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ON REYNOLDS STRESS, TURBULENT DIFFUSION AND VELOCITY PROFILE IN A STRATIFIED FLUID

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FREMLAGT I VIDENSKAPS-AKADEMIETS MØTE DEN 24DE MAI 1957 AV E. HØILAND

Summary. The two first sections contain a discussion of the Reynolds stresses in an adiabatic atmosphere. Triple correlations are cancelled compared with second order correlations, and

$$\tau_z = Q\,(U_{zz}\int_0^t \overline{w(s)w(t)}\;ds + U_z \frac{d}{dz}\int_0^t \overline{v(s)v(t)}\;ds)$$

is obtained. Here τ denotes the Reynolds stress, Q mean density, U mean velocity, t time, w vertical velocity, v horizontal velocity perpendicular to the mean velocity and subscript z indicates differentiation with respect to z. In section 3 the same way of reasoning is applied to turbulent diffusion, and in section 4 a derivation of a formula for the velocity profile in a stratified fluid is given. It is found that, approximately,

$$U_z = CZ^{-1+S\varepsilon}$$

where C and ε are constants, and $S = g\dfrac{\Theta_z}{\Theta}$ with Θ denoting potential temperature.

1. Introduction. In a discussion of turbulent motion we will of necessity be interested only in averaged quantities. Relations between these quantities are obtained by averaging the equations at our disposal. It turns out, however, that if the equations are non-linear, the relations connecting these quantities also contain terms which are the average product of two or more eddy terms. An example of such terms is the Reynolds terms introduced by averaging the Navier-Stokes equation.

To get a closed system of relations it is obviously necessary to find a way to express the Reynolds terms by the quantities describing the mean field and empirical functions. The most famous and successful attempts to solve this problem are those of Prandtl and Taylor. The importance of their results in theoretical as well as in applied hydrodynamics is well known to all who have been working in this field.

Their mode of procedure is similar in that both assume that the turbulent mixing is a discontinuous process, and that a conservation law is valid. The main difference is the choice of this conservation law: Prandtl's theory is based on the validity of conservation of momentum and Taylor's on conservation of vorticity. In an incompressible homogenous fluid with the mean (horizontal) velocity U only depending on the vertical coordinate z, Prandtl's approximation leads to

$$\tau_z = Q \frac{d}{dz} (K\, U_z). \tag{1.1}$$

Here τ denotes the Reynolds stress, Q the mean density, K the eddy viscosity (an empiric function which may be expressed by means of the mixing length and U) and subscript z indicates differentiation with respect to the vertical coordinate. On the other hand, Taylor obtains

$$\tau_z = Q K\, U_{zz} \tag{1.2}$$

where K is the eddy viscosity, but not defined in the same way as above (throughout this paper Taylor's work on two-dimensional vorticity transfer is referred to and not his work on three-dimensional transfer which seems to be too complex to have found much application [1]). It is readily seen that only if the eddy viscosity is independent of z, can the two expressions for τ_z be identical. Observations show that this is generally not the case. The question then arises which of the two theories gives the better description of the phenomenon. Do we have any decisive facts from observations?

Let us look at two such results. Firstly, it is usually agreed that in the atmosphere near the ground one may consider τ as a constant for the purpose of determining the velocity profile. Equation (1.2) would then lead to U_z = constant whereas equation (1.1) gives $K\, U_z$ = constant. It is well known that, in the case of turbulent motion and no static stability, U is very well described by a logarithmic increase with height. This law which may be obtained from equation (1.1) by choosing $K \sim z$, is in contradiction to equation (1.2). Secondly, in a recent work by Fjeldstad [2] it is proved that equation (1.2) applied to the theory of Ekman's spiral in the ocean leads to $\frac{d^2R}{dz^2} \geqq 0$ (R the magnitude of the horizontal velocity), in contradiction to what is often observed.

