sun we have $\gamma + n\omega = 0$ and

$$a = \left(1 - \frac{\omega}{\Omega}\right)\frac{n}{2} = 0{,}997\ \frac{n}{2}.$$

Thus $a \neq 1$ both in the diurnal and in the semidiurnal oscillation, but in the latter case the difference $a - 1$ is very small, and it seems therefore highly probable that the properties of the semidiurnal oscillation are substantially the same as those of the oscillation $(n = 2,\ a = 1)$.

We have seen that, when ε is given, p and the other variables can be completely determined in a region surrounding the Pole in the case $(n = 2,\ a = 1)$, whereas this cannot be done in the case $(n = 1,\ a = \frac{1}{2})$ except when ε satisfies very special conditions. Thus, in the model atmosphere, no obstacle prevents the development of the semidiurnal oscillation $A \sin(2\psi + \eta)$ but conditions seem unfavourable for the development of the diurnal oscillation $A \sin(\psi + \eta)$.

It is of interest to note the agreement existing between this theoretical result and certain facts established by observation. In the real atmosphere the semidiurnal oscillation preponderates, whereas the diurnal oscillation is indistinct and much disturbed by local influences. In a later paragraph we shall return to the above theory in connection with some physical properties of these oscillations and of the complex motion of which they are parts.

From a mathematical point of view the present theory is incomplete, since the region of convergence of the power series for p has not been determined. What has been called «vicinity of the Pole» may, on closer investigation, prove to be an extensive region.

The following problem also arises: Supposing that the function p can be evaluated for all values of θ between 0 and $\frac{\pi}{2}$ by analytical continuation of the series (6.6), we arrive at the equator from the North and from the South with a certain value $p = p_e$. Denoting by the subscripts N and S the values immediately north of and south of the equator, we have in virtue of the symmetry

$$\left(\frac{\partial p}{\partial \theta}\right)_N = -\left(\frac{\partial p}{\partial \theta}\right)_S$$

but is not evident a priori that the necessary condition

$$\left(\frac{\partial p}{\partial \theta}\right)_N = -\left(\frac{\partial p}{\partial \theta}\right)_S = 0 \tag{6.14}$$

is fulfilled.

For the study of this question one may consider trigonometrical series similar to those used by *Margules*. Assuming ε to be given in the form

$$\varepsilon = A_2 \sin^2 \theta + A_4 \sin^4 \theta + \ldots. \tag{6.15}$$

one would have to determine p in the form

$$p = B_4 \sin^4 \theta + B_6 \sin^6 \theta + \ldots. \tag{6.16}$$

A_i and B_i being functions of r. To determine B_i one would have to introduce the series (6.15—16) in Eqs. (4.6—8) and in the boundary condition obtained from (4.6).

It may also be convenient to use expansions in series of the Legendre associated functions of the first kind, e. g.

$$p = \sum_n a_n P_n^{(4)} (\cos \theta). \tag{6.17}$$

If it is possible to find a series (6.16—17) which is convergent for all values of θ, then p and its derivates will be continuous everywhere and (6.14) will be fulfilled.

This problem, however, must be left for later examination.

As we have seen, a semidiurnal oscillation can have the simple form $A \sin(2\psi + \eta)$ in all latitudes only when p has the form (6.6), i. e. $p = O(\theta^4)$. A rapid decrease of the pressure amplitude towards the Pole is largely in agreement with observations although these are not sufficiently accurate to decide if for instance $p^* = O(\theta^4)$ or $O(\theta^3)$.

Since the equations of perturbation are not exact it seems useless to determine their mathematical solutions with absolute exactitude. If for instance p is given by a convergent series, the complete series gives no better solution than a certain finite number of terms. Therefore Table 3 need not be extended very far.

Taking Table 3 in its present form as an example, it is seen that we have nine linear equations for the nine quantities

$$\left\{ p_4'\ p_4''\ p_4'''\ p_6\ p_6'\ p_6''\ p_8\ p_8'\ p_{10} \right\}_{r=r_0}$$

$(p_4)_0$ being known. The determinant D_1 of these equations contains

$$\left\{ T,\ \frac{\partial T}{\partial r},\ \frac{\partial^2 T}{\partial r^2},\ \frac{\partial^3 T}{\partial r^3} \right\}_{r=r_0}$$

For certain values of the function $T(r)$ the determinant may vanish, thus making $p_4', p_4'' \dots p_{10}$ infinite. This is the case of *resonance*. If $\varepsilon = 0$ the oscillation is only possible if the temperature $T(r)$ is such that $D_I = 0$. We then have «free oscillations».

If the table is extended by one step it will represent 14 equations with 14 unknown quantities, but a similar reasoning will still apply.

In the oscillation $(n = 2, a = 1)$ we have

$$(6.18) \qquad \begin{array}{lll} p^* = O(\theta^4) & v_r = O(\theta^2) & \varepsilon_s = O(\theta^2) \\ v_\theta = O(\theta) & v_\varphi = O(1) & \tau_s = O(\theta^2) \end{array}$$ [1]

Let us consider Eq. (2.6, I) and the second-order terms belonging to this equation, listed in the first row of Table 2. It is seen that

$$(6.19) \qquad T \frac{\partial p^*}{\partial r} = O_1(\theta^4) \quad r v_\theta^2 = O_2(\theta^2)$$

and of these terms the latter, which was disregarded when the equations of perturbation were formed, predominates when $\theta \to 0$ however small the perturbation. Corresponding results are found for each of the equations (2.6) when comparing the first-order terms containing pressure with certain terms of Table 2.

In view of this circumstance it may at first thought seem doubtful whether the value obtained for p^* belongs to an approximate solution of the complete hydrodynamical equations (Condition (4.11 c)). We shall return to this question in a later paragraph in connection with non-linear equations of perturbation.

7. Remark on the quasistatic equations.

In the preceding paragraphs the complete linearized equations have been used. Since the quasistatic equations of Laplace are used extensively in investigations of this kind it may be of interest to repeat the calculations of para. 6 on the basis of these equations in order to see if the same results could have been obtained by means of the simple quasistatic method.

The simplification of the equations in the latter method consists in omitting the two first terms of Eq. (2.6, I) and the third term of (2.6, II).[2]

[1] This is seen from Eqs. (6.6—8), (5.2), (4.4, II, V) having regard to (2.9).

[2] Solberg (1936) p. 288—289.

In Equations (4.4) we must then omit $(-2\Omega a w - 2\Omega r \sin^2\theta \cdot v)$ from the first and $\frac{2\Omega}{r} w$ from the second equation. Repeating the computations of para. 5 A on this assumption it can be shown that no essential change takes place in the formulae (5.7—8) whence it follows that the rows I, III, V of Table 3 remain unchanged. However it is easily seen that the boundary condition suffers an important change. Having omitted the two first terms of Eq. (4.4, I) we obtain

$$(7.1) \qquad \tau = \frac{T^2}{g} \frac{\partial p}{\partial r}$$

and from Eq. (4.4, V) it then follows that

$$(7.2) \quad w = \frac{T}{g(\delta_a - \delta)} \left(2\Omega a \delta_a p - 2\Omega a T \frac{\partial p}{\partial r} + \frac{\delta_a}{T} \varepsilon \right).$$

Introducing here the series (6.6—7) and putting $r = r_0$, $w = 0$ we obtain

$$(7.3) \qquad (\varepsilon_2)_{r=r_0} = 0$$

$$(7.4) \qquad \left\{ \delta_a p_m - T p_m' + \frac{\delta_a}{2\Omega a T} \varepsilon_m \right\}_{r=r_0} = 0,$$

where $m = 4, 6, 8 \dots 2i \dots$

Comparing this to the correct boundary condition (6.9) it is seen that the latter determines $(p_4)_0$ when ε is given, whereas Eq. (7.4)

Table 4.

	a)	b)	c)
I	p_4 p_6 p_4'	p_6' p_4''	p_6'' p_4'''
II $(k = k_1)$	p_6 p_6'		
III	p_4 p_6 p_8 p_4' p_6' p_4''	p_8' p_6'' p_4'''	
IV $(k = k_2)$	p_8 p_8'		
V	p_4 p_6 p_8 p_{10} p_4' p_6' p_8' p_4'' p_6''		
VI $(k = k_3)$	p_{10} p_{10}'		

gives a relation between $(p_4)_0$ and $(p_4')_0$ which makes it possible to choose one of these quantities arbitrarily.

Table 4 is analogous to Table 3, the boxes II, IV, VI... representing the equations (7.4) with $m = 6, 8, 10 \ldots$ respectively. If $(p_4)_0$ be chosen $(p_4')_0$ is determined and we can find

$(p_6, p_6')_{r=r_0}$ from Eqs. I (a) and II,
$(p_4'', p_8, p_8')_{r=r_0}$ from Eqs. I (b), III (a) and IV,
$(p_4''', p_6'', p_{10}, p_{10}')_{r=r_0}$ from Eqs. I (c), III (b), V and VI

etc.

Thus it is seen that the series (6.6) contains an arbitrary constant. This, however, is an erroneous result which is due to the quasistatic method.

Solberg (1936) has shown the importance of the complete linearized equations ("exact dynamical method") in the theory of tidal waves in a *homogenous* ocean. *Hylleràs* (1943) contends that for the theory of such waves in a *stable* ocean the exact method can give no advantage over the quasistatic method, which should be preferred for simplicity. The present investigation shows that this is not always the case. So far as the semidiurnal oscillation is concerned it seems necessary, in theoretical considerations, to apply the exact dynamical method of Solberg, however great the static stability of the oscillating medium.

For numerical considerations based on observational data the quasistatic equations can always be used.

8. Some remarks concerning non-linear equations of perturbation.

The formulae deduced in the foregoing paragraphs are based on the linearized hydrodynamical equations, but we have had to consider also certain second-order terms in connection with the condition (4.11 c). It has appeared that, in the case $\gamma = 0, n = 2$, when $\theta \to 0$, some of the second-order terms become greater than certain terms of Eqs. (2.6). This leads us to consider in greater detail the non-linear equations obtained by adding the terms of Table 2 to the terms of Eqs. (2.6).

