ON THE DYNAMIC EFFECT OF VARIATION IN DENSITY ON TWO-DIMENSIONAL PERTURBATIONS OF FLOW WITH CONSTANT SHEAR

BY

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1. The Simplified Perturbation Equation.

Our basic flow is a rectilinear, horizontal current of an incompressible fluid with a velocity U along the X-axis. The velocity U depends only upon the vertical coordinate z, and is given by the relation

$$(1, 1) \qquad U = az,$$

choosing $z = 0$, where $U = 0$.

In the basic flow the density Q is assumed to be given by an exponential function of z,

$$(1, 2) \qquad Q = Q_0 e^{-\beta z},$$

where β is a constant. Positive values of β correspond to static stability, negative values to static instability.

To simplify the equations describing a small perturbation of our basic system, we disregard the "kinematic" effect of the density variation, taking terms depending upon this variation into account only when they occur together with g. This simplification means physically that the forces which are released by a small perturbation of our basic flow are assumed to act on a fluid of constant density equalling some average value of the density of the fluid. The smaller the relative density variations in our fluid, the smaller will be the difference in acceleration of the different fluid particles produced by a given force, and therefore the closer will the simplified perturbation equations represent the complete perturbation equations. Thus our simplified system of equations may also quantitatively be applied with confidence for systems with a height h much smaller than $\frac{1}{\beta}$.

With the simplifications introduced above the equation for the streamfunction ψ for two-dimensional (in the XZ-planes) perturbations turns out to be

$$(1, 3) \qquad \left(\frac{\partial}{\partial t} + az\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right) = -\beta g\frac{\partial^2\psi}{\partial x^2}.$$

Assuming a wave-solution given by

$$(1, 4) \qquad \psi = Z(z)e^{-ik(x+ct)},$$

the amplitude function Z is given by

$$(1, 5) \qquad Z'' - \left(k^2 - \frac{\beta g}{(az + c)^2}\right)Z = 0.$$

Here k is the wavenumber and c the velocity of propagation (in negative x-direction).

2. The Solution of the Perturbation Equation for k=0.

The general solution of the differential equation (1, 5) (which may be transformed to a Bessel equation) has been given by Taylor [1]). The solution is

$$(2, 1) \qquad Z = \left(z - \frac{c}{a}\right)^{\frac{1}{2}} B_{\frac{\delta}{2}}\, ik\left(z - \frac{c}{a}\right)$$

[1]) G. I. Taylor: Effect of Variation in Density on the Stability of Superposed Streams of Fluid. Proc. of the Roy. Soc., A, Vol. 132, 1931.

where $B_{\frac{\delta}{2}}$ is the cylinder function of order $\frac{\delta}{2}$ with

$$\delta = \left(1 - \frac{4\beta g}{\alpha^2}\right)^{\frac{1}{2}}. \tag{2, 2}$$

In the following we will only discuss the case that $k = 0$, i.e. waves of infinite wavelength. In this case the solution of equation (1, 5) assumes the simple form

$$Z = A(\alpha z + c)^{\frac{1-\delta}{2}}[(\alpha z + c)^\delta - B], \tag{2, 3}$$

where A and B are constants of integration.

In the case that $\delta = 0$, i.e. that

$$\alpha^2 = 4\beta g, \tag{2, 4}$$

our solution (2, 3) fails to give the general solution. In that case the general solution is easily found to be given by

$$Z = A(\alpha z + c)^{\frac{1}{2}} \ln \frac{(\alpha z + c)}{B}. \tag{2, 5}$$

3. A Layer bounded by two Horizontal Planes.

We consider first wave motion in a layer bounded by two horizontal planes, $z = 0$ and $z = h$. This case was also to some degree discussed by Taylor[1]) for $\beta > 0$ (static stability). He arrived at the results: for δ real, i.e. $\alpha^2 < 4\beta g$, no solutions exist for infinite height of the layer, and Taylor maintains that it seems unlikely that solutions exist for any value of h in this case. For δ imaginary, i.e. $\alpha^2 > 4\beta g$, a series of wavelengths corresponding to stability waves can exist. Nothing is, however, proved about the existence or non-existence of instability waves.

Confining our considerations to so small values of the wavenumber that k^2 may be neglected in equation (1, 5), the frequency equation of our problem assumes the simple form

$$(\alpha z + c)^\delta - B = 0 \text{ for } z = 0 \text{ and } z = h, \tag{3, 1}$$

or

$$(\alpha h + c)^\delta = c^\delta. \tag{3, 2}$$

Without loss of generality we may assume $\alpha h > 0$, and the pure imaginary part c_i of c also greater than zero. This last assumption is justified because if $c = c_r + ic_i$ is a solution of our frequency equation, corresponding to the value $Z_r + iZ_i$ of Z, also $c = c_r - ic_i$ is a solution corresponding to the value $Z_r - iZ_i$ of Z.

With the above assumption we have

$$\arg c > \arg (\alpha h + c). \tag{3, 3}$$

Now if δ is real

$$\arg c^\delta = \delta \arg c, \arg (\alpha h + c)^\delta = \delta \arg (\alpha h + c),$$

and therefore

$$\arg c^\delta > \arg (\alpha h + c)^\delta. \tag{3, 4}$$

To have a solution we must then have

$$\delta[\arg c - \arg (\alpha h + c)] = 2n\pi, \tag{3, 5}$$

where n is a positive integer.

