ON TWO-DIMENSIONAL PERTURBATION OF LINEAR FLOW

BY

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1. The Perturbation Equation.

Our basic system is a rectilinear flow of an incompressible and homogeneous fluid bounded by two parallel rigid surfaces. The basic velocity U, along the X-axis of our system of references, depends only upon the perpendicular distance z from, for instance, the lower rigid surface. Thus we have

(1, 1) $$U = U(z),$$

and the rigid walls are given by

$$z = 0 \quad \text{and} \quad z = h.$$

Assuming the perturbations take place in planes parallel to the XY-plane, are independent of the direction perpendicular to this plane, and have a trigonometric dependence on time t and on x, we obtain for the vertical velocities w in the perturbed motion the well known equation

(1, 2) $$w'' - \left(\frac{U''}{U-c} + k^2\right) w = 0.$$

Here a prime denotes differentiation with regard to z, c is the velocity of propagation and k the wave number.

The perturbation equation (1, 2) has a singularity for $U - c = 0$ except when $U'' = 0$ at the same place, i.e. except when an inflection point appears in the velocity profile and the velocity of propagation is equal to the mean velocity at the point of inflection. The physical nature of this singularity will be discussed in a later paper.

An inflection point in the velocity profile is necessary if permanent waves (c real) without singularity in the added vorticities shall be possible. In the following section we will examine such solutions of equation (1, 2) more closely.

2. Permanent Waves as Solution of the Perturbation Equation when the Velocity Profile has a Point of Inflection.

We assume that U'' is equal to zero at one or more levels of our fluid, i.e. that the vorticity in the mean flow has a minimum or a maximum value at these levels. Now, consider the case that the velocity profile has only one point of inflection, and that the vorticity U' has numerically a minimum value at this point. It is then easily seen that $\frac{U''}{U-c} > 0$ throughout the fluid. Then the expression within the parentheses in equation (1, 2) will be positive for all values of z, and w can not be zero for two different values of z: $z = 0$ and $z = h$. In this case we have therefore no permanent wave solutions.

In the following discussion of permanent wave solutions we will therefore confine our consideration to the case that U' has numerically a maximum value where we have $U'' = 0$.

We introduce in the equation (1, 2)

(2, 1) $$V = U - c,$$

and obtain

(2, 2) $$w'' - \left(\frac{V''}{V} + k^2\right) w = 0.$$

V is the mean velocity in a coordinate system following the mean motion at the level where the velocity profile has a point of inflection. Since we have assumed c real, V is a real function of z. Now put

$$\frac{V''}{V} + k^2 = f(z),$$

or

$$V'' - (f(z) - k^2)V = 0, \tag{2, 3}$$

the equation (2, 2) takes the form

$$w'' - f(z)w = 0. \tag{2, 4}$$

For a given $f(z)$ the two last equations determine the vertical velocity w in the disturbed motion and the corresponding velocity profile V. To get permanent wave solutions the function $f(z)$ must satisfy the requirements introduced by the kinematic boundary conditions at the horizontal rigid planes, i.e. that here w must be equal to zero. As mentioned in the preceding section $f(z)$ must then in part of the interval from $z = 0$ to $z = h$ be negative. Considering the case that $f(z)$ is a constant throughout, this constant must necessarily be a negative constant,

$$f(z) = -\varkappa^2. \tag{2, 5}$$

For this form of $f(z)$ we obtain from equation (2, 4) the solution

$$w = b \sin \varkappa z, \tag{2, 6}$$

satisfying the boundary conditions at the lower boundary. b is a constant of integration assumed to be small. To satisfy the boundary conditions at the upper boundary we must have

$$\varkappa = \frac{n\pi}{h}, \tag{2, 7}$$

where n is an integer.

The basic velocity corresponding to the solution (2, 6) is found from equation (2, 3) to be given by

$$V = U - c = A \sin \sqrt{\varkappa^2 + k^2}\, z + B \cos \sqrt{\varkappa^2 + k^2}\, z. \tag{2, 8}$$

Denoting the wavelength of the permanent wave by λ and the "wavelength" of the velocity profile by L, we find the relation

$$n = \frac{2h}{\lambda L} \sqrt{\lambda^2 - L^2}. \tag{2, 9}$$

If n for all values of λ is less than 1, we get no permanent waves. This occurs when $L > 2h$, i.e. when half the profile wave length is greater than the height of the layer. If the maximum (real) value of n is greater or equal to 1 but less than 2, we get one wavelength. If for ininstance $L = 2h$, i.e. half the profile wavelength equals the height of the layer, we get one permanent wave with an infinitely large wavelength. In general, if the maximum (real) value of n is greater or equal to m but smaller than $m + 1$ (m an integer), i.e. if

$$m \leq \frac{2h}{L} < m + 1, \tag{2, 10}$$

we get a discrete set of m permanent waves. When the equality sign applies, one of the waves will have an infinite wavelength. From the equation (2, 9) we see that the wavelength of the permanent waves will always be greater than the profile wavelength.