Introducing the simplifying assumptions (4.1) we may write the first equation as follows

$$(8.1)\quad \frac{dv_r}{dt} - 2\Omega r \sin^2\theta \cdot v_\varphi + T\frac{\partial p^*}{\partial r} - \frac{g}{T}\tau_s = -\boldsymbol{v}\cdot\nabla v_r + rv_\theta^2 + r\sin^2\theta\cdot v_\varphi^2 - \tau_s\frac{\partial p^*}{\partial r}$$

and the other equations may be written in like manner. Denoting by

$$(8.2)\quad \bar{p}^*,\ \bar{\tau}_s,\ \bar{v}_\varphi,\ \bar{v}_r,\ \bar{v}_\theta,\ \varepsilon_s$$

a solution of these non-linear equations and by

$$(8.3)\quad p^*,\ \tau_s \ldots \varepsilon_s$$

a solution of the linear equations (2.6) we may write

$$(8.4)\quad \begin{aligned} \bar{p}^* &= p^* + x_p & \bar{v}_\theta &= v_\theta + x_\theta \\ \bar{v}_r &= v_r + x_r & \bar{\tau}_s &= \tau_s + x_\tau \\ \bar{v}_\varphi &= v_\varphi + x_\varphi & \bar{\varepsilon}_s &= \varepsilon_s + x_\varepsilon, \end{aligned}$$

where the "corrections" $x_p, x_r \ldots x_\varepsilon$ are of the same order as p^{*2} or of higher order of magnitude.

It is immediately apparent that if we replace for inst. $\bar{p}^*\,\bar{v}_r$ by $p^*\,v_r$ the error does not exceed the 3rd order of magnitude, and this rule holds for any product of first-order terms. Introducing (8.4) in Eq. (8.1) and neglecting 3rd-order terms this equation may therefore be written

$$(8.5)\quad \begin{aligned} &\frac{dx_r}{dt} - 2\Omega r\sin^2\theta\cdot x_\varphi + T\frac{\partial x_p}{\partial r} - \frac{g}{T}x_\tau = S_1, \\ &S_1 = -\boldsymbol{v}\cdot\nabla v_r + rv_\theta^2 + r\sin^2\theta\cdot v_\varphi^2 - \tau_s\frac{\partial p^*}{\partial r}. \end{aligned}$$

By the same procedure applied to the other equations we obtain 5 linear equations for $x_p, x_r \ldots x_\varepsilon$. These are formed by replacing in Eqs. (2.6) $p^*, v_r \ldots \varepsilon_s$ by $x_p, x_r \ldots x_\varepsilon$ and inserting on the right-hand side instead of zero the expressions $S_1, S_2 \ldots S_5$ taken from the horizontal rows of Table 2 (multiplied by -1).[1])

Having regard to Eqs. (2.9) it is easily seen that we may write

$$(8.6)\quad S_i = \alpha_i + \beta_i \sin 2\nu + \gamma_i \cos 2\nu \quad (i = 1, 2, 3, 4, 5)$$

where $\alpha_i\ \beta_i\ \gamma_i$ are functions of r and θ.

For simplicity we shall limit our application of the non-linear equations to the case $\gamma = 0$, $n = 2$ $(a = 1)$. We consider first the properties of the solutions when $\theta \to 0$. For this purpose

[1]) Method of successive approximations V Lamb Hydrodynamics 3rd. ed § 186.

we shall have to determine the order of magnitude in θ of the quantities S_i. In virtue of Eqs. (6.18) we find in Table 2, I that the terms

$$v_\varphi \frac{\partial v_r}{\partial \psi}, \; v_\theta \frac{\partial v_r}{\partial \theta}, \; -r v_\theta^2, \; -r \sin^2 \theta v_\varphi^2$$

are of the order $O_2(\theta^2)$ while the other terms are of higher order. Thus we have $S_1 = O_2(\theta^2)$. In Table 2, II we find

$$v_\varphi \frac{\partial v_\varphi}{\partial \psi} = O_2(1), \;\; 2 \operatorname{cotg} \theta \cdot v_\varphi v_\theta = O_2(1),$$

but the sum of these terms

$$v_\varphi \left(\frac{\partial v_\varphi}{\partial \psi} + 2 \operatorname{cotg} \theta \cdot v_\theta \right) \tag{8.7}$$

vanishes for $\theta = 0$. This is seen from Eq. (2.6, II) noting that

$$\frac{dv_\varphi}{dt} = \Omega \frac{\partial v_\varphi}{\partial \psi} \text{ and } \sigma_\theta v_\theta = 2 \operatorname{cotg} \theta \cdot \Omega v_\theta$$

are of the order $O_1(1)$. The sum of these two terms must vanish for $\theta = 0$ since the other terms in Eq. (2.6. II) are of the order $O_1(\theta^2)$. The sum (8.7) being zero we have $S_2 = O_2(\theta^2)$. In like manner we find $S_3 = O_2(\theta)$, $S_4 = O_2(\theta^2)$, $S_5 = O_2(\theta^2)$.

In (8.6) we may then write

$$\begin{aligned} S_i &= s_i^{(2)} \theta^2 + \dots \quad (i = 1, 2, 4, 5) \\ S_3 &= s_3^{(1)} \theta + \dots \end{aligned} \tag{8.8}$$

where the coefficients are of the form

$$s_i^{(m)} = a_i^{(m)} + b_i^{(m)} \sin 4\psi + c_i^{(m)} \cos 4\psi, \tag{8.9}$$

$a_i^{(m)}$ $b_i^{(m)}$ $c_i^{(m)}$ being functions of r.

Having regard to (6.18) we may seek to determine the following power series for the unknown corrections:

$$\begin{aligned} x_p &= x_p^{(4)}\theta^4 + x_p^{(6)}\theta^6 + \dots & x_\theta &= x_\theta^{(1)}\theta + x_\theta^{(3)}\theta^3 + \dots \\ x_r &= x_r^{(2)}\theta^2 + x_r^{(4)}\theta^4 + \dots & x_\tau &= x_\tau^{(2)}\theta^2 + x_\tau^{(4)}\theta^4 + \dots \\ x_\psi &= x_\psi^{(0)} + x_\psi^{(2)}\theta^2 + \dots & x_\varepsilon &= x_\varepsilon^{(2)}\theta^2 + x_\varepsilon^{(4)}\theta^4 + \dots \end{aligned} \tag{8.10}$$

where the coefficients $x_p^{(4)}$, $x_r^{(2)} \dots$ are assumed to be functions of the same form as $s_i^{(m)}$ (i. e. independent of t).[1])

Introducing the above series in Eq. (8.5) and the corresponding equations we obtain, putting $\frac{d}{dt} = \Omega \frac{\partial}{\partial \psi}$ and collecting in each equation the terms of the lowest order in θ:

$$\begin{aligned} &\text{I} && \Omega \frac{\partial x_r^{(2)}}{\partial \psi} - 2\Omega r x_\psi^{(0)} - \frac{g}{T} x_\tau^{(2)} = s_1^{(2)}, \\ &\text{II} && \Omega \left(\frac{\partial x_\psi^{(0)}}{\partial \psi} + 2 x_\theta^{(1)} \right) = s_2^{(0)} = 0, \\ &\text{III} && \Omega \left(\frac{\partial x_\theta^{(1)}}{\partial \psi} - 2 x_\psi^{(0)} \right) = s_3^{(1)}, \\ &\text{IV} && \frac{\partial x_\psi^{(0)}}{\partial \psi} + 2 x_\theta^{(1)} = s_4^{(0)} = 0, \\ &\text{V} && c_p \Omega \frac{\partial x_\tau^{(2)}}{\partial \psi} + (g - c_p \delta) x_r^{(2)} - x_\varepsilon^{(2)} = s_5^{(2)}. \end{aligned} \tag{8.11}$$

We note that Eqs. II and IV are identical. From II and III we obtain, having regard to (8.9)

$$\begin{aligned} x_\psi^{(0)} &= \varphi_1 \sin 2\psi + \varphi_2 \cos 2\psi \\ &\quad + \frac{1}{\Omega} \left(\tfrac{1}{6} b_3^{(1)} \sin 4\psi + \tfrac{1}{6} c_3^{(1)} \cos 4\psi - \tfrac{1}{2} a_3^{(1)} \right), \\ x_\theta^{(1)} &= \varphi_2 \sin 2\psi - \varphi_1 \cos 2\psi \\ &\quad + \frac{1}{\Omega} \left(\tfrac{1}{3} c_3^{(1)} \sin 4\psi - \tfrac{1}{3} b_3^{(1)} \cos 4\psi \right) \end{aligned} \tag{8.12}$$

where φ_1 and φ_2 are arbitrary functions of r. Inserting these values in Eqs. (8.11, I, V) an infinite number of solutions $x_r^{(2)}$ $x_\tau^{(2)}$ $x_\varepsilon^{(2)}$ can be obtained. By putting in I and V $x_r^{(2)} = 0$ it is seen that the solutions can be chosen so as to satisfy the boundary condition.

It is of interest to observe that $x_\psi^{(0)}$ may contain a non-periodic term, $-\frac{a_3^{(1)}}{2\Omega}$, whereas $x_\theta^{(1)}$ contains no such term. Thus, if $a_3^{(1)} \neq 0$, a slow constant motion along the circles of latitude is superposed on the oscillating motion.

Eqs. (8.11) were formed by collecting terms of the lowest order in θ. Repeating the procedure for the next power of θ we can deduce 5 new equations which contain, in addition to the variables of (8.11), the 6 new variables

$$x_r^{(4)} \; x_\psi^{(2)} \; x_p^{(4)} \; x_\tau^{(4)} \; x_\theta^{(3)} \; x_\varepsilon^{(4)}.$$

It is always possible to determine these so as to satisfy the 5 equations and the boundary condition.

As a consequence of this result we may affirm that it is possible to determine the corrections in the form (8.10) and since we have $\left| \frac{x_p}{p^*} \right| << 1$ for all values of θ, including $\theta = 0$, and corresponding inequalities for the other variables, the functions $p^* \dots \varepsilon_s$ having the property

[1]) Some of the first terms of the series (8.10) may possibly vanish.

(6.18) are then approximate solutions of the exact hydrodynamical equations.

The determination of the corrections $x_p \ldots x_\tau$ in the general case would involve rather lengthy computations. We shall treat here only a special case which, though very simple, presents some features of interest.