These two examples speak clearly in favour of the momentum transfer theory. On the other hand, it seems hard to understand why a theory based on conservation of momentum should give a strikingly better result than a theory based on conservation of vorticity. In Section 2 in this paper an expression for the Reynolds stresses will be derived, without introducing discontinuous processes, which should be well fitted for a discussion of the two hypothesis. The result which we end up with is neither equation (1.1) nor equation (1.2). It will be shown that by assumptions which seem to be of a much less restrictive nature than those applied by Prandtl and Taylor, τ_z is found to be given by

$$\tau_z = Q\,(K\,U_{zz} + G_z\,U_z). \tag{1.3}$$

Here K and G are defined by

$$K = \int_0^t \overline{w(s)w(t)}\,ds, \tag{1.4}$$

$$G = \int_0^t \overline{v(s)v(t)}\,ds \tag{1.5}$$

where a bar denotes a space average, w and v the components of the turbulent velocity in the z and y direction, and t (time) is assumed large. By an additional assumption this expression is equal to Prandtl's result.

In Section 3 the same way of reasoning will be applied to turbulent diffusion, and in Section 4 the effect of a non-homogenous stratification will be discussed.

2. The Reynolds stress in an adiabatic atmosphere. We introduce a system of references x, y, z with the z axis positive upwards. The mean velocity U is assumed to have no component in the y and z direction and to be only a function of z. It seems plausible to assume that friction does not play any dominant rôle in the diffusion process giving rise to the Reynolds stress. Friction is therefore neglected in the equations henceforth. The equation of motion may than be written:

$$(Q + q)\,[u_t + Uu_x + uu_x + vu_y + wU_z + wu_z] = -p_x - P_x \tag{2.1}$$

$$(Q + q)\,[v_t + Uv_x + uv_x + vv_y + wv_z] = -p_y \tag{2.2}$$

$$(Q + q)\,[w_t + Uw_x + uw_x + vw_y + ww_y] = -p_z - gq - P_z - gQ. \tag{2.3}$$

Here u, v, w are the components of the turbulent velocity, p the pressure fluctuations, P the mean pressure, Q the mean density, q the density fluctuation and g the acceleration of gravity. Subscripts t, x, y, z denote differentiation with respect to t (time), x, y and z.

We will assume that the fluid is incompressible. The equation of continuity is then

$$u_x + v_y + w_z = 0. \tag{2.4}$$

It will, below, also be assumed that mean quantities are independent of x, y and t. The mean is to be interpreted as a space average, obtained by integration in the x, y plane.

Taking the average of equation (2.1) then gives

$$\varrho\overline{vu_y} + \varrho\overline{wu_z} + \overline{qu_t} + U\overline{qu_x} + \overline{quu_x} + \overline{qvu_y} + U_z\overline{qw} + \overline{qwu_z} = -P_x. \tag{2.5}$$

Experience shows that in an atmosphere with adiabatic lapse rate the effect of the density fluctuations may be neglected. In this section equation (2.5) is therefore reduced to

$$\varrho\overline{vu_y} + \varrho\overline{wu_z} = -P_x, \tag{2.6}$$

and we conclude that the Reynolds force τ_z is given by

$$\tau_z = -\varrho\overline{vu_y} - \varrho\overline{wu_z}. \tag{2.7}$$

According to the assumptions made above, equation (2.7) may also be written in the form

$$\tau_z = -\varrho\frac{d}{dz}\overline{uw} \tag{2.8}$$

or

$$\tau_z = \varrho\overline{v\zeta} - \varrho\overline{w\eta} \tag{2.9}$$

where η and ζ are components of the vorticity and defined by

$$\begin{aligned}\eta &= u_z - w_x \\ \zeta &= v_x - u_y.\end{aligned} \tag{2.10}$$

We will take as a starting point equation (2.9) and try to express $\overline{w\eta}$ and $\overline{v\zeta}$ by the mean velocity field. Eliminating the pressure in equations (2.1) and (2.3) and neglecting terms containing q and ϱ_z leads to

$$\begin{aligned}\eta_t + U(u_{xz} - w_{xx}) + U_z(u_x + w_z) + U_{zz}w + u_zu_x + uu_{xz} + v_zu_y + vu_{yz} + \\ w_zu_z + wu_{zz} - u_zw_x - uw_{xx} - v_xw_y - vw_{xy} - w_xw_z - ww_{xz} = 0.\end{aligned} \tag{2.11}$$