It has been shown elsewhere [1] (where references to other authors are also given) that the solutions (2, 6) and (2, 8) are also valid for a finite value of b. Choosing for convenience a coordinate system such that we get stationary motion, the equation for finite disturbances is quite generally given by

$$\nabla^2 \Psi = -g(\Psi), \tag{2, 11}$$

expressing simply the fact that in a stationary motion of a homogeneous and incompressible fluid the vorticity must be constant along a streamline (Ψ is the stream function for the total velocity, $g(\Psi)$ an arbitrary function of Ψ). If wee choose

$$g(\Psi) = (n^2 + k^2)\Psi, \tag{2, 12}$$

we again obtain our solutions (2, 6) and (2, 8), now for finite values of b. In this case the equation for stationary motion is linear even for finite disturbances. Other choices of the function $f(z)$ than that given by equation (2, 5) will correspond to linearization of other forms of $g(\Psi)$ than that given by equation (2, 12).

Above we have studied the permanent waves occurring for a choice of $f(z)$ leading to a harmonic velocity profile. As a further illustration we will choose

$$f(z) = -e^{2z} + \nu^2. \tag{2, 13}$$

Introducing into equations (2, 3) and (2, 4) we obtain

(2, 14)
$$V'' + [e^{2z} - (\nu^2 - k^2)]V = 0,$$
$$w'' + (e^{2z} - \nu^2)w = 0.$$

Choosing $y = e^z$ as a new independent variable it is easily seen that we get the solutions

(2, 15)
$$V = BZ_{(\nu^2 - k^2)^{\frac{1}{2}}}(e^z),$$
$$w = bZ_\nu(e^z),$$

where Z_ν and $Z_{(\nu^2 - k^2)^{\frac{1}{2}}}$ are the cylinder functions of order ν and $(\nu^2 - k^2)^{\frac{1}{2}}$ respectively, b an arbitrary constant small of first order, and B an arbitrary finite constant. For real values of ν, we get periodic behaviour of w, and therefore horizontal planes where the vertical velocity is zero. With such horizontal planes as boundaries we get permanent wave solutions. Again, as in the case with a harmonic velocity profile, the permanent solutions can never consist of more than a discrete set of waves for discrete values of the wave number k. Confining our consideration to the case in which the velocity profile is given by cylinder functions of real order, we must have

$$k^2 < \nu^2,$$

which again gives a lower limit for the wavelength. Also for the height of the layer equal to about half the "wavelength" of the considered cylinder function, we get only one permanent wave with an infinite wavelength, and for more narrow layers, we get no permanent waves. Thus, we obtain results similar to those obtained for the harmonic profile.

3. Instability Waves for the Harmonic Velocity Profile.

It has been suggested (see for instance Fjørtoft [2]) that the permanent waves represent transitions from stability to instability waves in such a way that for a wave number smaller than that corresponding to a permanent wave, the waves will be instability waves. Since it may be shown that disturbances of sufficiently small wavelength will not release any instability while it appears possible that sufficiently long waves may be instability waves, the suggestion is quite plausible. It leaves, however, one interesting question open, namely: assume that it applies to the shortest of a set of permanent waves for a considered velocity profile, what kind of transition is represented by the other

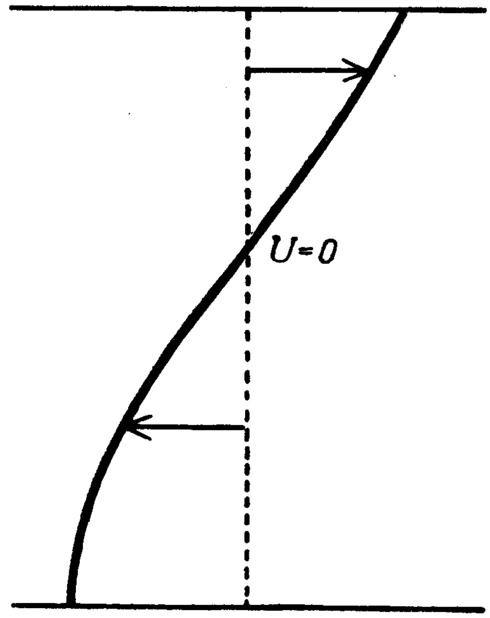

Fig. 1.

permanent wave solutions? It has also one important consequence which seems to have been overlooked, namely that *profiles giving no permanent waves must represent a stable motion.* Thus, for instance, the profile drawn in the diagram Fig. 1, with less than half a wavelength of a cosine curve, should, according to what has been deduced in the preceding section, represent a definitely stable motion.