Let us consider a region at the equator at the ground. Since the oscillation is assumed to be symmetrical to the equator we then have

$$v_r = 0, \quad v_\theta = 0.$$

Putting $V_\psi = \Omega$, $\theta = \frac{\pi}{2}$, supplementing Eq. (2.6, II) with the terms of Table 2, II and introducing the expressions (8.4), we obtain

$$\frac{dx_\psi}{dt} + \frac{T}{r^2}\frac{\partial x_p}{\partial \psi} = -v_\psi \frac{\partial v_\psi}{\partial \psi} - \frac{1}{r^2}\tau_s \frac{\partial p^*}{\partial \psi}. \tag{8.13}$$

In the following the periodic terms need not be studied separately whereas the non-periodic terms will be of special interest. For brevity we shall denote any sum of the form

$$\sum_m A_m \sin(m\psi + \gamma_m t + \eta_m), \tag{8.14}$$

where m is an integer ($m \neq 0$), by $\sum f_2$ if A_m is of the order of magnitude of p^2 or higher order. Differentiations with respect to ψ and t render similar expressions which will also be denoted by $\sum f_2$.

Let us assume that $x_p = \sum f_2$ and that $\bar{p}^*$ is given in the form

$$\bar{p}^* = A_p^{(1)} \sin(\psi + \eta_p^{(1)}) + A_p^{(2)} \sin(2\psi + \eta_p^{(2)}) + \sum f_2 \quad ^{1)} \tag{8.15}$$

in conformity with the pressure wave observed in the atmosphere. It is easily seen that we may write.

$$v_\psi \frac{\partial v_\psi}{\partial \psi} = \sum f_2. \tag{8.16}$$

Further we have

$$\tau_s = A_\tau^{(1)} \sin(\psi + \eta_\tau^{(1)}) + A_\tau^{(2)} \sin(2\psi + \eta_\tau^{(2)})$$

$$\frac{\partial p^*}{\partial \psi} = A_p^{(1)} \cos(\psi + \eta_p^{(1)}) + 2A_p^{(2)} \cos(2\psi + \eta_p^{(2)}) \tag{8.17}$$

whence it follows that

$$-\tau_s \frac{\partial p^*}{\partial \psi} = a_2 + \sum f_2, \text{ where} \tag{8.18}$$

$$a_2 = \tfrac{1}{2} A_p^{(1)} A_\tau^{(1)} \sin(\eta_p^{(1)} - \eta_\tau^{(1)}) + A_p^{(2)} A_\tau^{(2)} \sin(\eta_p^{(2)} - \eta_\tau^{(2)}) \tag{8.19}$$

1) $\sum f_2$ is zero or too small to be observed. The 8-hour oscillation, being negligible at the equinoxes, is omitted here.

is a function of r alone. Introducing the expressions (8.15, 16, 18) the equation (8.13) can be written in the form

$$\frac{dx_\psi}{dt} = \frac{a_2}{r^2} + \sum f_2.$$

Since, by definition, $\frac{d}{dt} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \psi}$ we obtain by integration

$$x_\psi = \frac{a_2}{r^2} t + \sum f_2 + \varphi(\psi - \Omega t)$$

where φ is an arbitrary function. We must have

$$\varphi(\psi + 2\pi - \Omega t) = \varphi(\psi - \Omega t)$$

so that φ must have the form

$$\varphi = A_0 + \sum_m A_m \sin m(\psi - \Omega t) + \sum_m B_m \cos m(\psi - \Omega t),$$

Including $\varphi - A_0$ in $\sum f_2$ we may finally write

$$r x_\psi = \frac{a_2}{r}(t - t_0) + \sum f_2, \tag{8.20}$$

where t_0 denotes an arbitrary constant.

Thus it appears that $r x_\psi$ contains a non-periodic term $\frac{a_2}{r}(t - t_0)$. Obviously an infinite number of cases may be imagined where $a_2 = 0$, the simplest being the case $\eta_p^{(1)} = \eta_\tau^{(1)}$, $\eta_p^{(2)} = \eta_\tau^{(2)}$, but in the most general case we have $a_2 \neq 0$ which means that the air parcel at the equator is subjected to a constant force directed west-east or v. v. according as a_2 is positive or negative. In the course of time $|v_\psi + x_\psi|$ will then be steadily increasing unless some counteracting force is introduced. In the real atmosphere such a force exists, viz. the friction at the ground. If the force corresponding to the acceleration $\frac{a_2}{r}$ were balanced by friction the result would be a slow but steady west-east or east-west motion superposed on the periodic motion given by v_ψ so that the oscillating air masses at the equator at the ground would gradually travel round the globe.

Let us introduce in (8.19) some values from observations in the atmosphere near the equator. Observations from Batavia give for the temperature

$$\eta_\tau^{(1)} = 232° \quad \eta_\tau^{(2)} = 63° \quad ^{1)}$$

Observations in Rurki (India) give for the pressure

$$\eta_p^{(1)} = 325° \quad \eta_p^{(2)} = 146° \quad ^{1)}$$

1) Hann-Süring: Lehrbuch der Meteorologie, 5. Aufl. 1939, p. 209 and 289.

This gives positive values for both terms in Eq. (8.19) so that a_2 is positive, which indicates the possibility of a slow west-east motion at the Equator at the ground. A closer investigation of this matter would require the inclusion of frictional terms in the original equations.

It would be of interest to examine the non-periodic motion of the air masses in other parts of the atmosphere as well, but the computations would be complicated owing to the large number of second-order terms.

9. Numerical values of the variables in the 12-hour oscillation.

The oscillation ($n = 2$, $\alpha = 1$), treated in para. 5 and 6 b, is of special interest because of its resemblance to the semidiurnal oscillation of the real atmosphere. Let us suppose that the model atmosphere described in the introduction performs an oscillation of the type ($n = 2$, $\alpha = 1$) with the significant variables

$$p^*, \tau_s, v_r, v_\varphi, v_\theta, \varepsilon_s$$

and let the values of the corresponding variables in the semidiurnal oscillation of the real atmosphere be

$$(p^*), (\tau_s) \dots (\varepsilon_s).$$

Choosing for p^* the value $p^* = (p^*)$ we may compute τ_s, $\boldsymbol{v}$ and ε_s by means of the mathematical theory of the preceding paragraphs. The numerical values thus obtained for τ_s and $\boldsymbol{v}$ may be compared to the observed values of (τ_s) and $(\boldsymbol{v})$. Within the limits set by observational errors it will thus be possible to ascertain whether the atmosphere oscillates more or less like the simple model. In the following we shall lay stress on a systematic computation of v_r and of ε_s.

A general method of finding the numerical values of τ_s, $\boldsymbol{v}$ and ε_s, when p^* is known, is the following:

Assuming p^* in Eqs. (2.8—9) to be given by the auxiliary quantities $p^{(1)}$, $p^{(2)}$, n, γ, we consider Eqs. (3.1) and the corresponding equations for v and τ. Putting $r = r_0$, $w = 0$ and denoting by the subscript nought the values of the variables for $r = r_0$ we find ε_0 from the first equation (provided that $d_0 \neq 0$) and u_0, v_0, τ_0 from the other equations. Introducing these quantities in Eq. (2.14, IV) we find $\left(\frac{\partial w}{\partial r}\right)_0$. The next step is based on the assumption that w may be considered as a linear function of r, viz. $w = k(r - r_0)$, from the ground upwards to a certain height h. Denoting by the subscript h the values for $r - r_0 = h$ we may write

$$w_h = \left(\frac{\partial w}{\partial r}\right)_0 h\,.$$

Introducing this in Eq. (3.1) and the corresponding equations for τ and v we find $(u, v, \tau, \varepsilon)_h$, and introducing the latter in Eq. (2.14, IV) we compute $\left(\frac{\partial w}{\partial r}\right)_h$. This quantity may be considered as a mean value of $\frac{\partial w}{\partial r}$ from $r = r_0$ to $r = r_0 + 2h$ so that we may put

$$w_{2h} = \left(\frac{\partial w}{\partial r}\right)_h \cdot 2h.$$

From Eqs. (3.1) and the corresponding equations for τ and v, and from (2.14, IV) we find

$$\left(u, v, \tau, \varepsilon, \frac{\partial w}{\partial r}\right)_{2h}.$$

Starting from the altitude $2h$ the whole procedure may be repeated, putting

$$w_{3h} = w_{2h} + \left(\frac{\partial w}{\partial r}\right)_{2h} \cdot h,$$

$$w_{4h} = w_{2h} + \left(\frac{\partial w}{\partial r}\right)_{3h} \cdot 2h$$

etc.

Having computed $u^{(i)}$ $v^{(i)}$ $w^{(i)}$ $\tau^{(i)}$ and $\varepsilon^{(i)}$ ($i = 1$, $i = 2$) by means of the known values of $p^{(i)}$ we thus know v_r, v_φ, v_θ, τ_s, ε_s from the ground upwards to a certain height H above which no accurate pressure observations exist.

In the following we shall give a numerical example of the above method, choosing the oscillation $n = 2$, $\alpha = 1$ in the model atmosphere and pressure observations from the semidiurnal oscillation of the real atmosphere. Although the quasistatic equations were found to be invalid near the Pole they will give sufficient accuracy in the following computations, which deal with observations from middle latitudes.

For simplicity we shall limit our considerations to the barotropic case treated in para. 4. The oscillations ($n = 1$, $\alpha = \frac{1}{2}$) and ($n = 2$, $\alpha = 1$)

will be called the «24-hour oscillation» and the «12-hour oscillation» respectively. These are included in the more general case $\gamma = 0$, $\alpha = \frac{n}{2}$. Applying the latter condition to Eqs. (4.4) and introducing the quasistatic simplification (omitting the first and second term of (4.4, I) and the third term of II) we obtain

$$(9.1)\quad \text{I} \qquad \tau = \frac{T^2}{g}\frac{\partial p}{\partial r},$$

$$\text{II} \qquad ru = \frac{-nT}{\Omega r\,(4\cos^2\theta - n^2)}\left(2\,\text{cotg}\,\theta\cdot p + \frac{\partial p}{\partial\theta}\right),$$

$$\text{III} \qquad r\sin\theta\cdot v = \frac{n^2 T}{\Omega r\,(4\cos^2\theta - n^2)}\left(\frac{1}{\sin\theta}p + \frac{2}{n^2}\cos\theta\frac{\partial p}{\partial\theta}\right),$$

$$\text{IV} \qquad \frac{n\Omega}{R}p - \frac{n\Omega}{T}\tau - \frac{\delta_b}{T}w + \frac{\partial w}{\partial r} + nv + \frac{\partial u}{\partial\theta} + \text{cotg}\,\theta\cdot u = 0,$$

$$\text{V} \qquad n\Omega c_p\tau - n\Omega Tp + c_p(\delta_a - \delta)\,w - \varepsilon = 0.$$

The first three equations are derived by a simple transformation of Eqs. (4.4, I, II, III) (after introduction of the quasistatic approximation). In the third term of Eq. IV $\frac{2}{r}$ has been neglected since we have $\frac{2}{r} << \frac{\delta_b}{T}$. By means of Eqs. (9.1) $\tau, u, v, w, \varepsilon$ can be computed when p is known. The following computations will be based on values of $p^{(i)}$ derived from pressure observations collected and studied by *Wagner* (1932). The stations from which his material was taken are situated in the Alps at heights ranging from 500 to 4500 m above sea level and are selected so as to be representative for the free atmosphere, as far as possible. Their middle latitude is about 47° corresponding to $\theta = 43°$.