The motion will be described below in the system of references where $U = 0$. Integration of equation (2.11) from $t = 0$ to $t = t$, multiplication with w and taking the average then gives

$$\overline{\eta w} - \overline{\eta_o w} + U_z\int_o^t \overline{(u_x(s) + w_z(s))w(t)}ds + U_{zz}\int_o^t \overline{w(s)w(t)}ds + \mathrm{I} = 0. \tag{2.12}$$

Here $\eta_o = \eta(o)$, and I is an abbreviation for the terms consisting of triple correlations. It seems reasonable to cancel these terms compared with the other terms in equation (2.12), and this has in fact been done by Prandtl and Taylor. This simplification is adopted also in the present work. Furthermore, t will be chosen so large that the term $\overline{\eta_0 w}$ is small. Equation (2.12) then reduces to

$$\overline{\eta w} = - U_{zz} \int_0^t \overline{w(s)w(t)} ds - U_z \int_0^t \overline{(u_x(s) + w_z(s))w(t)} ds. \tag{2.13}$$

By an analogous procedure we obtain from equations (2.1) and (2.2)

$$\overline{\zeta v} = U_z \int_0^t \overline{w_y(s)v(t)} ds. \tag{2.14}$$

Introducing these expressions in equation (2.9) and making use of the assumptions made above, τ_z may be written

$$\tau_z = Q [U_{zz} \int_0^t \overline{w(s)w(t)} ds + U_z \int_0^t \overline{w_y(s)v(t)} ds + U_z \int_0^t \overline{v(s)w_y(t)} ds]. \tag{2.15}$$

Let us stop here for a moment to see what the expression for τ_z would look like if we were to adopt the approximations applied by Taylor and Prandtl. Taylor assumes that the vorticity is conserved, which is true when the motion is two-dimentional. With this simplification equation (2.15) reduces to

$$\tau_z = Q\, U_{zz} \int_0^t \overline{w(s)w(t)} ds. \tag{2.16}$$

On the other hand, Prandtl assumes that the momentum is conserved, which is true when the pressure effect can be neglected. Cancelling $p_x + P_x$ in equation (2.1) and following the same line of reasoning as above,

$$\tau_z = Q \frac{d}{dz} (U_z \int_0^t \overline{w(s)w(t)} ds) \tag{2.17}$$

is found. We notice that these two formulas for τ_z are in accordance with equations (1.1) and (1.2), and that K here is defined by a continuous process.

Let us return to equation (2.15). From equations (2.2) and (2.3), neglecting triple correlations and terms containing q, we obtain

$$\overline{w_y(s)v(t)} = \overline{w_y(o)v(t)} - \overline{v_z(o)v(t)} + \overline{v_z(s)v(t)} + U_z \int_0^s \overline{v(t)v_x(a)}\, da. \tag{2.18}$$

Since t is large, the two first terms on the right hand side will be cancelled. Integration of this equation then gives

$$\int_0^t \overline{w_y(s)v(t)} ds = \int_0^t \overline{v_z(s)v(t)} ds + U_z \int_0^t ds \int_0^s \overline{v(t)v_x(a)} da. \tag{2.19}$$

Correspondingly,

$$\overline{w_y(t)v(s)} = \overline{w_y(o)v(s)} + \overline{v_z(t)v(s)} - \overline{v_z(o)v(s)} + U_z \int_0^t \overline{v(s)v_x(a)} da, \tag{2.20}$$

and by integration

$$\int_0^t \overline{w_y(t)v(s)}ds = \int_0^t \overline{w_y(o)v(s)}ds + \int_0^t \overline{v_z(t)v(s)}ds - \int_0^t \overline{v_z(o)v(s)}ds \tag{2.21}$$

since

$$\int_0^t ds \int_0^t \overline{v(s)v_x(a)}da = 0.$$

Equation (2.21) may also, be means of equation (2.19) and the assumption of stationariness be written

$$\int_0^t \overline{w_y(t)v(s)}ds = \int_0^t \overline{v_z(t)v(s)}ds + U_z \int_0^t ds \int_0^s \overline{v(t)v_x(a)}da. \tag{2.22}$$