In order to make an attempt to decide how the permanent wave solutions shall be interpreted, we will discuss in some detail the solutions of our perturbation equation (1, 2) for the harmonic velocity profile in the case of an infinite wavelength ($k = 0$). For infinite wavelength the equation can be integrated explicitely, giving the well known solution

(3, 1)
$$w = b(U - c)\int_{z_1}^{z} \frac{dz}{(U - c)^2},$$

satisfying the boundary conditions at the lower rigid surface, which is now, with later use in mind, given by

$$z = z_1.$$

If the upper rigid surface is given by

$$z = z_2, \quad \text{i.e.} \quad z_2 - z_1 = h,$$

we get the frequency equation

$$(3,\,2)\qquad \int\limits_{z_2}^{z_1} \frac{dz}{(U-c)^2} = 0.$$

Since the integrand for real values of c is definitely positive ($z_2 > z_1$), the frequency equation can never be satisfied for real values of c. The permanent wave solutions with real c will not satisfy the frequency equation (3, 2), in spite of the fact that they are solutions of equation (1, 2) and the boundary conditions. As solutions of equation (3, 2), however, we will obtain complex values of c differing infinitesimally from the value of c corresponding to the permanent wave solutions, and introducing these complex values of c into equation (3, 1) we will in the same approximation obtain the vertical velocity corresponding to the permanent solutions.

Introducing now into the frequency equation

$$(3,\,3)\qquad U = A\cos\gamma z,$$

we obtain

$$(3,\,4)\qquad \int\limits_{z_1}^{z_2} \frac{dz}{(A\cos\gamma z - c)^2} = 0.$$

We introduce a new variable y given by

$$(3,\,5)\qquad y = \cot\frac{\gamma z}{2}.$$

With this variable we get the frequency equation

$$(3,\,6)\qquad \int\limits_{y_1}^{y_2} \frac{(y^2+1)dy}{(y^2-a^2)^2} = 0,$$

where

$$(3,\,7)\qquad a = \left(\frac{A+c}{A-c}\right)^{\frac{1}{2}},$$

the real value of a being chosen as positive.

From equation (3, 6) we find by integration

$$(3,\,8)\qquad \left[(a^2-1)\ln\frac{y_2+a}{y_2-a} + a(a^2+1)\times \left(\frac{1}{y_2-a}+\frac{1}{y_2+a}\right)\right] - \left[(a^2-1)\ln\frac{y_1+a}{y_1-a} + a(a^2+1)\times \left(\frac{1}{y_1-a}+\frac{1}{y_1+a}\right)\right] = 0.$$

It must here be remembered that the ln-functions are many-valued functions. The difference between them is, however, uniquely determined by taking account of the "path" of integration from y_1 to y_2, or how the angle φ given by

$$(3,\,9)\qquad Re^{i\varphi} = \frac{y+a}{y-a},$$

varies from y_1 to y_2.

Consider only cases when $y_1 = \infty$ and when $y_1 = 0$. When $y_1 = \infty$, the expression within the last parantheses in equation (3, 8) may be put equal to zero, so that the frequency equation may be written

$$(3,\,10)\qquad \ln\frac{y_2+a}{y_2-a} + \frac{a(a^2+1)}{a^2-1}\left(\frac{1}{y_2+a}+\frac{1}{y_2-a}\right) = 0.$$

When $y_1 = 0$, we obtain instead of this equation the frequency equation

$$(3,\,10')\qquad \ln\frac{a+y_2}{a-y_2} + \frac{a(a^2+1)}{a^2-1}\left(\frac{1}{y_2+a}+\frac{1}{y_2-a}\right) = 0.$$

The expression on the left hand side of equation (3, 10) and of equation (3, 10′) will for $y_2 = \infty$ and $y_2 = 0$, respectively, have the value zero. This is also true for the corresponding expression (the expression in equation (3, 10) multiplied by $-\frac{4(a^2-1)}{a^3}$) having as derivative the integrand

$$\frac{(y^2+1)}{(y^2-a^2)^2}$$

in equation (3, 6). Introducing in this expression

$$(3,\,11)\qquad a = a_r + ia_i,$$

where a_r and a_i are real quantities, it becomes

$$(y^2+1)\frac{(y^2-a_r^2+a_i^2)^2-4a_r^2a_i^2+[4a_ra_i(y^2-a_r^2+a_i^2)]i}{[(y^2-a_r^2+a_i^2)^2+4a_r^2a_i^2]^2}.$$

Since both the real and the pure imaginary part of the integral (for $y = 0$ and for $y = \infty$) starts with the value zero for the lower limit, and it must for the value $y = y_2$ again be zero, the derivative must for both its real and imaginary part be equal to zero for a value of y between $y = y_1$ and $y = y_2$. From the expression for its pure imaginary part we then deduce

$$(3,\,12)\qquad a_r^2 > a_i^2,$$

or the real part of a must be numerically greater than its pure imaginary part.

