GEOFYSISKE PUBLIKASJONER
GEOPHYSICA NORVEGICA
VOL. XX NO. 3

TWO-DIMENSIONAL AND THREE-DIMENSIONAL MOUNTAIN WAVES

BY ENOK PALM

FREMLAGT I VIDENSKAPS-AKADEMIETS MØTE DEN 24DE MAI 1957
AV E. HØILAND

Summary. In Chapter I of this paper two-dimensional atmospheric waves behind a mountain ridge are discussed. Section 1 contains a non-mathematical description of the creation of lee waves. In Section 2, 3 and 4 models with a velocity and stability distribution which approximate the observed data, are discussed. It has been possible in two cases to compare the computed wave lengths with those observed in the Sierra Nevada region and in both cases good agreement is found. The streamline and pressure field due to the resonance waves have been drawn for one of these.

Chapter II contains a discussion of three-dimensional waves generated by a mountain top. The basic velocity is assumed to increase linearly with height in the troposphere and to be constant in the stratosphere. The stability is assumed constant in the troposphere as well as in the stratosphere. Isolines for the vertical velocity have been computed and drawn in a numerical example.

TWO-DIMENSIONAL MOUNTAIN WAVES

1. Introduction. The aim of Chapter I of this paper is to give theoretical discussion of the waves in the atmosphere set up by a two-dimensional mountain, and to compare the theoretical results with observations. This problem of two-dimensional waves produced by a corrugated bottom, is an old one and has already been attacked by Kelvin [1] and Rayleigh [2]. They studied surface waves on an incompressible, non-viscous and homogeneous fluid which had a finite depth but an infinite extent in the horizontal direction. Later, models more similar to the atmosphere have been studied by several writers. We will, however, begin with a discussion of surface waves, since, as will be shown, an understanding of these waves will be of great importance for the understanding of the more complicated lee waves in the atmosphere. Successively, more realistic models will be introduced and compared with observations. Throughout the paper the effect of the rotation of the earth will be neglected, since typical wave lengths are only of the order of 5—25 km, and the equations will be linearized.

Surface waves. Before mentioning the mathematical solution we give a short physical discussion of the creation of lee waves, similar to that given in the theory of ship waves [3]. Let us assume for the sake of simplicity that the bottom corrugation is a single isolated obstacle. Further, let the fluid be set into motion at time t equal to zero and (for simplicity) quickly obtain its constant mean velocity U. Every moment the obstacle produces an impulse on the free surface, vertically above the obstacle consisting of all possible wave lengths. We divide the entire wave spectrum into an infinite number of small groups where each group is represented by its mean wave length. The energy of such a small group of wave lengths is propagated with the group velocity corresponding to the mean wave length. The waves, once created, will behave independently of the obstacle and are therefore free waves, i. e. waves fulfilling the two boundary conditions: pressure constant along the free surface and the vertical velocity zero at the bottom. The effect at a distance x downstream from the obstacle at a time t due to the impulse produced at time τ will then be effect of the small group of waves characterized by the wave number k defined by

$$
(1.1)\qquad
\begin{aligned}
x &= \left(U - \frac{d\sigma}{dk}\right)(t-\tau) \qquad && x < U(t-\tau) \\
x &= \left(U + \frac{d\sigma}{dk}\right)(t-\tau) \qquad && x > U(t-\tau).
\end{aligned}
$$

Here σ is the frequency of the waves generated and thus $\frac{d\sigma}{dk}$ is the group velocity. This wave determined by equation (1.1), will in the vicinity of x be a small fraction of a sine wave travelling with the corresponding phase velocity.

The total deformation at x at time t is obtained by adding all waves produced at τ-values ranging from $\tau = 0$ to $\tau = t$. Most of these waves will neutralize each other, approximately, having different phases. The main contribution to the deformation is due to the groups of waves, if any, the phases of which at x are independent of a change in τ, or, since the deformation due to each wave is a function of $t - \tau$, of a change in t. These groups of waves are those travelling with a phase velocity equal to $-U$. Thus we obtain the following picture of the creation of waves behind an obstacle: The free waves with a phase velocity equal to $-U$ play a dominant role. These waves are usually called resonance waves. Before the groups advancing with the group velocities of the resonance waves have arrived, the deformation consists of waves approximately neutralizing each other. After the first resonance group has passed, the deformation is approximately a (stationary) sine wave with a wave length equal to the wave length of the resonance wave. When the second resonance group has passed, the deformation will approximately be two sine waves, and so on. When no resonance wave exists, the stationary solution will be a surface elevation which is very small beyond a certain distance from the seat of disturbance. Further, if the group velocity is less than the phase velocity, the corresponding wave will be located downstream. If the group velocity is greater than the phase velocity, the wave will be located upstream.