According to *Simpson* (1918) we have for the atmospheric 12-hour oscillation with good approximation

$$(9.2)\quad A_p = (A_p)_e \sin^3\theta,\ \frac{\partial\eta_p}{\partial\theta} = 0,\ \text{i. e.}\ \frac{\partial}{\partial\theta}\left(\frac{p^{(2)}}{p^{(1)}}\right) = 0,$$

whence it follows that

$$(9.3)\qquad p^{(1)} = p_e^{(1)}\sin^3\theta,\quad p^{(2)} = p_e^{(2)}\sin^3\theta,$$

$(A_p)_e$, $p_e^{(1)}$, $p_e^{(2)}$ denoting the values of these variables at the equator.

Introducing (9.3) in (9.1) and putting $n = 2$ we obtain for the 12-hour oscillation:

$$(9.4)\quad ru = \frac{5T\cos\theta}{2\,\Omega r\sin^3\theta}p,\quad \frac{\partial u}{\partial\theta} = \frac{-5T}{2\,\Omega r^2\sin^2\theta}p.$$

$$(9.5)\qquad r\sin\theta\cdot v = \frac{-Tp}{\Omega r\sin^3\theta}(1 + \tfrac{3}{2}\cos^2\theta),$$

$$(9.6)\qquad \frac{\partial w}{\partial r} - \frac{\delta_b}{T}w = \frac{2\,\Omega T}{g}\frac{\partial p}{\partial r} + Bp$$

where

$$(9.7)\qquad B = \frac{5 + 4\sin^2\theta}{2\,\Omega r^2\sin^4\theta}T - \frac{2\,\Omega}{R}.$$

For $\theta = 43°$ we then have

$$(9.8)\quad ru = \frac{5{,}73\,T}{\Omega r}p,\quad r\sin\theta\cdot v = -\frac{5{,}80\,T}{\Omega r}p,$$

$$(9.9)\qquad B = 0{,}536\cdot 10^{-8}\,T - 0{,}51\cdot 10^{-6},$$

using the Meter-Ton-Second system of units.[1])

Choosing

$$T = T_0 = 283°\ \text{for}\ r = r_0,\ \delta = -\frac{\partial T}{\partial r} = 6\cdot 10^{-3},$$

we find for the coefficients of Eq. (9.6) the values set out in Table 5 below.

Table 5.

$h = r - r_0$	T	$-\frac{\delta_b}{T}$	$\frac{2\,\Omega T}{g}$	B
0	283	-1.00.10⁻⁴	4.21.10⁻³	1.00.10⁻⁶
2000	271	-1 04 »	4.03 »	0.94 »
4000	259	-1.09 »	3.85 »	0.88 »
8000	235	-1.20 »	3 49 »	0.76 »

The numerical computation of the vertical velocity v_r and the supplied heat ε_s in the 12-hour oscillation is presented in Tables 6, 7 and 8.

[1]) (9 2—3) are empirical relations which hold in low and middle latitudes In high latitudes A_p becomes so small that the relations can no longer be verified by observation. We have found, however, that $p = O(\theta^4)$ when $\theta \to 0$, which shows that Eqs. (9 2—9) do not hold when $\theta \to 0$.

Table 6.

1	2	3	4	5	6	7	8	9	10	11	12	13
h (km).	$R \frac{a_2}{b}$	A_2	A_p	η_p	$p^{(1)} = A_p \cos \eta_p$	$\frac{\partial p^{(1)}}{\partial r}$	$p^{(2)} = A_p \sin \eta_p$	$\frac{\partial p^{(2)}}{\partial r}$	$\frac{2\Omega T}{g} \frac{\partial p^{(1)}}{\partial r}$	$Bp^{(1)}$	$\frac{2\Omega T}{g} \frac{\partial p^{(2)}}{\partial r}$	$Bp^{(2)}$
0			$10.80\ 10^{-2}$	149°	$-9.25.10^{-2}$	$1.8.10^{-5}$	$5.56\ 10^{-2}$	$1.6.10^{-5}$	$7\ 6.10^{-8}$	$-9.3\ 10^{-8}$	$6\ 7\ 10^{-8}$	$5\ 6\ 10^{-8}$
0.5	$10\ 38.10^{-2}$	144.8	10.44	143	–8.35	1.9	6.29	1 3	7.9	–8.3	5 4	6 3
1	10.12	135.7	10.07	137	–7.35	2.0	6.85	1.0	8 2	–7.1	4 1	6 6
1.5	9 85	131.1	9.70	131	–6.36	2 2	7.32	0.8	9.0	–6.1	3.3	7.0
2	9.27	123.6	9.33	124	–5.21	2 3	7.74	0.6	9.3	–4 9	2 4	7 3
2.5	8.92	117.2	8.95	117	–4.06	2 4	7.97	0.4	9 6	–3.7	1 6	7.3
3	8.65	109.0	8.65	109	–2 82	2 4	8.16	0 3	9 5	–2.6	1.2	7.4
3.5	8.44	100.6	8 44	101	–1.61	2.2	8.30	0 3	8.6	–1.4	1.2	7 5
4	8.60	94.0	(8.55)	(94)	(–0.60)	(1.9)	(8.54)	(0 4)	(7 3)	(–0.5)	(1 5)	(7.5)
4.5	8.74	87.9	(8.80)	(88)	(0.31)	(1.8)	(8.80)	(0.5)	(6.9)	(0.3)	(1.9)	(7 5)

Table 7.

h	$w^{(1)}$	$w^{(2)}$	η_r	A_r
1000	$-1.7.10^{-5}$	$12.3.10^{-5}$	98°	$12.4.10^{-5}$
2000	1.8 »	23.8 »	86°	23.9 »
3000	6 4 »	35.9 »	80°	36.4 »
4000	17.0 »	48.6 »	71°	51.5 »

Table 8.

h	$-2\Omega Tp^{(1)}$	$\frac{2\Omega c_p T^2}{g} \frac{\partial p^{(1)}}{\partial r}$	$c_p(\delta_a - \delta)\, w^{(1)}$	$\varepsilon^{(1)}$	$-2\Omega Tp^{(2)}$	$\frac{2\Omega c_p T^2}{g} \frac{\partial p^{(2)}}{\partial r}$	$c_p(\delta_a - \delta)\, w^{(2)}$	$\varepsilon^{(2)}$	η_ε
0	$3.82.10^{-3}$	$21.5.10^{-3}$	0	$25.3.10^{-3}$	$-2.30.10^{-3}$	$19\ 1.10^{-3}$	0	$16\ 8.10^{-3}$	34°
1000	2.98 »	22.8	$-0.06.10^{-3}$	25.8 »	–2.78 »	11.4 »	$0.5.10^{-3}$	9.1 »	20°
2000	2.06 »	25 2	$0\ 07\ 10^{-3}$	27.3 »	–3 06 »	6.5 »	0.9 »	4.3 «	9°
3000	1.09 »	25 0	0 24 »	26 3 »	–3 16 »	3 1 »	1.4 »	1.3 »	3°
4000	0.23 »	19.0	0.64 »	19.9 »	–3.23 »	4 0 »	1.9 »	2.7 »	8°

Table 9.

h	$\tau^{(1)}$	$\tau^{(2)}$	η_τ	$ru^{(1)}$	$ru^{(2)}$	$r \sin\theta\, v^{(1)}$	$r \sin\theta\, v^{(2)}$	rA_θ	$r \sin\theta A_\psi$
0	0.15	0.13	42°	–0.32	0 19	0.32	–0.19	0.37	0.37
1000	0 16	0.08	27°	–0.25	0.23	0.25	–0.23	0.34	0.34
2000	0.17	0.04	15°	–0 17	0.26	0.17	–0.25	0.31	0.31
3000	0.17	0.02	7°	–0.09	0.27	0.09	–0.26	0.28	0.28
4000	0.13	0.03	12°	–0.02	0.27	0.02	–0.27	0.27	0 27

Explanation of Tables 6—9.

In the paper by *Wagner* (1932) mentioned above we find on page 323 a table giving the values of $\frac{a_2}{b}$ at different levels, a_2 being the amplitude of the 12-hour pressure oscillation and b the mean pressure. In the notation of the present paper we may write

$$b = P \quad R\frac{a_2}{b} = A_p.$$

In *Table 6* the 1st column gives the height above sea level in kilometres. The 2nd and 3rd column give $R\frac{a_2}{b}$ and the phase angle A_2 taken from Wagner's tables. In fig. 1 and 2 these values are plotted against the height and a smoothed curve is drawn through the points obtained. The values extracted from these smoothed curves are set out in the 4th and 5th column. The value for $h = 0$ has been found by extrapolation. The columns 6—9 contain the values of $p^{(1)}, \frac{\partial p^{(1)}}{\partial r}, p^{(2)}, \frac{\partial p^{(2)}}{\partial r}$. Columns 10—11 and 12—13 give the quantities on the right-hand side of Eq. (9.6) for $w^{(1)}$, $p^{(1)}$ and for $w^{(2)}$, $p^{(2)}$ respectively.

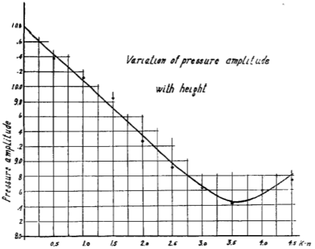


Fig. 1.

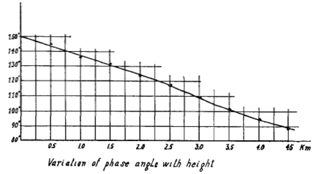


Fig 2.