Further if $y_1 = \infty$, i.e. $\gamma z_1 = 0$ (or $2n\pi$), we must have

$$(3,\,13)\qquad y_2^2 < a_r^2 - a_i^2,$$

if a solution shall exist when $0 < \gamma z_2 < \pi$, i.e. for less than half a profile wavelength between the rigid surfaces. In this interval

$$y > 0.$$

For the next half profile wavelength

$$y < 0,$$

and so on.

If we had started with $y_1 = 0$, i.e. $\gamma z_1 = \frac{\pi}{2}$ (or $\frac{\pi}{2} + 2n\pi$), we must have

$$y_2^2 > a_r^2 - a_i^2, \tag{3, 14}$$

if a solution shall exist for a height of the layer less than half a profile wavelength. For this starting value of y,

$$y < 0,$$

for the first half profile wavelength from z_1, and

$$y > 0,$$

for the next half profile wavelength, and so on.

Splitting the terms in equation (3, 10) into real and pure imaginary parts, we obtain the two equations

$$\text{arc tan} \frac{2a_i y_2}{y_2^2 - (a_r^2 + a_i^2)} = -2a_i y_2 \frac{(y_2^2+a_r^2+a_i^2)[(a_r^2+a_i^2)^2-1]-4a_r^2[y_2^2-(a_r^2+a_i^2)]}{[(a_r^2-a_i^2-1)^2+4a_r^2a_i^2][(y_2^2-a_r^2+a_i^2]+4a_r^2a_i^2]} \tag{3, 15}$$

$$\tfrac{1}{2}\ln\frac{(y_2+a_r)^2+a_i^2}{(y_2-a_r)^2+a_i^2} = -2a_r y_2 \frac{[y_2^2-(a_r^2+a_i^2)][(a_r^2+a_i^2)^2-1]+4a_i^2(y_2^2+a_r^2+a_i^2)}{[(a_r^2-a_i^2-1)^2+4a_r^2a_i^2][(y_2^2-a_r^2+a_i^2)^2+4a_r^2a_i^2]}$$

We see immediately that if a_r and a_i are solutions of these equations, a_r and $-a_i$ will also be solutions. We may therefore assume

$$a_i > 0.$$

Above we have chosen also

$$a_r > 0.$$

Then we have

$$\text{arc tan}\frac{2a_i y_2}{y_2^2 - (a_r^2 + a_i^2)} > 0,$$

both for positive and negative values of y_2 and whether we start from $y_1 = \infty$ or $y_1 = 0$.

Further we have

$$\ln\frac{(y_2+a_r)^2+a_i^2}{(y_2-a_r)^2+a_i^2} \begin{matrix} > 0, \text{ for } y_2 > 0, \\ < 0, \text{ for } y_2 < 0. \end{matrix}$$

Thus, since the denominator on the right hand side of equations (3, 15) is positive, we must have:

1. *For $y_2 > 0$:*

$$\begin{aligned} &(y_2^2 + a_r^2 + a_i^2)[(a_r^2 + a_i^2)^2 - 1] - 4a_r^2[y_2^2 - (a_r^2 + a_i^2)] < 0, \\ &[y_2^2 - (a_r^2 + a_i^2)][a_r^2 + a_i^2)^2 - 1) + 4a_i^2(y_2^2 + a_r^2 + a_i^2) < 0. \end{aligned} \tag{3, 16}$$

Assume:

a. $\quad a_r^2 + a_i^2 > 1.$

Then from the first of our relations it follows

$$y_2^2 > a_r^2 + a_i^2,$$

and from the second of our relations

$$y_2^2 < a_r^2 + a_i^2,$$

so that the assumption $a_r^2 + a_i^2 > 1$ leads to contradictory results when $y_2 > 0$. From the last of our relations we also see that $a_r^2 + a_i^2 = 1$ is impossible in this case.

Assume then:

b. $\quad a_r^2 + a_i^2 < 1.$

Then the first of our relations may be satisfied without any definite further requirements concerning the quantities y_2^2 and $a_r^2 + a_i^2$. From the second relation, however, we get the condition

$$y_2^2 > a_r^2 + a_i^2.$$

Thus, we must have

$$a_r^2 + a_i^2 < 1, \ a_r^2 + a_i^2 < y_2^2 \text{ for } y_2 > 0. \tag{3, 17}$$

Comparing the last of these two relations with the relation (3, 13) which had to be fulfilled in this case ($y_2 > 0$) when the layer had a height less than half the profile wavelength, we arrive at contradicting results. Hence we have proved:

A linear flow of an inviscid fluid bounded by rigid horizontal surfaces with a velocity profile given by $U = A \cos \gamma z$ ($z = 0$ the lower rigid surface) is not exponentially unstable for small perturbations of infinite wavelength when the height of the layer is less than half the profile wavelength.