Since we also have

$$\arg c - \arg (\alpha h + c) \leq \pi, \tag{3, 6}$$

equation (3, 5) cannot be fulfilled for $\delta < 2$.

Thus for

$$0 < 1 - \frac{4\beta g}{\alpha^2} < 4,$$

or for

$$1 > \frac{4\beta g}{\alpha^2} > -3, \tag{3, 7}$$

no solutions of the form (1, 4) exist. It is easily seen that this is also so when we have $\delta = 0$ or $\frac{4\beta g}{\alpha^2} = 1$.

For statically stable stratification, $\beta > 0$, we then have no solutions of the form (1, 4), when the dimensionless number

$$\frac{\beta g}{\alpha^2} \leq \frac{1}{4}. \tag{3, 8}$$

For statically unstable stratification, $\beta = -\beta^* < 0$, we have no waves, either stability or instability waves, of the form (1, 4), when the dimensionless number

$$\frac{\beta^* g}{\alpha^2} < \frac{3}{4}. \tag{3, 9}$$

What is quite remarkable with this last result is that a sufficiently strong shear will neutralize, or at least weaken, a static instability. This result may probably be of some interest in connection with the problem of cumulus-development in a situation with shear.

1) Taylor: loc. cit.

When

$$\frac{\beta^* g}{a^2} = \frac{3}{4} \quad \text{or} \quad \delta = 2, \tag{3, 10}$$

the solution of the frequency equation (3, 2) is

$$c = -\frac{ah}{2}, \tag{3, 11}$$

or the velocity of propagation, remembering that c is considered positive in negative x-direction, is equal to the basic velocity in the middle of the layer. But this value of c gives a singularity of Z at this level, so that the solution cannot be used. Thus, for $\delta = 2$, we again have no solutions of the form (1, 4).

When

$$\frac{\beta^* g}{a^2} > \frac{3}{4} \quad \text{or} \quad \delta > 2, \tag{3, 12}$$

equation (3, 5) can always be fulfilled. For

$$2 < \delta \leq 4$$

it can for complex c be fulfilled for one value of $\arg c - \arg(ah + c)$, for

$$4 < \delta \leq 6,$$

for two values of the same quantity, and so on. Since we also must have

$$\operatorname{mod} c = \operatorname{mod}(ah + c), \tag{3, 13}$$

we must have

$$c_r = -\frac{ah}{2}. \tag{3, 14}$$

Thus the instability waves must be propagated with a velocity equal to the basic velocity in the middle of the layer. From this it follows that

$$\arg(ah + c) = \pi - \arg c, \tag{3, 15}$$

where both of the arguments may be considered smaller than π and positive. This equation together with (3, 5) gives

$$\arg c = \left(\frac{n}{\delta} + \frac{1}{2}\right)\pi, \tag{3, 16}$$

where the positive integer n can assume any value satisfying the relation

$$\frac{n}{\delta} < \tfrac{1}{2}. \tag{3, 17}$$

The sign of equality cannot be applied since then we get a singularity in Z. The relation gives the number of solutions corresponding to given values of δ or β^*.

For the imaginary part of c we obtain

$$c_i = -\frac{ah}{2}\tan\left(\frac{n}{\delta} + \frac{1}{2}\right)\pi = \frac{ah}{2}\cot\text{an}\,\frac{n}{\delta}\pi. \tag{3, 18}$$

The formula shows that the amplification decreases with n. The maximum amplification corresponds to $n = 1$, so that we have

$$c_{i\,\max} = \frac{ah}{2}\cot\text{an}\,\frac{\pi}{\delta} \tag{3, 19}$$

$$= \frac{ah}{2}\cot\text{an}\,\frac{\pi}{\left(1 + \frac{4\beta^* g}{a^2}\right)^{\frac{1}{2}}}.$$

This value of c_i corresponds to the smallest "tilt" of the waves. Increasing values of n give increasing tilt.

Consider now the case when δ is imaginary,

$$\delta = i\nu, \quad \nu = \left(\frac{4\beta g}{a^2} - 1\right)^{\frac{1}{2}}, \quad \frac{\beta g}{a^2} > \frac{1}{4}. \tag{3, 20}$$

Our frequency equation (3, 2) then assumes the form

$$(ah + c)^{i\nu} = c^{i\nu}. \tag{3, 21}$$

Assuming again ah and c_i positive, we still have the relation (3, 3). Further we have

$$\operatorname{mod}(ah + c)^{i\nu} = e^{-\nu \arg(ah + c)},$$
$$\operatorname{mod} c^{i\nu} = e^{-\nu \arg c},$$

and therefore for complex c,

$$\operatorname{mod}(ah + c)^{i\nu} > \operatorname{mod} c^{i\nu}, \tag{3, 22}$$

so that for complex values of c, equation (3, 21) can never be fulfilled. Thus in this case we will never have any instability waves.