Furthermore, changing the order of integration leads to

$$\int_0^t ds \int_0^s \overline{v(t)v_x(a)}da = \int_0^t \overline{v(t)v_x(s)}\,(t-s)ds. \tag{2.23}$$

Making use of these results, equation (2.15) takes the form

$$\tau_z = Q[U_{zz}\int_0^t \overline{w(s)w(t)}ds + U_z \int_0^t \overline{v_z(s)v(t)}ds + \tag{2.24}$$

$$+ U_z \int_0^t \overline{v(s)v_z(t)}ds + 2U_z^2 \int_0^t \overline{v(t)v_x(s)}\,(t-s)ds].$$

This expression may be further simplified by introducing the derivative of $\int_0^t \overline{v(s)v(t)}ds$ with respect to z. Since U is function of z, v is referred to a system of references which moves with a velocity depending on z. Taking account of this,

$$\frac{d}{dz}\int_0^t \overline{v(s)v(t)}ds = \int_0^t \overline{v_z(s)v(t)}ds + \int_0^t \overline{v(s)v_z(t)}ds - U_z \int_0^t \overline{v_x(s)v(t)}\,(t-s)ds \tag{2.25}$$

is obtained. Equation (2.24) may then be written

$$\tau_z = Q[U_{zz}\int_0^t \overline{w(s)w(t)}ds + U_z \frac{d}{dz}\int_0^t \overline{v(s)v(t)}ds + 3U_z^2 \int_0^t \overline{v_x(s)v(t)}\,(t-s)ds]. \tag{2.26}$$

The integrands in the two first terms on the right hand side in this equation obviously have their largest value in $s = t$, and decrease for increasing $t - s$ values. The integrand in the third term, however, is zero of at least second order when $s \to t$. It is difficult to draw any conclusions from the equations of motion and the assumptions adopted here about the behaviour of $\overline{v_x(s)v(t)}$ for arbitrary values of $t - s$. On the other hand, it seems likely that the coefficient of correlation between $v_x(s)$ and $v(t)$ is very small, and the last term in the equation above will be cancelled. Equation (2.26) then reduces to

$$\tau_z = Q[U_{zz} \int_0^t \overline{w(s)w(t)}ds + U_z \frac{d}{dz} \int_0^t \overline{v(s)v(t)}ds]. \tag{2.27}$$

It should be remembered that t is assumed large so that the right hand side is in fact independent of time.

The formula just obtained for τ_z, (2.27), should be compared with formulas (1.1) and (1.2) (or (2.16) and (2.17)). It is noticed that if

$$\frac{d}{dz} \int_0^t \overline{v(s)v(t)}ds = \frac{d}{dz} \int_0^t \overline{w(s)w(t)}ds, \tag{2.28}$$

the formula derived here is identical with Prandtl's result. We may at once conclude that very close to a fixed plane (or the ground) equation (2.28) cannot be fulfilled. This follows from the fact that approaching the plane, w tends towards zero as z^2 whereas v only as z. On the other hand, observations seem to indicate that a little above the ground (plane) we have to a fair approximation

$$\overline{v^2} = \overline{w^2} + c \tag{2.29}$$

where c is independent of z (for observations in the atmosphere see the tables in [3] and [4]). If this is correct, we are led to consider equation (2.28) as approximately true, and we obtain

$$\tau_z = Q \frac{d}{dz} (KU_z) \tag{2.30}$$

with

$$K = \int_0 \overline{w(s)w(t)}ds. \tag{2.31}$$

Near the ground τ may be considered as a constant, and equation (2.30) gives

$$U_z = \frac{C}{K} \tag{2.32}$$

where C is a constant. Let the origin ($z = 0$) be placed just over the ground but still in the region of full turbulence. We develop K in a Taylor series and retain the two first terms

$$K = a + bz. \tag{2.33}$$

Equation (2.32) then leads to the logarithmic profile with a (which, usually, may be neglected) as zero displacement.