2. *For $y_2 < 0$:*

$$\begin{aligned} &(y_2^2 + a_r^2 + a_i^2)[(a_r^2 + a_i^2)^2 - 1] - 4a_r^2[y_2^2 - (a_r^2 + a_i^2)] > 0, \\ &[y_2^2 - (a_r^2 + a_i^2)][(a_r^2 + a_i^2)^2 - 1] + 4a_i^2(y_2^2 + a_r^2 + a_i^2) < 0. \end{aligned} \tag{3, 18}$$

Assume:

a. $\quad a_r^2 + a_i^2 > 1.$

The first of our relations may then be fulfilled without any further definite requirements. From the second relation it follows as before

$$y_2^2 < a_r^2 + a_i^2.$$

Assume then:

b. $$a_r^2 + a_i^2 < 1.$$

From the first of our relations it follows

$$y_2^2 < a_r^2 + a_i^2,$$

and from the second as before

$$y_2^2 > a_r^2 + a_i^2,$$

two relations leading to contradictory results.

Thus we are left with the possibilities

$$(3, 19) \quad a_r^2 + a_i^2 > 1, \; a_r^2 + a_i^2 > y_2^2 \text{ for } y_2 < 0.$$

Comparing the last of these two relations with the relation (3, 14) which for $y_2 < 0$ had to be valid when the layer had a height less than half the profile wavelength, we again arrive at a contradictory result, leading again to the result emphasized above.

Since y changes sign when y_1 is changed from ∞ to 0, the above results should have as a consequence that if solutions representing instability waves exist for a height h_1 given by

$\frac{n+1}{2} L > h_1 > \frac{n}{2} L$ (L is the profile wavelength),

solutions representing instability waves should also exist for heights h_2 satisfying the relation

$$\frac{n}{2} L > h_2 > \frac{L}{2},$$

excluding the cases when h_2 is equal to an integral number of $\frac{L}{2}$, when we get the permanent waves.

From equation (3, 7) we find c_r and c_i (the real and imaginary part of c respectively) given by

$$(3, 20) \quad \begin{aligned} c_r &= A \frac{(a_r^2 + a_i^2)^2 - 1}{[1 + (a_r^2 - a_i^2)]^2 + 4a_r^2 a_i^2}, \\ c_i &= A \frac{4a_r a_i}{[1 + (a_r^2 - a_i^2)]^2 + 4a_r^2 a_i^2}. \end{aligned}$$

Since a_r and a_i, determined from the frequency equation (3, 8), are independent of A (the amplitude of the profile curve), we see that the velocity of propagation and the amplification factor (velocity of flight) are proportional to A. Thus, the larger the variation in the mean velocity with height, the larger the amplification of the wave. The layer where c_r is equal to the mean velocity will, however, be independent of A.

Starting with $z_1 = 0$ at the lower rigid plane, so that here we have a maximum positive velocity in the basic motion, we deduced above that

$$a_r^2 + a_i^2 > 1 \text{ when } (2n-1)\frac{L}{2} < h < 2n\frac{L}{2},$$

$$a_r^2 + a_i^2 < 1 \text{ when } \quad 2n\frac{L}{2} < h < (2n+1)\frac{L}{2},$$

where n is an integer equal to or larger than 1. From the first of equations (3, 20) we then deduce

$$(3, 21) \quad \begin{aligned} c_r > 0 &\text{ when } (2n-1)\frac{L}{2} < h < 2n\frac{L}{2}, \\ c_r < 0 &\text{ when } \quad 2n\frac{L}{2} < h < (2n+1)\frac{L}{2}. \end{aligned}$$

Thus, for $n = 1$ in the first relation we must have a c_r as shown in the diagram Fig. 2 (the mean velocity is zero at the inflection points of the profile curve). The same is the case for all heights satisfying the first relation. For $n = 1$ in the second relation, we must have a c_r as shown in the diagram Fig. 3. The same is the case for all heights satisfying the second relation.

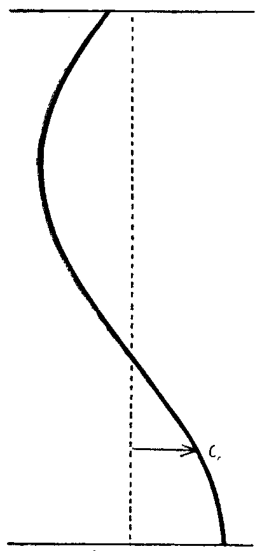

Fig. 2.