For real values of c and

$$1.\ c > 0, \quad \text{or} \quad 2.\ c < -ah, \tag{3, 23}$$

the right-hand and left-hand side of equation (3, 21) will have the same modulus. The equation for the argument is

$$\ln|ah + c| = \ln|c| + \frac{2n\pi}{\nu},$$

where n is an integer (positive or negative). The equation may also be written

$$ah + c = ce^{\frac{2n\pi}{\nu}},$$

or

$$c = \frac{ah}{e^{\frac{2n\pi}{\nu}} - 1}. \tag{3, 24}$$

We see that for n positive, we have $c > 0$, and for n negative $c < -ah$. Excluding the singular case $n = 0$, the greatest positive value for c (the greatest velocity of propagation in negative x-direction) is given by

$$c_1 = \frac{ah}{e^{\frac{2\pi}{\nu}} - 1},$$

increasing with increasing value of ν. For a sufficiently large static stability or a sufficiently small shear ($\nu \gg 2\pi$), we have the approximate formula

$$c_1 = \frac{ah\nu}{2\pi} = \frac{ah}{2\pi}\left(\frac{4\beta g}{\alpha^2} - 1\right)^{\frac{1}{2}}.$$

The velocity of propagation decreases with increasing values of n. Now the number of nodal planes from $z = 0$ to $z = h$ is $n - 1$. Thus the value of the velocity of propagation decreases with increasing number of nodal planes, and has its maximum value when no nodal planes exist. When the number of nodal planes increases without limit, the velocity of propagation decreases towards zero, i.e. towards the basic velocity at the lower rigid plane.

When n is negative, $n = -m$, we obtain for the velocity of propagation in positive x-direction, $c' = -(ah + c)$, relative to the fluid at the upper rigid plane

$$c' = \frac{ah}{e^{\frac{2m\pi}{\nu}} - 1}, \tag{3, 25}$$

i.e. the same expression as (3, 24). Thus we get the obvious result that to a wave with a certain velocity of propagation relative to the fluid at the lower rigid plane, there must correspond a wave with the same velocity of propagation relative to the fluid at the upper rigid plane propagated in opposite direction.

Inserting the value we have found for c in formula (2, 3), we obtain as solution of our problem the two sets of amplitude functions

$$Z = A_n \left[az + \frac{ah}{e^{\frac{2n\pi}{\nu}} - 1}\right]^{\frac{1-i\nu}{2}} \left\{\left[az + \frac{ah}{e^{\frac{2n\pi}{\nu}} - 1}\right]^{i\nu} - \left[\frac{ah}{e^{\frac{2n\pi}{\nu}} - 1}\right]^{i\nu}\right\}, \text{ and}$$

(3, 26)

$$Z = B_n \left[a(z-h) - \frac{ah}{e^{\frac{2n\pi}{\nu}} - 1}\right]^{\frac{1-i\nu}{2}} \left\{\left[a(z-h) - \frac{ah}{e^{\frac{2n\pi}{\nu}} - 1}\right]^{i\nu} - \left[-ah\left(1 + \frac{1}{e^{\frac{2n\pi}{\nu}} - 1}\right)\right]^{i\nu}\right\},$$

where n now is a positive integer.

4. The Upper Boundary a Free Surface.

We will now discuss the case that the layer has a free surface instead of a rigid plane as an upper boundary. For $k = 0$, we still have the solution (2, 3). Further we also at the lower rigid plane $z = 0$ must have $Z = 0$, so that the solution can be written

$$Z = A(az + c)^{\frac{1-\delta}{2}}[(az + c)^{\delta} - c^{\delta}]. \tag{4, 1}$$

The condition which must be fulfilled at the surface, $z = h$, is the dynamic boundary condition, leading to the equation

$$(az + c)^2 Z' - [g + a(az + c)]Z = 0 \quad \text{for } z = h. \tag{4, 2}$$

Inserting from equation (4, 1) we find the frequency equation

$$\left[\frac{g}{a^2 h} + \frac{1-\delta}{2}\left(1 + \frac{c}{ah}\right)\right]\left(1 + \frac{c}{ah}\right)^{\delta} = \left[\frac{g}{a^2 h} + \frac{1+\delta}{2}\left(1 + \frac{c}{ah}\right)\right]\left(\frac{c}{ah}\right)^{\delta}. \tag{4, 3}$$

For convenience we introduce the nondimensional quantities

$$\gamma = \frac{g}{a^2 h}, \qquad c^* = \frac{c}{ah}, \tag{4, 4}$$

and obtain

$$\left[\gamma + \frac{1-\delta}{2}(1 + c^*)\right](1 + c^*)^{\delta} = \left[\gamma + \frac{1+\delta}{2}(1 + c^*)\right]c^{*\delta}. \tag{4, 5}$$

Here γ is a positive quantity, c^* is the velocity of propagation (in negative x-direction) measured in terms of the basic velocity at the free surface.

$$0<\delta\leq 1.$$

We consider first the case that δ is real and smaller than or equal to 1, $\delta=0$ excluded. Then we have, assuming c_i different from zero and positive

$$\arg\frac{1-\delta}{2}(1+c^*)=\arg\frac{1+\delta}{2}(1+c^*),$$

$$\mathrm{mod}\frac{1-\delta}{2}(1+c^*)<\mathrm{mod}\frac{1+\delta}{2}(1+c^*).$$

Therefore we also get

$$\arg\left[\gamma+\frac{1-\delta}{2}(1+c^*)\right] < \arg\left[\gamma+\frac{1+\delta}{2}(1+c^*)\right].$$

We have further

$$\arg(1+c^*)<\arg c^*,$$

and therefore

$$\arg(1+c^*)^\delta<\arg c^{*\delta}.$$

Thus we finally get

$$\arg\left\{\left[\gamma+\frac{1-\delta}{2}(1+c^*)\right](1+c^*)^\delta\right\} < \arg\left\{\left[\gamma+\frac{1+\delta}{2}(1+c^*)\right]c^{*\delta}\right\}.$$

It follows then that for

$$(4,6)\qquad 0<\delta\leq 1 \quad\text{or}\quad 4>\frac{\beta g}{\alpha^2}\geq 0,$$

our frequency equation has no complex solutions. We have no instability waves.