3. Turbulent diffusion. In this section the same way of reasoning as above will be applied to turbulent diffusion. We neglect molecular diffusion. Let $\theta + \Theta$ (θ the fluctuation, Θ the mean) denote a quantity which is conserved. If we assume that Θ is only a function of z and t, we then have

$$\theta_t + \Theta_t + u\theta_x + v\theta_y + w\theta_z + w\Theta_z = 0, \tag{3.1}$$

where the motion is referred to a system of references with the mean velocity zero. Taking the average of the equation gives

$$\Theta_t + \overline{u\theta_x} + \overline{v\theta_y} + \overline{w\theta_z} = 0, \tag{3.2}$$

or, by assuming homogenity in the x and y direction and applying the equation of continuity

$$\Theta_t + \frac{\partial}{\partial z}\,\overline{\theta w} = 0. \tag{3.3}$$

To find an expression for $\overline{\theta w}$ equation (3.1) is integrated with respect to t, multiplied by w, and the average is taken. Neglecting triple correlations we obtain

$$\overline{\theta w} = \overline{\theta_o w} - \int_o^t \Theta_z(s)\overline{w(s)w(t)}\,ds, \tag{3.4}$$

where $\theta_o = \theta(o)$. The term $\overline{\theta_o w}$ will be cancelled either by assuming $\theta_o = 0$ or t large. Equation (3.3) then takes the form

$$\Theta_t - \frac{\partial}{\partial z}\int_o^t \Theta_z(s)\overline{w(s)w(t)}\,ds = 0, \tag{3.5}$$

which is an integro-differential equation in Θ if $\overline{w(s)w(t)}$ is considered as known. $\overline{w(s)w(t)}$ obviously has its largest value for $s = t$. If it is correct that $\overline{w(s)w(t)}$ decays rapidly for increasing $t - s$ values, whereas the variation of $\Theta_z(s)$ is small, equation (3.5) may approximately be written

$$\Theta_t = \frac{\partial}{\partial z}\,(K\Theta_z). \tag{3.6}$$

K is here a diffusion coefficient and is defined by (2.31).

It should be noted that K is formally analoguous to the diffusion coefficient introduced by Taylor [5] from a statistical approach to the problem and which is also studied by Sutton [6] and Batchelor [7]. The difference is that in the papers referred to the diffusion coefficient is defined in Langrangian coordinates whereas it is here defined in an Eulerian frame of references where the mean velocity is zero.

4. The velocity profile in a stratified fluid*. A natural extension of the discussion of the Reynolds stress given in Section 2 is obtained by taking into account the effect of density stratification. It will soon be clear that a density stratification complicates the equations considerably, and to obtain expressions which we can handle it will be necessary to make assumptions in addition to those made above.

The Reynolds force is according to equation (2.5) given by

$$\tau_z = -\,Q\overline{vu_y} - Q\overline{wu_z} - \overline{qu_t} - U_z\overline{qw}, \tag{4.1}$$

neglecting triple correlations. In this model Q and U are functions of t, and we will get integrals of the type

$$\int_0^t Q_z(s)\,\overline{w(s)w(t)}\,ds,\ \int_0^t U_z(s)\,\overline{w(s)w(t)}\,ds,$$

i. e. Q_z and U_z are parts of the integrands. If $\overline{w(s)w(t)}$ (or the equivalent) decays rapidly with increasing $t - s$ values, Q_z and U_z (or the equivalent) may with a good approximation be taken outside the integral. This simplification will be adopted below.