Our above statements are true only if solutions really exist. To prove the existence of solutions we have solved the equation (3, 15) for some positive and negative values of y_2. The results with a_r and a_i as functions of the height of the layer with half the profile wavelength as a unit are given in the diagrams Fig. 4. The points marked with a cross are the computed points. The corresponding diagrams for c_r and

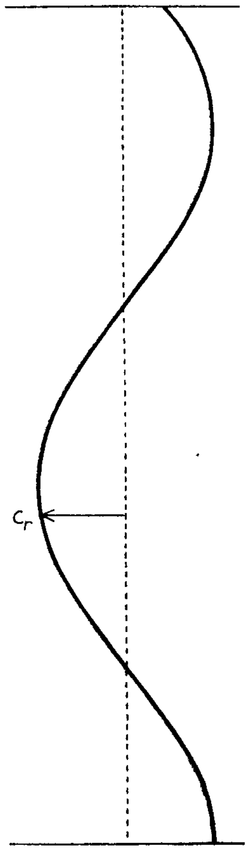

Fig. 3.

c_i with A as unit are given in Fig. 5. We have only given the results for layers of heights between 1 and 3 half profile wavelengths. From the numerical computation it appeared, however, that solutions existed for all higher layers.

Thus, excepting the cases when the height of the layer is an integer in the units applied, we obtain instability waves of infinite wavelength for heights larger than half a profile wavelength. For heights between 1 and 2 we get a maximum instability with c_i equal to about $0.26\,A$ for a height equal to about 1.4, and for heights between 2 and 3 we get a maximum instability with c_i equal to about $0.22\,A$ for a height equal to about 2.45.

Comparing what we have deduced above with what we found about the existence of permanent solutions, excepting again the cases when the height of the layer is an integer, we see that as soon as permanent solutions exist for a finite value of the wavelength, the system will be unstable for perturbations of infinite wavelength. A reasonable conclusion should then be: the permanent solution with the shortest wavelength corresponds to a transition from stability (for smaller wavelengths) to instability (for larger wavelengths), whereas all other permanent solutions do not correspond to such a transition, giving instability waves for larger as well as for smaller wavelengths. If no permanent solutions exist, the flow is stable for all perturbations of an inviscid fluid. Thus, *a flow of an inviscid fluid with a harmonic velocity profile and bounded by parallel rigid walls less than half a profile wavelength apart should be stable for all perturbations.* This result should be valid also for hori-

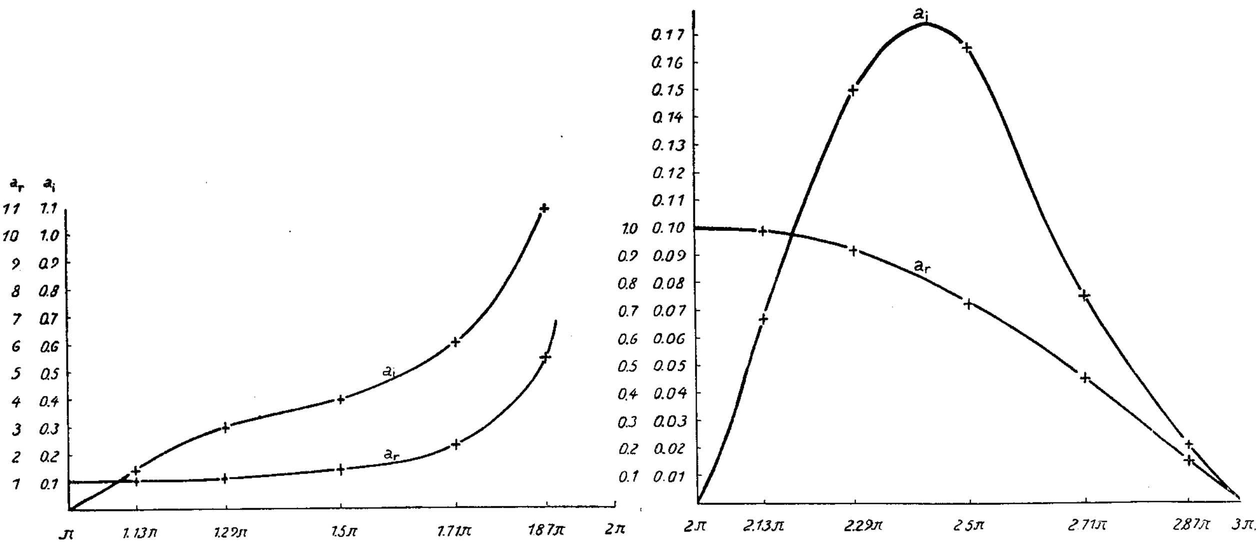

Fig. 4.