The values of $w^{(1)}$ and $w^{(2)}$ in *Table 7* are found in the following manner. Starting with $r - r_0 = h = 0$, $w = 0$ we obtain from Eq. (9.6) by means of Table 5 and 6

$$\left(\frac{\partial w^{(1)}}{\partial r}\right)_0 = -1{,}7 \cdot 10^{-8}$$

Putting

$$w^{(1)}{}_{1000} = \left(\frac{\partial w^{(1)}}{\partial r}\right)_0 \cdot 1000 = -1{,}7 \cdot 10^{-5}$$

and introducing this in Eq. (9.6) together with the values for $h = 1000$ taken from Table 6, 10th—11th column, we obtain

$$\left(\frac{\partial w^{(1)}}{\partial r}\right)_{1000} = 0{,}9 \cdot 10^{-8}.$$

Taking this as a mean value in the interval $h = 0$ to $h = 2000$ we put

$$w^{(1)}{}_{2000} = \left(\frac{\partial w^{(1)}}{\partial r}\right)_{1000} \cdot 2000 = 1{,}8 \cdot 10^{-5}.$$

Introducing this value in Eq. (9.6) together with the value of the quantities on the right-hand side for $h = 2000$ taken from Table 6, 10—11th column, we find

$$\left(\frac{\partial w^{(1)}}{\partial r}\right)_{2000} = 4{,}6 \cdot 10^{-8}$$

and hence

$$w^{(1)}{}_{3000} = w^{(1)}{}_{2000} + \left(\frac{\partial w^{(1)}}{\partial r}\right)_{2000} \cdot 1000 = 6{,}4 \cdot 10^{-5}$$

By means of Eq. (9.6) we then compute

$$\left(\frac{\partial w^{(1)}}{\partial r}\right)_{3000} = 7{,}6 \cdot 10^{-8}$$

which is taken as a mean value in the interval from $h = 2000$ to $h = 4000$ so that we have

$$w^{(1)}{}_{4000} = w^{(1)}{}_{2000} + \left(\frac{\partial w^{(1)}}{\partial r}\right)_{3000} \cdot 2000 = 17{,}0 \cdot 10^{-5}.$$

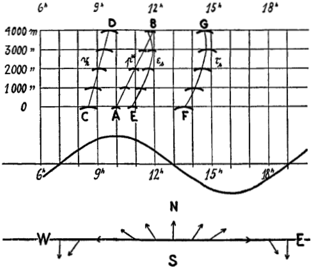
In the same manner $w^{(2)}$ has been found, using the columns 12—13 of Table 6. η, and A_r have been computed from the formulae

$$\operatorname{tg} \eta_r = \frac{w^{(2)}}{w^{(1)}} \qquad A_r = \sqrt{w^{(1)2} + w^{(2)2}} \quad ^{1)}$$

Table 8 shows the computation of $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ by means of Eq. (9.1, I and V). The values of $p^{(i)}$ and $w^{(i)}$ used in this table are taken from Table 6 and 7.

Table 9 gives temperature and horizontal velocity computed from the formula (9.1, I) and (9.8) using the values of Table 6.

On the basis of the values η_r, η_ε, η_τ given in Table 7—9 and the observed values of η,



Phase angles of the variables in the semidiurnal oscillation

Fig. 3.

Fig. 3 is plotted so as to show the time of maximum of the variables at different levels. As abscissa is chosen local time expressed in hours from midnight. The sinusoidal curve represents the pressure wave at sea level. The time of pressure maximum at differents levels is shown by the line AB, A giving the time of maximum at sea level and B at 4000 metres. In like manner the time of maximum of v_r, ε_s and τ_s is given by the lines CD, EB and FG respectively. Thus, for instance, the maximum of $v_r = A_r \cos(2\psi + \eta_r)$ takes place when $\psi = \psi_m = -\frac{1}{2}\eta_r$. From Table 7 we have, for $h = 2000$, $\psi_m = -43°$ corresponding to about 3 hours before 0^h and 12^h (i. e. 9^h and 21^h).

At the bottom of Fig. 3 the phase of the horizontal wind wave at sea level is shown. At the latitude considered the wind force is practically constant.

It will be of interest to compare the values of τ_s, v_θ, v_ψ computed above to observed values. From Tables 6 and 9 we have for $h = 2500$ m since $\eta_\theta = \eta_p$, $\eta_\psi = \eta_p + 180°$, by (9.8):

$$\text{(9.10)} \qquad r v_\theta = 0{,}29 \cos(2\psi + 117°)$$
$$r \sin\theta \;\; v_\psi = 0{,}29 \sin(2\psi + 297°).$$

J. v. Hann (1903) gave the following annual mean values for the Säntis (47° N, 2500 m above sea level)

$$\text{(9.11)} \qquad r v_\theta = 0{,}27 \cos(2\psi + 149°{,}9),$$
$$r \sin\theta \cdot v_\psi = 0{,}29 \sin(2\psi + 329°{,}4).$$

It is seen that the computed amplitude gives a very good fit to the observed values. In phase there is a difference between (9.10) and (9.11) amounting to about an hour.

As regards τ_s it is difficult to obtain representative values from the free atmosphere by direct observations. The amplitude at the Equator has been estimated by *Chapman* (1924) to 0°.4. This seems to agree fairly well with the values of Table 9 which gives a temperature amplitude of $\sqrt{0{,}15^2 + 0{,}13^2} = 0{,}2$ in the 12-hour oscillation at 47° N at sea level. However, observations at Lindenberg indicate a greater value (Table 11).

The method of computing the temperature oscillation from Eq. (9.1, I) is due, in principle, to *Hergesell* (1919). The horizontal wind wave was computed by *Margules*, whose results were of the same order of magnitude as the values of Table 9, although the assumptions underlying his investigation were somewhat different from those made here.

As regards the vertical velocity no great accuracy can be ascribed to the values of Table 7

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1) The numerical computations necessary for the evaluation of the tables have been made by means of an ordinary slide rule. The errors due to this method of computation are unimportant in comparison with observational errors. The numerical values presented in Table 6—9 must be considered as rather rough approximations. It may be noted, for instance, that $w^{(1)}$ at h = 1000—2000 m is found as difference between two small quantities so that its value is very inaccurate. However, this is of minor importance since the approximations obtained suffice to show that $|w^{(1)}|$ is small compared to $|w^{(2)}|$ which means, (a) that η_r is not far from $\frac{\pi}{2}$ and, (b) that the amplitude A_r is nearly equal to $|w^{(2)}|$.

but it follows from the equation of continuity (9.1, IV) that, when p, τ, v, u are determined with the right order of magnitude, the computed value of w cannot be very far from the real value. However, since the latter cannot be observed, no direct verification is possible. It is seen that v_r is extremely small below 4000 m, but v_r is likely to increase rapidly with height.

The heat supplied to a unit mass of air per unit time in the 12-hour oscillation is given in Table 8. The amplitude $A_\varepsilon = \sqrt{\varepsilon^{(1)^2} + \varepsilon^{(2)^2}}$ is approximately 0.03 at sea level and decreases to about 0.02 at 4000 m. Since we have assumed $W = \text{const}$, $\frac{dW}{dt} = 0$ in the fundamental state, we have in the state of perturbation

$$\varepsilon_s = \triangle\left(\frac{dW}{dt}\right) = \frac{\partial}{\partial t}(W + \triangle W) - \frac{dW}{dt} = \frac{d}{dt}\triangle W$$
$$= A_\varepsilon \cos(2\psi + \eta_\varepsilon).$$

During the time interval $t = t_1$ to $t = t_2$ the heat supplied to a unit mass is

$$\triangle W\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} A_\varepsilon \cos(2\psi + \eta_\varepsilon)\, dt.$$

Let us choose for t_1 the instant when ε_s has its maximum value at sea level, viz. $t_1 = 11^h$ local time (Fig. 3), and for t_2 the value $t_2 = t_1 + 3^h = 14^h$. The corresponding values of $\psi = \Omega t$ are

$$\psi_1 = -\tfrac{1}{2}\eta_\varepsilon \quad \psi_2 = \psi_1 + \frac{\pi}{4}.$$

Thus the heat supplied to a unit mass from 11^h to 14^h may be written

$$\triangle W\Big|_{11^h}^{14^h} = \frac{A_\varepsilon}{2\Omega}\int_{-\eta_\varepsilon}^{\frac{\pi}{2}-\eta_\varepsilon} \cos(2\psi + \eta_\varepsilon)\, d(2\psi) = \left(\frac{A_\varepsilon}{2\Omega c_p}\right) c_p.$$

Inserting here the numerical value $A_\varepsilon = 0.03$ we obtain

$$\triangle W\Big|_{11^h}^{14^h} = 0{,}20\ c_p \text{ at sea level.}$$

In like manner we find

$$\triangle W\Big|_{12^h}^{15^h} = 0{,}14\ c_p \text{ at the 4000 m. level.}$$

Thus the heat supplied at sea level in the 3-hour interval considered is equivalent to the amount of heat necessary to increase the temperature by 0.2 degrees at constant pressure.

The above computations refer to 47° latitude. At any other middle or low latitude $\frac{\pi}{2} - \theta$ the values of $p^{(i)}$, $\frac{\partial p^{(i)}}{\partial r}$ can be found by multiplying the values of Table 6 by $\frac{\sin^3\theta}{\sin^3 43°}$. The other variables are then deduced from Eqs. (9.1) putting $n = 2$ and having regard to (9.3). It is easily seen that

$$\frac{u^{(2)}}{u^{(1)}} = \frac{v^{(2)}}{v^{(1)}} = \frac{p^{(2)}}{p^{(1)}}, \text{ i. e. } \operatorname{tg}\eta_\theta = \operatorname{tg}\eta_\varphi = \operatorname{tg}\eta_p,$$

and since η_p is independent of θ this is also the case with η_θ and η_φ, the phase angles of the north-south and the west-east wind component. The same applies to the phase η_τ of temperature and also to η_ε of the "heat oscillation" in the lower layers where the latter is determined chiefly by the temperature.

As regards the vertical velocity, v_r, the case is different. We find here an appreciable variation of phase with latitude. This can be seen by computing $w^{(1)}$ and $w^{(2)}$ at the equator. From Eq. (9.7) it follows for $\theta = 90°$:

$B = -0.08$ at sea level,
$B = -0.12$ at the 4000 m. level.