The fluid will be assumed to be incompressible so that

$$u_x + v_y + w_z = 0 \tag{4.2}$$

and

$$q_t + Q_t + uq_x + vq_y + wq_z + wQ_z = 0. \tag{4.3}$$

The first term on the right side of equation (4.1) will now take the same form as in Section 2 so that

$$-\,Q\overline{vu_y} = QU_z\int_0^t \overline{w_y(s)v(t)}\,ds. \tag{4.4}$$

The second term is in the same way found to be

$$-\,Q\overline{wu_z} = QU_{zz}\int_0^t \overline{w(s)w(t)}\,ds + QU_z\int_0^t \overline{v(s)w_y(t)}\,ds + \\ + U_zQ_z\int_0^t \overline{w(s)w(t)}\,ds + Q_z\overline{uw} + g\int_0^t \overline{q_x(s)w(t)}\,ds. \tag{4.5}$$

The two last terms in equation (4.1) may be rewritten by means of the equation of continuity (4.3). We find

$$-\,\overline{qw} = Q_z\int_0^t \overline{w(s)w(t)}\,ds \tag{4.6}$$

* This section has been changed during the proof reading of the manuscript (September 1957).

and

$$(4.7)\qquad -\overline{qu_t} = Q_z \int_0^t \overline{w(s)u_t(t)}ds = Q_z \int_0^t \overline{w_s(s)u(t)}ds = -Q_z\overline{uw}.$$

In the last equation it is assumed that the turbulence is stationary.

Introducing the expressions above, equation (4.1) takes the form

$$(4.8)\qquad \tau_z = QU_{zz} \int_0^t \overline{w(s)w(t)}ds + QU_z \int_0^t \overline{v(s)w_y(t)}ds + QU_z \int_0^t \overline{w_y(s)v(t)}ds +$$

$$2\,U_z Q_z \int_0^t \overline{w(s)w(t)}ds + g\int_0^t \overline{q_x(s)w(t)}ds.$$

We notice that this equation contains two new terms compared with equation (2.15), namely

$$2\,U_z Q_z \int_0^t \overline{w(s)w(t)}ds,\quad g\int_0^t \overline{q_x(s)w(t)}ds.$$

These terms are due to the kinematical and dynamical effect of the density stratification, respectively. The last expression may, by applying the equation of continuity and the assumptions made above, be written

$$(4.9)\qquad g\int_0^t \overline{q_x(s)w(t)}ds = -g\,Q_z \int_0^t ds \int_0^t \overline{w_x(a)w(t)}da =$$

$$-g\,Q_z \int_0^t \overline{w_x(s)w(t)}\,(t-s)ds.$$

The integral on the right side is of the same kind as the integral we cancelled in Section 2 (see the remarks in connection with formula (2.26)), and will also be cancelled here. With this simplification equation (4.8) may be written

$$(4.10)\qquad \tau_z = QKU_{zz} + QLU_z + 2\,Q_z K\,U_z$$

where K is defined by equation (2.31) and

$$(4.11)\qquad L = \int_0^t \overline{v(s)w_y(t)}ds + \int_0^t \overline{w_y(s)v(t)}ds.$$

Applying equations (2.2) and (2.3), neglecting triple correlations and assuming t large, leads to

$$(4.12)\qquad Q\int_0^t \overline{w_y(s)v(t)}ds = Q\int_0^t \overline{v_z(s)v(t)}ds + Q_z\int_0^t \overline{v(s)v(t)}ds -$$

$$g\int_0^t ds \int_0^s \overline{q_y(a)v(t)}da + QU_z \int_0^t ds \int_0^s \overline{v_x(a)v(t)}da.$$

The last integral in this equation has been cancelled in Section 2, and will also be cancelled here. Correspondingly, making use of the fact that $\overline{v(s)w_y(t)}$ is, due to the assumption of stationarity, a function only of $t - s$,

$$Q\int_o^t \overline{v(s)w_y(t)}ds = Q\int_o^t \overline{w_y(-s)v(-t)}ds = Q\int_o^t \overline{v_z(-s)v(-t)}ds \tag{4.13}$$

$$+ Q_z\int_o^t \overline{v(-s)v(-t)}ds - g\int_o^t ds\int_o^{-s} \overline{q_y(a)v(-t)}da.$$