Assuming now a real c^*, it is evident that we must have either $c^*>0$ or $c^*<-1$, i.e. the stability wave must be propagated with a velocity different from any velocity of the basic flow.

Considering first the possibility

a. $c^*>0$, *i.e. the wave is propagated in negative x-direction.*

We write the frequency equation in the form

$$(4,7)\qquad \frac{\gamma+\frac{1-\delta}{2}(1+c^*)}{\gamma+\frac{1+\delta}{2}(1+c^*)}=\left(\frac{c^*}{1+c^*}\right)^\delta.$$

The left-hand side of this equation represents a function of c^*, decreasing monotonically from the value

$$\frac{\gamma+\frac{1-\delta}{2}}{\gamma+\frac{1+\delta}{2}}$$

for $c^*=0$ to the value

$$\frac{1-\delta}{1+\delta}$$

for $c^*\to\infty$, both values smaller than 1.

On the other hand the right-hand side of the equation represents a function of c^* monotonically increasing from the value zero for $c^*=0$ to the value 1 for $c^*\to\infty$. Thus there will always be one and only one positive value of c^* satisfying the frequency equation. This value of c^* must satisfy the relation

$$(4,8)\qquad \left(\frac{2\gamma+1-\delta}{2\gamma+1+\delta}\right)^{\frac{1}{\delta}}>\frac{c^*}{1+c^*}>\left(\frac{1-\delta}{1+\delta}\right)^{\frac{1}{\delta}},$$

where, of course, only real positive roots are considered.

We so pass to the discussion of the other possibility

b. $c^*<-1$, *i.e. the wave is propagated in positive x-direction with a velocity greater than the basic velocity at the free surface.*

We introduce in equation (4, 5)

$$(4,9)\qquad (c^*+1)=-c^{**},$$

where c^{**} is the dimensionless velocity of propagation in positive x-direction relative to the fluid at the free surface. The equation then may be written in the form

$$(4,10)\qquad \frac{\gamma-\frac{1+\delta}{2}c^{**}}{\gamma-\frac{1-\delta}{2}c^{**}}=\left(\frac{c^{**}}{1+c^{**}}\right)^\delta.$$

The function on the right-hand side of this equation has exactly the same behaviour as the function on the right-hand side of equation (4, 7), increasing monotonically from zero to 1 when c^{**} increases from zero to infinite values. The function on the left-hand side starts with the value 1 for $c^{**}=0$, and decreases monotonically towards infinite negative values in the interval $0<c^{**}<\frac{2\gamma}{1-\delta}$. Thus, in this interval we have always one and only one solution of our frequency equation. Since this solution cannot appear for negative values of the left-hand side, the corresponding value of c^{**} must obey the relation

$$0 < c^{**} < \frac{2\gamma}{1+\delta}$$

or

$$(4,\ 11) \qquad 0 < c^{**} < \frac{\gamma a^2}{2\beta g}\left[1 - \left(1 - \frac{4\beta g}{a^2}\right)^{\frac{1}{2}}\right],$$

$$\text{with } 0 \leq \frac{4\beta g}{a^2} < 1.$$

For $c^{**} > \frac{2\gamma}{1-\delta}$ no solutions exist. For such values of c^{**} the left-hand side of equation (4, 10) will be greater than 1. Thus for $c^{**} > 0$, we have one and only one solution of our frequency equation.

If $\beta = 0$, homogeneous fluid, δ is equal to 1, and we get

$$c^{**} < \gamma,$$

or introducing for c^{**} and γ in terms of dimensional quantities

$$-(c + ah) < \frac{g}{a},$$

giving an upper limit for the velocity of propagation relative to the fluid at the surface of the wave propagated in positive direction. The same upper limit is for $\beta = 0$ (from formula (4, 8)) found for the velocity of propagation relative to the fluid at the lower rigid plane of the wave propagated in negative x-direction.

For a very small value of δ, ($\frac{4\beta g}{a^2}$ almost equal to 1) we find for the velocity of propagation of the wave propagated in positive direction

$$c^{**} < 2\gamma,$$

or

$$-(c + ah) < \frac{2g}{a}.$$

$$\delta = 0.$$

In the special case that $\delta = 0$, we had the solution (2, 5) and we get the frequency equation

$$(4,\ 12) \quad (c^* + 1) - [\gamma + \tfrac{1}{2}(c^* + 1)] \ln \frac{c^* + 1}{c^*} = 0.$$

Assume first c^* complex. As before we study only the case that $c^*_i > 0$. Then we have

$$\arg(1 + c^*) > \arg[\gamma + \tfrac{1}{2}(1 + c^*)],$$

and also

$$\arg(1 + c^*) - \arg[\gamma + \tfrac{1}{2}(1 + c^*)] < \pi.$$

The pure imaginary part of $\frac{1+c^*}{c^*}$ will be negative. Therefore we must have either

$$\arg\left(\ln \frac{1+c^*}{c^*}\right) < 0$$

or

$$\arg\left(\ln \frac{1+c^*}{c^*}\right) > \pi,$$

dependent upon what value we choose for the multivalued ln-function. Comparing the first of these relations with the first of the above relations, and the second with the second of the above relations, we see that we will never get complex solutions of equation (4, 12).