In Eq. (9.6) we have to introduce for $p^{(i)}$ and $\frac{\partial p^{(i)}}{\partial r}$ the values of Table 6 multiplied by $\frac{1}{\sin^3 43°} = 3.16$. By the method described above, in connection with Table 7, we then find the following values:

Table 10.

h	$w^{(1)}$	$w^{(2)}$	η_r
1000	$26.3.10^{-5}$	$19\,8.10^{-5}$	37°
2000	61.6 »	26 2 »	23°
3000	99.0 »	34.1 »	19°
4000	144.6 »	35.4 »	14°

From the last column it is seen that the maximum of v_r takes place at 10.8^h at sea level and 11.5^h at the 4000 m. level, i. e. about two hours later at the equator than at 47° latitude.

The computations of this paragraph all refer to levels from which reliable pressure observations are available. At higher levels the properties of the 12-hour oscillation can be studied only on the basis of hypothetical assumptions regarding one of the variables. Thus we may for instance choose values of $\varepsilon^{(i)}$ tending towards zero with height, which means that the oscillation is supposed to become adiabatic above a certain level. The value of ε being prescribed, the other variables of Eqs. (9.1) may be computed as follows:

Let H be the highest level where p^* and $\frac{\partial p^*}{\partial r}$ are determined by observations. The values of the other variables at this level can be found by computation using the method described above. We then compute $\left(\frac{\partial w}{\partial r}\right)_H$ from (9.1, IV). Putting

$$w_{H+h} = w_H + \left(\frac{\partial w}{\partial r}\right)_H \cdot h \qquad p_{H+h} = p_H + \left(\frac{\partial p}{\partial r}\right)_H h$$

and introducing w_{H+h} and p_{H+h} in (9.1, II, III, V) we find u_{H+h} and v_{H+h}. Since ε is given (9.1, V) determines τ_{H+h}. $\left(\frac{\partial p}{\partial r}\right)_{H+h}$ and $\left(\frac{\partial w}{\partial r}\right)_{H+h}$ are deduced from Eq. I and IV respectively. Now we may write

$$w_{H+2h} = w_H + \left(\frac{\partial w}{\partial r}\right)_{H+h} \cdot 2h\,,$$

$$p_{H+2h} = p_H + \left(\frac{\partial p}{\partial r}\right)_{H+h} \cdot 2h\,,$$

and Eqs. (9.1) then give $(u, v, \tau)_{H+2h}$. Thus all the variables are known at the level $H + 2h$, and we may proceed from this to higher levels repeating the procedure.

Some computations of this kind for the equatorial region have been attempted by the author in a previous paper. A feature of interest is the steady increase of the amplitude of vertical velocity with height, while the time of maximum remains near to 12^h.

10. The heat transport and the heat converted to work in the 24-hour and 12-hour oscillation.

In the following the indices 24 and 12 will be used to denote variables of the 24-hour and the 12-hour oscillation respectively.

It has been shown in previous paragraphs that, if the pressure wave p^*_{12} is considered as a given function of r, ψ, θ, t, the other variables are then determined and can be computed by means of the five hydrodynamical equations, having regard to the boundary condition at the ground. It appeared that only one solution is possible and that the "heat oscillation", $(\varepsilon_s)_{12}$, of this solution furnishes a heat supply of 0.2 c_p in a 3-hour interval at 47° latitude and sea level.

Although $(\varepsilon_s)_{12}$ is thus a small quantity it is important in the theory of the oscillation since the "tilting" of the pressure wave would not be possible in the model atmosphere if we had $(\varepsilon_s)_{12} = 0$, as shown in para. 3.

The equations give mathematical relations between the significant variables but can give no information concerning cause and effect. The latter must be deduced from physical considerations. Most authors, among them Lord Kelvin, have regarded the temperature wave $(\tau_s)_{12}$ as the cause of the pressure wave. However, since the supplied heat depends more directly upon radiation and the physical properties of the atmosphere, it seems preferable to consider the "heat oscillation", $(\varepsilon_s)_{12}$, as the primary phenomenon and the variations in pressure, temperature and velocity as secondary phenomena.

It was shown in para. 3 that, if $(\varepsilon_s)_{12}$ is given, the hydrodynamical equations and the boundary condition at the ground are not sufficient to determine p^*_{12}, but if we add also certain conditions which must hold in the vicinity of the Pole, as stated in para. 6 B, p^*_{12} is then completely determined in the polar region and may possibly be determined all over the globe by analytical continuation. If this is the case, the small "heat oscillation" will provide an explanation of the 12-hour wave in the model atmosphere. This leads us to search for the causes that determine $(\varepsilon_s)_{12}$.

The problem must be closely related to the flow of heat through the atmosphere connected with radiation from the sun and outgoing radiation from the atmosphere. When considering the flow of heat it must be borne in mind that we are here concerned solely with a model atmosphere having no motion apart from the combined 24-hour and 12-hour wave.

In the simple oscillation each individual parcel of air moves in an elliptic orbit. Thus, in the oscillation examined in para. 9 the parcels of air at 47° latitude move in circles with radius 2.5 km corresponding to the speed of 0.37 m/sec (Table 9). Heat is absorbed by the parcel in one part of its orbit and rejected in another part. Since we have $\varepsilon_s = A_\varepsilon \cos(n\psi + \eta_\varepsilon)$ there is no net loss or gain of heat during a period. If the atmosphere had no motion other than the pure 12-hour oscillation $(\varepsilon_s)_{12}$ might be determined solely by radiation and absorption of heat in the air masses and, $(\varepsilon_s)_{12}$ being given, the oscillation would probably be determined.

In the atmosphere, however, conditions are not so simple. So far we have considered the 24-hour and 12-hour oscillation separately, allowing for no interaction between them, but ultimately it will be necessary to consider also the complex motion of which these two simple oscillations are parts. If the heat supplied to the atmosphere is the cause of this complex motion the air masses must move in such a manner that the necessary flow of heat is maintained and simultaneously the kinematical conditions at the ground and in the vicinity of the Pole must be fulfilled.

It is easily seen that, in the complex motion, the paths of air parcels must be largely as indicated in Fig. 4 which shows the resultant of two motions in circular paths with the periods t_0 and $2t_0$ respectively.[1])

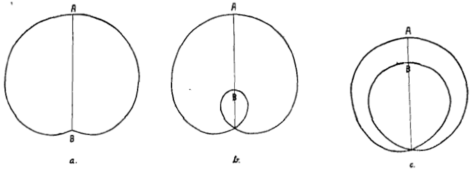

Fig. 4.

The air parcel moves from A to B during 12 hours and continues (in the clock-wise direction) from B to A during the next 12 hours. (a) shows the case of a small 12 hour wave superposed on a predominating 24-hour wave. In (b) the 12-hour component is greater and in (c) it is predominating.

In the complex oscillation we have

$$\varepsilon_s = A_\varepsilon^{(1)} \cos(\psi + \eta_\varepsilon^{(1)}) + A_\varepsilon^{(2)} \cos(2\psi + \eta_\varepsilon^{(2)}).$$

Fig. 5 indicates a possible form of ε_s. The areas S_1 and S_2 represent the amount of heat absorbed by a unit mass during certain parts of the 24-hour period and F_1 and F_2 represent the heat rejected during the rest of the period. We have $S_1 + S_2 = F_1 + F_2$.

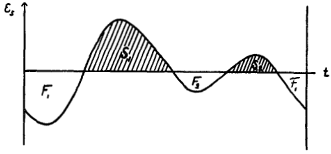

Fig. 5

Since the amplitudes and phase angles of ε_s can be altered in an infinity of ways without altering $S_1 + S_2$ it is seen that the same amount of heat can be transported by a parcel of air in an infinity of different motions of the types described in Fig. 4. Therefore, if kinematical conditions are unfavourable for the development of the 24-hour component, as indicated by the investigation made in para. 6, it seems probable that the 12-hour component will take over a greater part of the heat transport. $(\varepsilon_s)_{12}$ will then be comparatively great, the resulting path of the air parcels will be more like Fig. 4 (c)

[1]) A special case with regular paths has been chosen to illustrate the principal features of the motion An infinity of less regular paths may occur.

than 4 (a), and the 12-hour oscillation will thus be strengthened at the expense of the 24-hour oscillation.

This hypothesis is based upon the computations of para. 6 which must, however, be regarded as preliminary and incomplete since they are valid only in the vicinity of the Pole. It would be of interest to study the problem by means of more advanced mathematical methods. Pending a more complete investigation no proof can be given, but it may be pointed out that the hypothesis seems to conform with observational facts. It is important to observe that it is entirely independent of periods of free oscillations of the atmosphere. On the other hand, resonance is not in any way precluded by the above theory. In fact, the effect described would be added to a possible resonance effect.

Assuming that the flow of heat plays an important part in the theory of the complex oscillation, it is of interest to inquire whether any information about $(\varepsilon_s)_{12}$ and $(\varepsilon_s)_{24}$ can be derived from observations. It was mentioned in para. 9 that, in the lower layers, ε_s can be deduced with fairly good approximation from τ_s alone. Table 8 shows that, up to 1000 m, the terms $2\Omega Tp$ and $c_p(\delta_a - \delta)w$ in Eq. (4.4, V) are comparatively small so that we may put

$$\varepsilon^{(i)}{}_{12} = 2\Omega c_p \tau^{(i)}{}_{12} \quad (A_\varepsilon)_{12} = 2\Omega c_p (A_\tau)_{12}.$$

For the 24-hour oscillation the corresponding relations are

$$\varepsilon^{(i)}{}_{24} = \Omega c_p \tau^{(i)}{}_{24} \quad (A_\varepsilon)_{24} = \Omega c_p (A_\tau)_{24}.$$

Putting

$$\varrho_\varepsilon = \frac{(A_\varepsilon)_{12}}{(A_\varepsilon)_{24}} \qquad \varrho_\tau = \frac{(A_\tau)_{12}}{(A_\tau)_{24}}$$

we have, as a rough approximation

$$\varrho_\varepsilon \approx 2\varrho_\tau$$

in the lower layers of the atmosphere.

In Table 11 some observed values of temperature amplitudes are set out.[1])

Table 11.

	$(A_\tau)_{24}$	$(A_\tau)_{12}$	ϱ_τ	ϱ_ε	$(A_\tau)_{24}$	$(A_\tau)_{12}$	ϱ_τ	ϱ_ε
	Lindenberg				Batavia			
Surface	2.97	0.50	0.2	0.4	2.79	0.88	0.3	0.6
500 m	1.11	0.31	0 3	0.6	0.39	0.31	0.8	1.6
1000–2000	0.46	0.36	0 8	1.6	0 16	0.58	3 5	7 0

1) Hann-Süring, Lehrbuch der Meteorologie, 5. Aufl. p. 209.