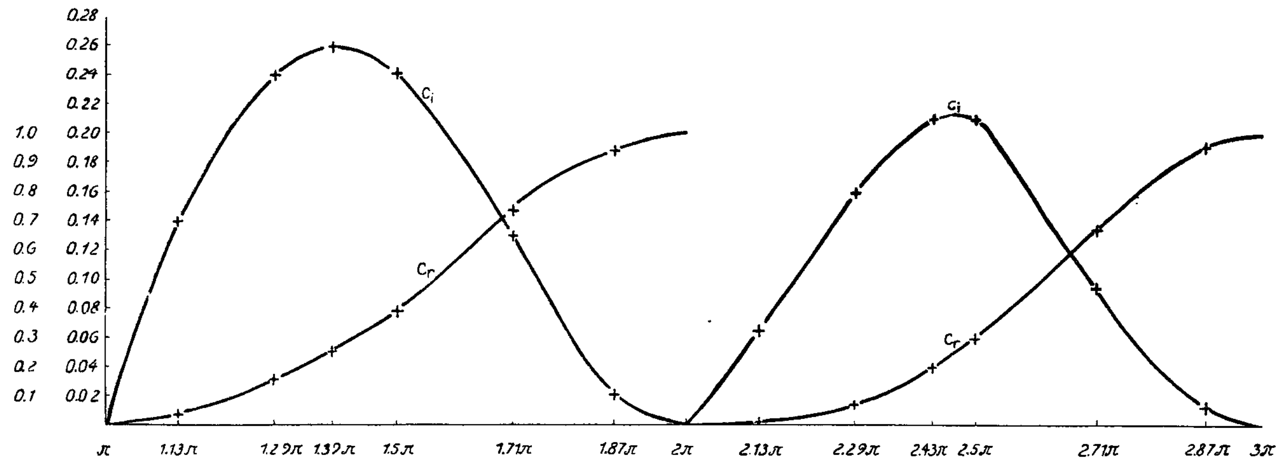

Fig. 5.

zontal motion of an inviscid rotating (homogeneous and incompressible) fluid, since we have then permanent solutions for the same heights and wavenumbers.

From the above discussion of a special case it should be permissible to conclude: *The existence of an inflection point corresponding to a maximum (numerically) value of the vorticity in the velocity profile, is not a sufficient although it is a necessary criterion for instability of a linear flow of an inviscid fluid.*

4. Some General Stability Theorems.

If we put $c = c_r + ic_i$ and $w = w_r + iw_i$ in equation (1, 2) and divide in real and imaginary parts we obtain the two equations

$$(4,\ 1)\qquad \begin{aligned} w''_r - \left(\frac{U''(U-c_r)}{(U-c_r)^2+c_i^2} + k^2\right)w_r + \frac{U''c_i}{(U-c_r)^2+c_i^2}w_i &= 0, \\ w''_i - \left(\frac{U''(U-c_r)}{(U-c_r)^2+c_i^2} + k^2\right)w_i - \frac{U''c_i}{(U-c_r)^2+c_i^2}w_r &= 0. \end{aligned}$$

Multiplying the first of these equations by w_i, the second by w_r, and subtracting, we obtain an equation which may be written

$$(w_r'w_i - w_rw_i')' + \frac{U''c_i}{(U-c_r)^2+c_i^2}(w_i^2 + w_r^2) = 0.$$

Integrating from $z = z_1$ to $z = z_2$, we obtain, since w_r and w_i at these levels must disappear

$$(4,\ 2)\qquad c_i\int_{z_1}^{z_2}\frac{U''}{(U-c_r)^2+c_i^2}(w_i^2 + w_r^2)dz = 0.$$

To have a c_i different from zero, i.e. instability waves, we see that U'' must change sign between the two rigid surfaces. Thus, if the velocity profile has no points of inflection, we get no instability waves. This result is given by Rayleigh [3].

Now, multiplying the first of equations (4, 1) by w_r, the second by w_i, adding, and integrating from $z = z_1$ to $z = z_2$, we obtain

$$\int_{z_1}^{z_2}(w_rw_r'' + w_iw_i'')dz - \int_{z_1}^{z_2}\left(\frac{U''(U-c_r)}{(U-c_r)^2+c_i^2} + k^2\right) \times (w_r^2 + w_i^2)dz = 0.$$

By partial integration, again utilizing the boundary conditions, we get the relation

$$(4,\ 3)\qquad \int_{z_1}^{z_2}\frac{U''(U-c_r) + k^2(U-c_r)^2 + k^2c_i^2}{(U-c_r)^2+c_i^2}(w_r^2+w_i^2)dz = -\int_{z_1}^{z_2}(w_r'^2 + w_i'^2)dz < 0.$$