Thus, also for $\delta = 0$, we have only real roots of the frequency equation. For real roots we must have either $c^* > 0$ or $c^* < -1$.

a. $c^* > 0$. We write the frequency equation in the form

$$(4,\ 13) \qquad \frac{2}{\frac{2\gamma}{c^*+1} + 1} = \ln \frac{c^*+1}{c^*}.$$

For c^* positive, the functions on both sides of this equation vary monotonically with c^*, the left-hand side increasing from the value $\frac{2}{2\gamma+1}$ for $c^* = 0$ to the value 2 for $c^* \to \infty$, the right-hand side decreasing from the value ∞ for $c^* = 0$ to zero for $c^* \to \infty$. Thus, we always get one and only one solution for $c^* > 0$. For this solution we must have

$$(4,\ 14) \qquad e^2 > \frac{c^*+1}{c^*} > e^{\frac{2}{2\gamma+1}}.$$

b. $c^* < -1$. We introduce as before $c^{**} = -(c^* + 1)$, and write the frequency equation

$$\frac{1}{\frac{\gamma}{c^{**}} - \frac{1}{2}} = \ln \frac{c^{**} + 1}{c^{**}}.$$

The function on the right-hand side again decreases monotonically from the value $+\infty$ for $c^{**} = 0$ to zero for $c^{**} \to \infty$. For $c^{**} > 2\gamma$, the left-hand side is negative and can not be equal to the right-hand side. In the interval from $c^{**} = 0$ to $c^{**} = 2\gamma$, the left-hand side increases monotonically from zero for $c^{**} = 0$ to the value $+\infty$ for $c^{**} \to 2\gamma$. In this interval of c^{**} we therefore always have one and only one solution, with

$$(4,\ 15) \qquad 0 < c^{**} < 2\gamma,$$

the same relation which we have found above for $\delta \to 0$.

$$\delta > 1.$$

We so consider the case that β is negative (the layer statically unstable), $\beta = -\beta^*$.

We introduce

$$\delta = 1 + \varepsilon \qquad \text{with } \varepsilon > 0,$$

and our frequency equation (4, 7) takes the form

$$(4, 16) \qquad \frac{\gamma - \frac{\varepsilon}{2}(1 + c^*)}{\gamma + \frac{2+\varepsilon}{2}(1 + c^*)} = \left(\frac{c^*}{1+c^*}\right)^{1+\varepsilon}.$$

For sufficiently small values of γ this equation will for all values of $\varepsilon > 0$ have complex roots. Considering the limiting case $\gamma \to 0$, the equation reduces to

$$(4, 17) \qquad \left(\frac{c^*}{1+c^*}\right)^{1+\varepsilon} = -\frac{\varepsilon}{2+\varepsilon}.$$

Introducing

$$c^* = R_1 e^{i\varphi_1}, \qquad 1 + c^* = R_2 e^{i\varphi_2},$$

we get the four equations

$$(4, 18) \qquad \begin{aligned} &\varphi_1 - \varphi_2 = \frac{(2n+1)\pi}{1+\varepsilon}, \quad R_1 = R_2\left(\frac{\varepsilon}{2+\varepsilon}\right)^{\frac{1}{1+\varepsilon}}, \\ &R_1 \sin \varphi_1 = R_2 \sin \varphi_2, \\ &R_1 \cos \varphi_1 = R_2 \cos \varphi_2 - 1, \end{aligned}$$

with the condition

$$0 < \varphi_1 - \varphi_2 < \pi,$$

or

$$2n < \varepsilon.$$

n is a positive integer. The last relation determines the number of solutions for given values of ε. Thus, for instance, for $\varepsilon < 2$, $(\delta < 3)$ we get only one solution. For $2 < \varepsilon < 4$, we get two solutions, and so on.

From the equations (4, 18) we find

$$(4, 19) \qquad \begin{aligned} &\frac{\sin\left(\varphi_1 - \frac{2n+1}{1+\varepsilon}\pi\right)}{\sin \varphi_1} = \left(\frac{\varepsilon}{2+\varepsilon}\right)^{\frac{1}{1+\varepsilon}}, \\ &\frac{\sin \varphi_2}{\sin\left(\varphi_2 + \frac{2n+1}{1+\varepsilon}\pi\right)} = \left(\frac{\varepsilon}{2+\varepsilon}\right)^{\frac{1}{1+\varepsilon}}, \\ &R_1 = \frac{\sin \varphi_2}{\sin \frac{2n+1}{1+\varepsilon}\pi}, \\ &R_2 = \frac{\sin \varphi_1}{\sin \frac{2n+1}{1+\varepsilon}\pi}. \end{aligned}$$