In view of the marked 24-hour period in radiation one might expect to find $\varrho_\varepsilon < 1$ but it appears from the table that this holds only in the surface layer. The comparatively great values of ϱ_ε at the 1000 m level are at first thought surprising. However, they seem to be in accordance with the hypothesis advanced above concerning the heat transport and its relation to the kinematical conditions and the form of the motion.

In the literature on the 24-hour and 12-hour oscillation attention is always drawn to the fact that

$$\varrho_p = \frac{(A_p)_{12}}{(A_p)_{24}}$$

is surprisingly great. Observations from small tropical islands, where local influences are negligible, give the approximate value $\varrho_p \approx 5$. It should be noted that ϱ_ε is not necessarily of the same magnitude. In fact it appears from Table 8 that $\varepsilon^{(i)}$ depends not so much upon $p^{(i)}$ as upon $\frac{\partial p^{(i)}}{\partial r}$ so that no proportionality exists between A_p and A_ε. Although ϱ_ε is greater than might be expected we may still have $\varrho_\varepsilon < \varrho_p$.

It would be of interest to carry out the computations of Table 6—9 also for the 24-hour oscillation in order to compare the two oscillations. This would present no problem if p^*_{24} were known with sufficient accuracy, but here we meet with the difficulty that the global 24-oscillation is much disturbed by local influences (land- and seabreeze, mountain-valley circulation) so that representative p^*_{24}-values are hard to obtain. Thus no law of latitudinal variation corresponding to Eq. (9.2) is known for p^*_{24}.

In the above considerations we have fixed our attention on the flow of heat through the atmosphere. We shall now study the heat transport in greater detail.

If a mass of air, acting as working substance in an ideal-gas cycle, absorbs heat at high pressure and rejects heat at low pressure, heat is converted to work. An example is afforded by the oscillation described in Fig. 3 where it is seen that the maximum of $(\varepsilon_s)_{12}$, the heat supplied per unit time, nearly coincides with the pressure maximum at low levels. At first thought this is surprising since we have seen that all significant variables of this oscillation are simple, periodical functions of time and since, therefore, the heat absorbed by a parcel of air during a part of the period of oscillation seems to be rejected completely during the rest of the period. Thus there seems to be no heat left which might be converted to work.

The explanation is afforded by the fact that the amount of work e_2 performed by a unit mass on its surroundings is of the order of magnitude of p^2 and therefore does not appear in the solutions of the linearized equations, where second-order terms have been neglected. The order of magnitude of e_2 becomes apparent when a P—S diagram is drawn (S denoting specific volume). Apart from 3rd-order terms the point representing the state of the unit mass describes an ellipse during a period of oscillation. The linear dimensions of this ellipse are of the order of p and s, and its area, representing the work done, is of the order of ps, i. e. of the second order.

The work performed by the unit mass during m periods is me_2 and is thus steadily increasing with time. If we had for instance $e_2 > 0$ throughout the atmosphere, the oscillation would then have to undergo some lasting change as regards temperature and velocity. In a stationary oscillation e_2 must either vanish everywhere or e_2 must be positive in some parts of the atmosphere and negative in other parts so that a suitable balance is attained.

In the 12-hour oscillation we have seen that $(e_2)_{12}$ is positive at the ground at 47° N. At the 4000 m level $(e_2)_{12}$ is probably negative. This can be shown as follows.

An individual parcel of air has maximum or minimum pressure when

$$\frac{\delta}{\delta t}(P^* + p^*) = \frac{\partial p^*}{\partial t} + \Omega\frac{\partial p^*}{\partial \psi} + v_r\frac{\partial P^*}{\partial r} = 0.$$

Introducing here

$$p^* = p^{(1)}\sin 2\psi + p^{(2)}\sin\left(2\psi + \frac{\pi}{2}\right).$$

$$v_r = w^{(1)}\cos 2\psi + w^{(2)}\cos\left(2\psi + \frac{\pi}{2}\right),$$

$$\frac{\partial P^*}{\partial r} = -\frac{g}{T},$$

we obtain

$$\frac{\delta}{\delta t}(P^* + p^*) = \left(2\Omega p^{(1)} - \frac{g}{T}w^{(1)}\right)\cos 2\psi$$
$$-\left(2\Omega p^{(2)} - \frac{g}{T}w^{(2)}\right)\sin 2\psi = 0.$$

Introducing numerical values from Table 6 and 7 for $h = 4000$ m we find

$$\frac{\delta}{\delta t}(P^* + p^*)$$
$$= -7{,}32 \cdot 10^{-6}\cos 2\psi + 5{,}90 \cdot 10^{-6}\sin 2\psi = 0,$$

whence it follows that $P^* + p^*$ has a minimum for $\psi = 25°.5$ and a maximum for $\psi = 115°.5$. The parcel then has minimum pressure at 1.7^h and 13.7^h (local time) and maximum pressure at 7.7^h and 19.7^h. Since the maximum of $(\varepsilon_s)_{12}$ at 4000 m occurs at 12^h, $(\varepsilon_s)_{12}$ is positive at the time of minimum pressure, and work is converted to heat ($(e_2)_{12}$ negative).

When considering the work e_2 we are concerned with second-order terms and it is then generally not permissible to study the complex oscillation merely by treating each of the component oscillations separately and combining them afterwards, since this method presupposes linear terms. However, it can be shown that the work performed in the complex oscillation is the sum of the amounts of work in the 24-hour and in the 12-hour oscillations considered separately, provided that the time interval considered is 24 m hours, m being an integer. This is seen in the following manner.

Let $S + s + x_s$ and $v_r + x_r$ denote specific volume and vertical velocity in the state of perturbation, x_s and x_r being second-order corrections as defined in (8.4), and let P_M, S_M denote pressure and spesific volume of an individual parcel of air, M, of unit mass. The work done by M on its surroundings during $t_0 = 24$ hours may be written as follows

$$(10.1)\qquad e_2 = \int P_M \delta S_M = P\left[\frac{dS}{dr}y_r + x_s\right]_{t=0}^{t=t_0} + \int_0^{t_0}\left(\frac{dS}{dr}\bar{p}v_r + \frac{dP}{dr}\varrho_1\frac{ds}{dt} + \bar{p}\frac{ds}{dt}\right)dt \text{ [1]}$$

[1] v Appendix.

where

$$\frac{dy_r}{dt} = x_r, \quad \frac{d\varrho_1}{dt} = v_r.$$

Considering now the complex oscillation we may write each of the first-order variables in a form similar to (8.17). If from (8.17) we compute the integral

$$\int_0^{t_0} \tau_s \frac{\partial p^*}{\partial \psi} dt = \frac{1}{\Omega} \int_0^{2\pi} \tau_s \frac{\partial p^*}{\partial \psi} d\psi,$$

we find that the terms containing $\sin(\psi + \eta_\tau^{(1)})$. $\cos(2\psi + \eta_p^{(2)})$ and $\cos(\psi + \eta_p^{(1)}) \sin(2\psi + \eta_\tau^{(2)})$ vanish and therefore the result of the said integration is the sum of the integrals obtained if the 24-hour and the 12-hour oscillation were considered separately. Obviously this applies to all integrals in (10.1), the integration being extended over one period or a whole number of periods. Thus the work done by M in the complex oscillation may be written $e_2 = (e_2)_{24} + (e_2)_{12}$.

$(e_2)_{12}$ has been considered above. In order to estimate $(e_2)_{24}$ we note that p^*_{24} has its maximum at about 5^h and its minimum about 17^h (local time).[1]) Since we have approximately $(\eta_\tau)_{24} = 195°$,[1]) $(\tau_s)_{24}$ is minimum at 5^h and maximum at 17^h, and $(\varepsilon_s)_{24}$ is zero at these hours, whence it follows that no heat is converted to work. Thus we may infer that the total work performed by the air masses at the ground is done in the 12-hour oscillation, no contribution being rendered by the 24-hour oscillation.

It would be of interest to examine the work done in the 24-hour oscillation at higher levels as well, but the observational data available are not sufficiently accurate to allow the necessary computations.

If we may assume that the work e_2 is negative at higher levels, as indicated by the computation made above for the 4000 m level, the thermodynamical process may be described as follows. Each parcel M in the lower atmosphere absorbs heat during a part of the period of oscillation. This amount of heat, which is of the first order of magnitude, is rejected during the rest of the period, save for a small amount of the second order of magnitude. The latter is converted to work which the lower atmosphere performs on the upper atmosphere. Here the work is absorbed and converted to heat which probably leaves the atmosphere as radiation.

In the real atmosphere a part of the work of the lower layers must be done against friction in order to maintain the motion.

The integral (10.1) depends on the phase angles of the functions (2.8). It may vanish in an infinity of cases, the simplest being the case where all the phase angles of (2.8) are equal, so that v_r, v_θ, ε_s pass through zero at the moment when p^*, τ_s, v_ψ have extreme values. Then all the integrands in (10.1) are periodic, $e_2 = 0$, and no work is performed by the air masses. If friction is introduced such an oscillation must gradually disappear, since no work is available to overcome friction. It is of interest to note in this connection that a phase difference between the pressure and temperature wave is observed in the atmosphere.

In conclusion the writer wishes to emphasize that the foregoing discussion has no pretension of completeness. An attempt has been made at explaining the broad features of the phenomenon without postulating any appropriate period of free oscillation of the atmosphere, thus neglecting the possibility of resonance. It should be observed, however, that the theory presented above and the resonance theory are not mutually exclusive.

Appendix.

Some details concerning the mathematical computations.

I. *Ad para. 2.*

To illustrate the method leading from Eqs. (1.1—5) to (2.6) we consider the variation of $\frac{dF}{dt}$ in (2.4). We have

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \boldsymbol{V} \cdot \nabla F.$$

If the variation of F is denoted by $\triangle F = f$ the first variation of $\frac{dF}{dt}$ may be written

$$\triangle\left(\frac{dF}{dt}\right) = \frac{\partial f}{\partial t} + \boldsymbol{V} \cdot \nabla f + \boldsymbol{v} \cdot \nabla F = \frac{df}{dt} + \boldsymbol{v} \cdot \nabla F.$$

[1]) Hann-Süring, Lehrbuch der Meteorologie, 5 Aufl. p. 288—289.