Applying equation (4, 2) and removing on the left-hand side only positive quantities, we obtain

$$(4,\ 4)\qquad \int_{z_1}^{z_2}\frac{U''U}{(U-c_r)^2+c_i^2}(w_r^2 + w_i^2)dz < 0.$$

This relation must be valid for every inertia system and therefore also for the system having a velocity equal to the velocity at the level where the velocity profile has a point of inflection. In this system of references

$$U''U > 0,$$

when the vorticity is a minimum (numerically) at the inflection point. We thus refind the result that this flow is not exponentially unstable. If the vorticity is a maximum (numerically) at one level

$$U''U < 0,$$

in a reference system having a velocity equal to the mean velocity at the level both at a region below and above this level. The inequality (4, 4) may be satisfied.

Corresponding results can be deduced for horizontal perturbation of a rotating homogeneous and incompressible fluid. Disregarding the kinematic effect of the curvature, we obtain the equation (1, 2) for the meridional velocity component with $U''-\beta$ substituted for U''. β is the "variation of the Coriolis' parameter". In this case we find that $U''-\beta$ must have a maximum to obtain instability waves. This result has been deduced by Fjørtoft [2]. Also the case when the density varies (no external force) is easily discussed by the above method.

From (4, 3) together with (4, 2) we also deduce

$$\int_{z_1}^{z_2} \frac{UU'' + k^2U^2 - 2k^2c_rU}{(U-c_r)^2 + c_i^2}(w_r^2 + w_i^2)dz < 0.$$

Assuming a harmonic velocity profile

$$U'' = -\gamma^2 U,$$

this inequality may further by use of (4, 2) be written

$$(4, 5) \qquad \int_{z_1}^{z_2} \frac{(k^2-\gamma^2)U^2}{(U-c_r)^2 + c_i^2}(w_r^2 + w_i^2)dz < 0.$$

Thus we can have no instability waves for

$$(4, 6) \qquad k > \gamma,$$

i.e. *no instability waves with wavelengths shorter than the wavelength of the velocity profile.*

When the induced vorticity or the effect of the variation of the Coriolis' parameter is taken into account we obtain exactly the same result by now introducing the velocity profile $U = A\cos\gamma z + \frac{\beta}{\gamma^2}$, $U'' = -\gamma^2 A\cos\gamma z$, i.e. again using a system of references so that the velocity is equal to zero where the maximum value of the absolute vorticity occurs, i.e. where $U''-\beta = 0$. It is interesting to note that γ also gives the upper limit for the wave numbers of permanent stability waves.

If we multiply equation (1, 2) with $(U-c)$ and divide it into real and imaginary parts, we obtain

$$(4, 7) \qquad \begin{aligned} (U-c_r)(w_r''-k^2w_r) + c_i(w_i''-k^2w_i) & \\ -U''w_r &= 0, \\ (U-c_r)(w_i''-k^2w_i) - c_i(w_r''-k^2w_r) & \\ -U''w_i &= 0. \end{aligned}$$

Adding these equations after having multiplied the first of them by w_i, the second by $-w_r$, and integrating from z_1 to z_2 utilizing the boundary conditions, we obtain

$$\int_{z_1}^{z_2} \{U'(w_r'w_i - w_rw_i') + c_i[w_r'^2 + w_i'^2 + k^2(w_r^2 + w_i^2)]\}dz = 0.$$

This equation may also be written

$$\int_{z_1}^{z_2}\left[\left(\frac{U'}{2q}-c_i\right)(w_r'^2+w_i'^2)+\left(\frac{U'q}{2}-k^2c_i\right)(w_r^2+w_i^2)\right]dz = \int_{z_1}^{z_2}\frac{U'}{2q}[(w_r'+qw_i)^2 + (w_i'-qw_r)^2]dz,$$

where q is an arbitrary function of z. If we choose q numerically equal to k, but everywhere having the same sign as U', i.e.

$$(4, 8) \qquad q = \frac{U'}{|U'|}k,$$

the right-hand side of our equation will be positive, and we obtain

$$(4, 9) \qquad \int_{z_1}^{z_2}\left\{\left(\frac{|U'|}{2k}-c_i\right)[w_r'^2+w_i'^2+k^2(w_r^2+w_i^2)]\right\}dz > 0.$$

Thus we must have

$$(4, 10) \qquad kc_i = \sigma_i < \frac{|U'|\max}{2},$$

giving an absolute upper limit for "the frequency of flight" or the amplification factor σ_i.

In our case with the harmonic profile of wavenumber γ and amplitude A, we obtain

$$\sigma_i < \frac{\gamma A}{2}. \tag{4, 11}$$

The upper limit increases with the wave number of the profile, or decreases with increasing profile wavelength.

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