Introducing the equation of continuity, changing the order of integration and applying the assumption of stationarity then gives

$$\tau_z = Q\,K\,U_{zz} + (Q\frac{d}{dz}\int_o^t \overline{v(s)v(t)}ds + 4\,Q_zK' + g\,Q_zM)\,U_z. \tag{4.14}$$

Here

$$K' = \frac{1}{2}(K + \int_o^t \overline{v(s)v(t)}ds) \tag{4.15}$$

and

$$M = \frac{1}{2}\int_{-t}^{+t} T(u)\,u^2du \tag{4.16}$$

where $T(u)$ is defined by

$$T(t-\beta) = \overline{w_y(\beta)v(t)}. \tag{4.17}$$

If the non-adiabatic stratification is not too strong,

$$\overline{w_y(s)v(t)} \approx \overline{v_z(s)v(t)}, \tag{4.18}$$

and we infer that M is positive.

For the purpose of determining the velocity profile, τ_z may in equation (4.14) be put equal to zero. It will, as in Section 2, be assumed that

$$\frac{d}{dz}\int_o^t \overline{v(s)v(t)}ds = \frac{d}{dz}K, \tag{4.19}$$

and that

$$\frac{K'}{K} = \gamma \tag{4.20}$$

where γ is a constant. We then obtain

$$U_{zz} + (\frac{K_z}{K} + 4\frac{Q_z}{Q}\gamma + g\frac{Q_z}{Q}\frac{M}{K})\,U_z = 0 \tag{4.21}$$

By integration, assuming $\frac{Q_z}{Q}$ independent of height,

$$U_z = C\,K^{-1} \exp\left(-4\,\frac{Q_z}{Q}\,\gamma z - g\,\frac{Q_z}{Q}\,N\right) \tag{4.22}$$

where C denotes an arbitrary constant and

$$N = \int \frac{M}{K}\,dz.$$

As in the neutral case we assume that

$$K = a + bz \tag{4.23}$$

where a (which is a small quantity) and b are positive constants. In order to get some information about N, we notice that, by making use of (4.18) together with the other assumptions introduced above,

$$\int_{-t}^{+t} T(u)\,du \approx b, \tag{4.24}$$

i. e. approximately independent of height. Comparing (4.24) with the expression for M(4.16) we are led to the conclusion that M increases somewhat with height since, apparently, bigger eddies contribute relatively more to M than to the integral (4.24). We will, however, as a first approximation let M be independent of z. The variation with height may then be introduced in the final result.

Furthermore, it turns out that the influence of the kinematical effect of the density variation on the velocity profile is small. With these simplifications (4.22) may be written

$$U_z = C(a + bz)^{-1 + S\varepsilon} \tag{4.25}$$

where

$$S = -g\,\frac{Q_z}{Q},$$

and ε is a positive quantity and defined by

$$\varepsilon = \frac{M}{b}. \tag{4.26}$$

The velocity profile is only sensibly affected by the non-adiabatic stratification if this is strong. In these cases

$$S = -g\frac{Q_z}{Q} \approx g\frac{\Theta_z}{\Theta} \tag{4.27}$$

with Θ denoting the potential temperature. Equation (4.25) is the final expression for the velocity profile with S defined by (4.27).

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1. Golds
2. Fjelds
3. Scrasi
4. Best,
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5. Taylo
6. Sutto
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Final remarks. It should be noted that (4.25) is essentially the same formula as found by Deacon [8] from an extensive set of observations. In the same paper, Deacon suggests that ε should increase somewhat with height, rather than being a constant, which is in good agreement with what is pointed out above.

It is believed that the assumptions made above are fairly good if the non-adiabatic stratification is not too strong. In this case new effects enter which are not taken in account here.

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