Since we have (from the second of the equations (4, 18))

$$R_2 > R_1,$$

the two last equations show that

$$\sin \varphi_1 > \sin \varphi_2.$$

As before we only consider positive values of c_i^*, i.e. $\sin \varphi_1$ and $\sin \varphi_2$ are positive. From the two first of equations (4, 19) we then deduce

$$(4, 20) \qquad \varphi_1 > \frac{2n+1}{1+\varepsilon}\pi, \qquad \varphi_2 > \frac{\varepsilon - 2n}{1+\varepsilon}\pi.$$

If $\varepsilon \leq 1$, it then follows that φ_1 must lie in the second quadrant, φ_2 in the first, i.e.

$$\varphi_1 > \frac{\pi}{2}, \qquad \varphi_2 < \frac{\pi}{2}.$$

Further, from the last of the equations (4, 19) and the first of the relations (4, 20), we obtain

$$R_2 < 1,$$

i.e.

$$c^*_i < \sin \varphi_2 < 1, \text{ when } \varepsilon \leq 1.$$

In the special case that $\varepsilon = 1$ $(\delta = 2)$, the equations (4, 19) reduces to

$$(4, 21) \qquad \begin{aligned} &\operatorname{cotan} \varphi_1 = -\frac{\sqrt{3}}{3}, \ \tan \varphi_2 = \frac{\sqrt{3}}{3}, \\ &R_1 = \tfrac{1}{2}, \qquad R_2 = \frac{\sqrt{3}}{2}. \end{aligned}$$

We have also

$$(4, 22) \qquad \varphi_1 - \varphi_2 = \frac{\pi}{2}.$$

Further we obtain

$$(4, 23) \qquad c^*_r = R_1 \cos \varphi_1 = -\frac{1}{4},$$

$$c^*_i = R_1 \sin \varphi_1 = \frac{\sqrt{3}}{4}.$$

These values of c^*_r and c^*_i could also have been obtained directly from the equation (4, 17), which for $\varepsilon = 1$ reduces to an algebraic equation of second degree.

For arbitrary values of ε, the equations determining φ_1 and φ_2 (the two first of equations (4, 19)) may be written

$$\operatorname{cotan} \varphi_1 = \frac{1}{\sin \frac{2n+1}{1+\varepsilon}\pi}\left[\cos \frac{2n+1}{1+\varepsilon}\pi - \left(\frac{\varepsilon}{2+\varepsilon}\right)^{\frac{1}{1+\varepsilon}}\right],$$

$$\cot \varphi_2 = \frac{1}{\sin \frac{2n+1}{1+\varepsilon}\pi} \left[\left(\frac{2+\varepsilon}{\varepsilon} \right)^{\frac{1}{1+\varepsilon}} - \cos \frac{2n+1}{1+\varepsilon}\pi \right].$$

It follows immediately that

$$\cot \varphi_2 > 0, \tag{4, 24}$$

and therefore that φ_2 lies in the first quadrant. To discuss the sign of $\cot \varphi_1$, we introduce

$$\tau = \frac{1}{1+\varepsilon}.$$

In terms of τ, φ_1 is given by

$$\cot \varphi_1 = \frac{1}{\sin [(2n+1)\pi\tau]} \left\{ \cos [(2n+1)\pi\tau] - \left(\frac{1-\tau}{1+\tau} \right)^{\tau} \right\}.$$

Since $\sin [(2n+1)\pi\tau] > 0$, the sign of $\cot \varphi_1$ is the same as the sign of the expression

$$N = \cos [(2n+1)\pi\tau] - \left(\frac{1-\tau}{1+\tau} \right)^{\tau},$$

in the interval $0 < \tau < \frac{1}{2}$, $(\infty > \varepsilon > 1)$. That $\cot \varphi_1$ is negative for $0 < \varepsilon \leq 1$, $(1 > \tau \geq \frac{1}{2})$ is shown above. For $\tau = 0$, $N = 0$, for $\tau = \frac{1}{2}$, $N = \cos \frac{2n+1}{2}\pi - \left(\frac{1}{3} \right)^{\frac{1}{2}} < 0.$

The functions

$$N_1 = \cos [(2n+1)\pi\tau], \quad N_2 = \left(\frac{1-\tau}{1+\tau} \right)^{\tau},$$

both have the value 1 for $\tau = 0$, and decrease monotonically when τ increases from 0 to $\frac{1}{2(2n+1)}$. To obtain a positive value of $\cot \varphi_1$, in the interval of τ, we must for some value of τ have $N = 0$, $\left| \frac{dN_1}{d\tau} \right| < \left| \frac{dN_2}{d\tau} \right|$, $\left(\frac{dN_1}{d\tau} = \frac{dN_2}{d\tau} = 0 \text{ for } \tau = 0 \right)$ or

$$(2n+1)\pi \tan [(2n+1)\pi\tau] < \frac{2\tau}{1-\tau^2} + \ln \frac{1+\tau}{1-\tau}.$$

For $\tau = 0$, the function on each side of this inequality starts with the value zero. The derivative with regard to τ of the left-hand side is equal to $\frac{(2n+1)^2\pi^2}{\cos^2[(2n+1)\pi\tau]}$ with the minimum value $(2n+1)^2\pi^2$ while the derivative of the right-hand side is equal to $\frac{4}{(1-\tau^2)^2}$ with the maximum value $\frac{64}{9}$. Since for all positive values of the integer n

$$(2n+1)^2\pi^2 > \frac{64}{9},$$

our inequality relation cannot be fulfilled. Thus, we must have

$$\cot \varphi_1 < 0, \tag{4, 25}$$

or φ_1 must lie in the second quadrant. The real velocity of propagation of the instability waves must in the limiting case that $\gamma \to 0$ be equal to the basic velocity at some level of the fluid. As shown above, we have also $R_2 > R_1$. Therefore the velocity of propagation is equal to the basic velocity at some level below the middle of the layer.