In virtue of (2.1—2) this reduces to

$$\triangle\left(\frac{dF}{dt}\right)=\frac{df}{dt}+v_r\frac{\partial F}{\partial r}+v_\theta\frac{\partial F}{\partial\theta}.$$

where

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+V_\psi\frac{\partial f}{\partial\psi}.$$

Thus we have, considering the first term of Eq. (1.4) and having regard to (1.6),

$$\triangle\left(\frac{1}{Q}\frac{dQ}{dt}\right)=\frac{1}{R}\triangle\left(\frac{dP^*}{dt}\right)-\frac{1}{T}\triangle\left(\frac{dT}{dt}\right)$$
$$-\triangle\left(\frac{1}{T}\right)\frac{dT}{dt}.$$

The last term vanishes according to (2.4) and we obtain

$$\triangle\left(\frac{1}{Q}\frac{dQ}{dt}\right)=\frac{1}{R}\left(\frac{dp^*}{dt}+v_r\frac{\partial P^*}{\partial r}+v_\theta\frac{\partial P^*}{\partial\theta}\right)$$
$$-\frac{1}{T}\left(\frac{d\tau_s}{dt}+v_r\frac{\partial T}{\partial r}+v_\theta\frac{\partial T}{\partial\theta}\right).$$

The following example may explain the terms in Table 2. In Eq. (1.4) we consider the term $-\frac{1}{T}\frac{dT}{dt}$, having regard to (1.6). If we pass from the variables of the first column to those of the last column in Table 1 this term becomes

$$-\frac{1}{T+\tau_s}\frac{\delta}{\delta t}(T+\tau_s)$$
$$=\left(-\frac{1}{T}+\frac{\tau_s}{T^2}-\dots\right)\left(\frac{d\tau_s}{dt}+\boldsymbol{v}\cdot\nabla T+\boldsymbol{v}\cdot\nabla\tau_s\right).$$

Here we have,

1st-order terms $-\frac{1}{T}\left(\frac{d\tau_s}{dt}+\boldsymbol{v}\cdot\nabla T\right)$,

2nd-order terms $-\frac{1}{T}\left(\boldsymbol{v}\cdot\nabla\tau_s-\frac{\tau_s}{T}\frac{d\tau_s}{dt}-\frac{\tau_s}{T}\boldsymbol{v}\cdot\nabla T\right)$.

3rd-order terms $\frac{\tau_s}{T^2}\boldsymbol{v}\cdot\nabla\tau_s+\dots$

II. *Ad para. 5 A.*

R_m is computed as follows: In Eqs. (5.1—2) we compute a_{-2}, γ_{-2}, δ_{-1}. Since $a^2=1$ it follows from (4.5) that $D=H\sin^2\theta$. When $\theta\to 0$ the predominating term in the coefficient a_1 in (4.9) is

$$\frac{(H-1)\,nT\operatorname{cotg}\theta}{2\Omega r^2H\sin^2\theta}.$$

Comparing (4.7) and (5.2) it is seen that

$$a_{-3}=\frac{(H-1)\,nT}{2\Omega r^2H}.$$

The predominating term in c_1, (4.9), is

$$\frac{aT\,(H-1)}{2\Omega r^2H\sin^2\theta}.$$

Comparing again (4.7) and (5.2) we find

$$\gamma_{-2}=\frac{aT\,(H-1)}{2\Omega r^2H}.$$

Considering d_1 in (4.9) and comparing (4.7) and (5.2) it is seen that

$$\delta_{-1}=\frac{\delta_a}{4\Omega^2rTH}.$$

Comparing Eqs. (5.2) and (5.4) it is immediately apparent that

$$a_{-2}=\gamma_{-2}.$$

In (5.3) the only terms containing $\frac{1}{\theta^3}\frac{\partial p}{\partial\theta}$ are

$$\frac{\partial u}{\partial\theta}+\frac{1-na}{\theta}u$$

whence it follows that

$$f_{-3}=a_{-3}-\gamma_{-2}\,(1+na)=-\gamma_{-2}.$$

In (5.3) the only terms containing $\frac{1}{\theta^4}p$ are

$$\frac{\partial u}{\partial\theta}+\frac{1-na}{\theta}u,$$

whence it follows that

$$g_{-4}=-3a_{-3}+(1-na)\,a_{-3}=-(2+na)\,a_{-3}.$$

We then have

$$R_m=m\,(m-1)\,a_{-2}+mf_{-3}+g_{-4}$$
$$=(m^2-2m)\,\gamma_{-2}-(2+na)a_{-3}$$
$$=\frac{T\,(H-1)}{2\Omega r^2H}\,(m-na-2)\,(ma+n).$$

Further we find, inserting u and $\frac{\partial u}{\partial\theta}$ from (5.2) in (5.3) and comparing with (5.4),

$$j_{-1}=\delta_{-1}\qquad k_{-2}=-\delta_{-1}+(1-na)\,\delta_{-1}=-na\delta_{-1}.$$

$$S_\mu=k_{-2}+\mu j_{-1}=\frac{(\mu-na)\,\delta_a}{4\Omega^2rTH}.$$

III. *Ad para. 5 B.*

Bearing in mind that

$$a^2\neq 1,\ D=(H-a^2)\,(a^2-1)\ \text{for}\ \theta=0,$$

and having regard to (5.10) we easily find γ_0 and $\overline{a}_{-1}$, comparing (5.2) with (4.7) and taking the predominating part of a_1 and c_1 in (4.9). We get

$$\overline{\gamma}_0=\frac{aT}{2\Omega r^2\,(a^2-1)}\qquad\overline{a}_{-1}=\frac{nT}{2\Omega r^2\,(a^2-1)}.$$

From (5.2—4) it is seen that

$$\bar{a}_0 = \bar{\gamma}_0, \quad \bar{g}_{-2} = -\bar{a}_{-1} + (1 - na)\,\bar{a}_{-1} = -na\bar{a}_{-1} = -n^2\bar{\gamma}_0.$$

The terms in (5.3) that contain $\frac{1}{\theta}\frac{\partial p}{\partial \theta}$ are

$$\frac{\partial u}{\partial \theta} \text{ containing } \frac{\bar{a}_{-1}}{\theta}\frac{\partial p}{\partial \theta},$$

$$\frac{1 - na}{\theta}u \text{ containing } \frac{1 - na}{\theta}\frac{\partial p}{\partial \theta},$$

$$\text{and} \quad \frac{nT}{2\Omega r^2}\frac{1}{\theta}\frac{\partial p}{\partial \theta},$$

whence it follows that

$$\bar{f}_{-1} = \bar{a}_{-1} + (1 - na)\,\bar{\gamma}_0 + \frac{nT}{2\Omega r^2} = \bar{\gamma}_0.$$

Finally we get

$$\bar{R}_m = m\,(m - 1)\,\bar{a}_0 + m\bar{f}_{-1} + \bar{g}_{-2} = \frac{aT}{2\Omega r^2\,(a^2 - 1)}\,(m^2 - n^2).$$

IV. *Ad para. 6 B.*

If in Eq. (6.3) $m \neq 4$, $\mu \neq 2$, we have the case (5.9 b), p_m, $p_{m+2}\ldots$ can be determined successively from Eq. (5.4), and the boundary condition leads to conditions which must be fulfilled by the coefficients ε_μ, $\varepsilon_{\mu+2}\ldots$ This case is then analogous to the case ($a = \frac{1}{2}$, $n = 1$).

If in (6.3—4) $m \neq 4$, $\mu = 2$ we have the case (5.9 c). Taking for instance $m = 6$, $\mu = 2$ it is seen from (5.6) that we have

$$(R_6 p_6 + C_2)\,\theta^2 + \ldots = 0,$$

whereby p_6 can be determined and the reasonings of the foregoing case may be repeated.

We consider next the case

$$m = 4, \ R_m = R_4 = 0.$$

Since we must always have $\mu \gtrless 2$ the condition (5.9 c) cannot hold in this case. Let us consider the condition (5.9 a), i. e. $\mu > 2$. Putting in Eq. (5.1) $r = r_0$, $w = 0$ we obtain on the right-hand side the series

$$(a_{-2} + 4\gamma_{-1})\,p_4\theta^2 + \ldots = 0$$

but this is not possible since it follows from Eq. (4.6) and the expressions for a and c in (4.9) with $a = 1$, $n = 2$ that we have

$$a_{-2} = \frac{-2T}{2\Omega rH}, \quad \gamma_{-1} = \frac{-T}{2\Omega rH}$$

which shows that $a_{-2} + 4\gamma_{-1} \neq 0$.

Thus there remains only the case $m = 4$, $\mu = 2$, which is treated in para. 6.

V. *Ad para. 10.*

The formula (10.1) is found in the following manner. We may write

$$S_M = S + s + x_s + \frac{dS}{dr}(\varrho_1 + \varrho_2 + y_r) + \tfrac{1}{2}\frac{d^2S}{dr^2}\varrho_1{}^2 + s_2,$$

where $(\varrho_1 + \varrho_2 + y_r)$ is the elevation of M above its mean level (P, S) and we have

$$\frac{d\varrho_1}{dt} = v_r, \quad \frac{dy_r}{dt} = x_r, \quad \varrho_2 = \frac{\partial \varrho_1}{\partial r}\varrho_1 + \frac{\partial \varrho_1}{\partial \psi}\triangle\psi + \frac{\partial \varrho_1}{\partial \theta}\triangle\theta,$$

$$s_2 = \frac{\partial s}{\partial \psi}\triangle\psi + \frac{\partial s}{\partial \theta}\triangle\theta + \frac{\partial s}{\partial r}\varrho_1.$$

Further we may write

$$\int P_M \delta S_M = P\int \delta S_M + \int\left(\bar{p} + \frac{\partial P}{\partial r}\varrho_1\right)\delta S_M.$$

Since s, ϱ_1, ϱ_2, s_2 are periodical functions with a 24-hour period we obtain, extending the integration over $t_0 = 24$ hours,

$$P\int \delta S_M = P\left[\frac{dS}{dr}y_r + x_s\right]_{t=0}^{t=t_0}.$$

Neglecting 3rd-order terms we have

$$\int_0^{t_0}\left(\bar{p} + \frac{dP}{dr}\varrho_1\right)\frac{\delta S_M}{\delta t}\,dt = \int_0^{t_0}\left(\bar{p} + \frac{dP}{dr}\varrho_1\right)\left(\frac{ds}{dt} + v_r\frac{dS}{dr}\right)dt.$$

Combining these expressions and noting that $\int_0^{t_0}\varrho_1 v_r dt = 0$, we arrive at the expression (10.1).

The second-order quantities y_r, x_s may be periodic or they may possibly contain non-periodic terms like x_ψ in (8.20).

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