We will now discuss for arbitrary values of γ the real solutions of the frequency equation. Considering first $c^* \geq 0$, the function on the left-hand side of the frequency equation (4, 16) starts for $c^* = 0$, with the value

$$\frac{\gamma - \frac{\varepsilon}{2}}{\gamma + 1 + \frac{\varepsilon}{2}},$$

and decreases monotonically with increasing c^* to the value

$$\frac{-\frac{\varepsilon}{2}}{1 + \frac{\varepsilon}{2}}$$

for $c^* \to \infty$. The function on the right-hand side of our equation starts for $c^* = 0$ with the value zero, and increases monotonically for increasing values of c^*, reaching the value 1 for $c^* \to \infty$. Thus if

$$\gamma \geq \frac{\varepsilon}{2},$$

we have one, and only one solution of our frequency equation for $c^* \geq 0$. If

$$\gamma < \frac{\varepsilon}{2},$$

the frequency equation has no solution with $c^* > 0$.

Consider next $c^* < -1$, we introduce again in our frequency equation $c^* = -(c^{**} + 1)$, so that it may be written

$$(4, 26) \qquad \frac{\gamma - \frac{2+\varepsilon}{2} c^{**}}{\gamma + \frac{\varepsilon}{2} c^{**}} = \left(\frac{c^{**}}{c^{**}+1}\right)^{1+\varepsilon}.$$

A similar reasoning as that applied above shows that for $c^{**} > 0$, the frequency equation has always one and only one solution.

For c^* real and $-1 < c^* < 0$, $\frac{c^*}{1+c^*}$ is negative, and has the argument $-\pi$. The right-hand side of the frequency equation (4, 16) has then the argument $-\pi(1+\varepsilon)$, while the left-hand side has the argument $n\pi$, where n is an integer (positive or negative). Thus only when ε is an integer will the frequency equation have a real solution in the considered interval for c^*. This solution can, however, not be accepted since it is connected with a singularity.

For very large values of γ, the frequency equation has complex roots always when $\varepsilon > 1$, $(\delta > 2)$, (the limiting form of the equation is $\left(\frac{c^*}{c^*+1}\right)^{1+\varepsilon} = 1$). One complex root (with $c_i > 0$) when $3 > \varepsilon > 1$, two complex roots when $5 > \varepsilon > 3$, and so on. For $\varepsilon = 2$, we obtain, for instance, for $\gamma \to \infty$ the root

$$c^* = \tfrac{1}{2}\left(-1 + i\frac{\sqrt{3}}{3}\right).$$

Quite generally we must have

$$c^*_r \to -\tfrac{1}{2} \quad \text{when } \gamma \to \infty.$$

For large values of γ, the argument of the left-hand side of the frequency equation will be a little smaller than 2π. While the argument of the right-hand side will be smaller than $\pi(1+\varepsilon)$. Therefore, when $0 < \varepsilon < 1$, we have no complex roots of the frequency equation for large values of γ. It is, however, not so easy to determine the value of γ below which the equation has complex solutions. Only in the singular case that $\varepsilon = 1$, this value of γ may be readily determined. In this case the frequency equation reduces to

$$(4, 27) \qquad 2c^{*3} + 3c^{*2} - (2\gamma - \tfrac{3}{2})c^* - (\gamma - \tfrac{1}{2}) = 0.$$

This equation will have complex equations only when

$$\gamma < \tfrac{3}{8} 2^{\frac{1}{3}}.$$

$$\delta = i\nu.$$

Finally we will discuss the case that δ is imaginary

$$(4, 28) \qquad \delta = i\nu,$$

with

$$(4, 29) \qquad \nu = \left(\frac{4\beta g}{\alpha^2} - 1\right)^{\frac{1}{2}}, \quad \alpha^2 < 4\beta g.$$

The frequency equation (4, 5) then assumes the form

$$(4, 30) \qquad \left[\gamma + \frac{1-i\nu}{2}(1+c^*)\right](1+c^*)^{i\nu} = \left[\gamma + \frac{1+i\nu}{2}(1+c^*)\right]c^{*i\nu}.$$

Assuming again $c_i^* > 0$, we have

$$(4, 31) \qquad \arg(1+c^*) < \arg c^*.$$

Further we have

$$\text{Mod}\left[\frac{1-i\nu}{2}(1+c^*)\right] = \text{Mod}\left[\frac{1+i\nu}{2}(1+c^*)\right],$$

$$\arg\left[\frac{1-i\nu}{2}(1+c^*)\right] < \arg\left[\frac{1+i\nu}{2}(1+c^*)\right],$$

considering only arguments which are smaller than π.

From the two last equations we deduce

$$(4, 32) \qquad \text{Mod}\left[\gamma + \frac{1-i\nu}{2}(1+c^*)\right] > \text{Mod}\left[\gamma + \frac{1+i\nu}{2}(1+c^*)\right].$$

Now

$$\text{Mod}(1+c^*)^{i\nu} = e^{-\nu \arg(1+c^*)}, \qquad \text{Mod}\, c^{*i\nu} = e^{-\nu \arg c^*}.$$

Therefore on account of relation (4, 31)

$$(4, 33) \qquad \text{Mod}(1+c^*)^{i\nu} > \text{Mod}\, c^{*i\nu}.$$

This equation together with equation (4, 32) shows that when $c^*_i > 0$, the modulus of the left-hand side of the equation is greater than the modulus of the right-hand side. Thus, the frequency equation has no complex solutions.

Assuming then c^* to be real, the frequency equation (4, 30) may be written

$$(4, 34) \qquad \tan\left(\frac{\nu}{2}\ln\frac{1+c^*}{c^*}\right) = \frac{\nu(1+c^*)}{2\gamma + (1+c^*)}.$$

Again we have no solutions in the interval

$$-1 < c^* < 0.$$

Considering positive values of c^*, the right-hand side of our equation is positive and increases from the value $\frac{\nu}{2\gamma+1}$ for $c^* = 0$ monotonically to the value ν for $c^* \to \infty$. The curve representing the right-hand side will then cut all the branches of the tan-function, so that we get infinitely many solutions. These solutions must satisfy the relation

$$\frac{2}{\nu}(n\pi + \operatorname{arc\,tan} \nu) > \ln \frac{1+c^*}{c^*} > \frac{2}{\nu}\left(n\pi + \operatorname{arc\,tan} \frac{\nu}{2\gamma+1}\right), \tag{4, 35}$$

where the arc-functions are considered positive and smaller than $\frac{\pi}{2}$. [n is any positive integer including $n = 0$. The smaller n, the larger the corresponding value of c^*. For $n = 0$, we have the largest value of c^*. For this largest c^* value we have

$$\frac{2}{\nu} \operatorname{arc\,tan} \nu > \ln \frac{1+c^*_{max}}{c^*_{max}} > \frac{2}{\nu} \operatorname{arc\,tan} \frac{\nu}{2\gamma+1}.$$

If, for instance, $\nu = 1$, we get

$$c^*_{max} < \frac{1}{e^{\frac{\pi}{2}} - 1}.$$

Considering next $c^* < -1$, we introduce as before $c^{**} = -(c^* + 1)$. Our frequency equation then takes the form

$$\tan\left(\frac{\nu}{2} \ln \frac{c^{**}+1}{c^{**}}\right) = \frac{\nu c^{**}}{2\gamma - c^{**}}. \tag{4, 36}$$

Again we get infinitely many solutions for c^{**} positive.

If

$$\nu \ln \frac{\gamma+1}{\gamma} < \pi,$$

we have no solutions with

$$c^{**} > \gamma.$$

5. Final Remarks.

It will be of interest to study waves with finite wavelength. Preliminary investigations by E. Riis[1]) seem to suggest that the principle results of the investigations carried through in the preceding sections apply also for finite values of k. The boundary problem discussed by Taylor[2]), which for $\beta > 0$ led to the same results concerning the existence or non-existence of ordinary waves, is quite different from the boundary problem considered in section 3 of this paper. Taylor has as a boundary condition at the upper "plane" that the amplitude function approaches zero when the height of the layer increases without limit, whereas we have as a boundary condition that the amplitude function is exactly zero at both boundaries. It is easy to show by a discussion of the zeros of the Hankel function involved[1]) that for $\beta = -\beta^* < 0$, Taylor's boundary problem leads to the criterion:

when $\frac{\beta^* g}{\alpha^2} < 2$, no ordinary waves,

,, $\frac{\beta^* g}{\alpha^2} > 2$, ordinary instability waves exist.

This criterion is valid for all values of the wave-number k.

Another question is to discuss how the fluid reacts upon perturbations in the cases that no solutions of the type (1, 4) exist, and in the cases when the existing solutions of this type do not form a complete set of functions. Probably the motion will then be a "wave"-propagation with different velocities at different levels, and with changing amplitudes in time, similar to the motion described by Kelvin for the Couette-flow without stability.

As well known viscosity will do away with the singularity appearing in the perturbation equation for the general linear flow of an incompressible and homogeneous fluid. That is not so when static stability occurs. The general perturbation equation for wave-disturbances in this case when viscosity is taken into account, is

$$\begin{aligned} &(U-c)[\psi'''' - 2k^2\psi'' + k^4\psi] \\ &- i\frac{k}{\nu}\{(U-c)^2\psi'' - \beta(U-c)^2\psi' \\ &+ [\beta g + (\beta U' - U'')(U-c) \\ &- k^2(U-c)^2]\psi\} = 0, \end{aligned} \tag{5, 1}$$

where ψ is the stream function for the perturbation motion and ν the kinematic coefficient of viscosity. The occurance of a singularity even when viscosity is considered, shows that stability will introduce a very important feature in the discussion of the disturbances of linear flow.

[1]) The author wishes to express his thanks to mr. Riis for many helpful discussions.

[2]) Loc. cit.

[1]) G. N. Watson: A Treatise on the Theory of Bessel Function, §§ 3.6, 3.7, 15.7, Cambridge 1948.

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