For surface waves the group velocity is always less than the corresponding phase velocity. The maximum phase velocity is $\sqrt{gh}$ where g is the acceleration of gravity and h the depth of the fluid. Thus, if $U > \sqrt{gh}$, no resonance wave will exist and therefore no lee wave. On the other hand, if $U < \sqrt{gh}$ we will get just one resonance wave and therefore one wave in the lee of the obstacle.

Assuming the motion to be stationary, it is easily proved mathematically that it is precisely the existence or non-existence of a stationary free wave which determines whether the solution will have the form of a wave, or just decay rapidly with the distance from the obstacle.

Let us assume that the vertical velocity w is

(1.2) $$w = g(z)e^{ikx}.$$

Here x and z are the horizontal and the vertical coordinates, k the wave number and $g(z)$ a function which is determined by a second order equation (since we have no friction) and the boundary conditions. $g(z)$ may therefore be written

(1.3) $$g(z) = c_1 g_1(z) + c_2 g_2(z)$$

where $g_1(z)$ and $g_2(z)$ are independent solutions of the linear differential equation and c_1 and c_2 arbitrary constants. The upper boundary condition will in mountain wave problems be a homogenous one so that we may write

(1.4) $$g(z) = c_3 g_3(z)$$

where $g_3(z)$ is a linear combination of $g_1(z)$ and $g_2(z)$, and c_3 is an arbitrary constant. If the corrugation of the bottom is given by

(1.5) $$\zeta = \gamma e^{ikx},$$

we get

(1.6) $$w = U_o ik\gamma e^{ikx} \qquad z = h_o$$

with U_o denoting the velocity of the stream at the bottom ($z = h_o$). w is therefore

(1.7) $$w = \frac{U_o\, ik\, \gamma\, g_3(z)}{g_3(h_o)}\, e^{ikx}.$$

To find the motion set up by an arbitrary bottom corrugation we integrate over k. We notice that we will get two different cases, according to whether $g_3(h_0)$ which is a function of k, is zero for any k-value or not. If $g_3(h_0)$ is zero for a certain value of k, i. e. a stationary free wave exists with that value of k, the integral will have a singularity and the main contribution to the integral will be due to this. In this case we get a sine wave, either upstream or downstream. On the other hand, if $g_3(h_0)$ is not zero for any value of k, the stationary solution will not be a wave solution.

The arguments and results above are of course not restricted to surface waves, we investigate below some other models applying the results just obtained.

The model with constant static stability and no wind shear. In this section we discuss the possibility of getting lee waves in a two-dimensional model with an incompressible fluid having a constant stability and no wind shear. This model has been studied mathematically by Lyra [4] and Queney [5] among others.

In an incompressible fluid the stability is given by $\frac{g}{Q}\frac{dQ}{dz}$ where g is the acceleration of gravity and Q the density in the undisturbed state. Since the stability is assumed constant, we have

(1.8) $$Q = Q_o e^{-\beta z}$$

where β is a (positive) constant. Expression (1.8) for the density will be used as an approximation also when the fluid is compressible. Introducing

(1.9) $$w = e^{\frac{1}{2}\beta z}\omega(z)e^{ikx},$$

we have

(1.10) $$\frac{d^2\omega}{dz^2} + \left(\frac{\beta g}{U^2} - \frac{\beta^2}{4} - k^2\right)\omega = 0.$$

For a free wave ω must be equal to zero at the ground ($z = h_0$). Further, when $z \rightarrow \infty$, the wave energy must be finite. The general solution of equation (1.10) is

(1.11 a) $$\omega = c_1 \exp - \sqrt{k^2 - k^2_a}\,z + c_2 \exp \sqrt{k^2 - k^2_a}\,z \qquad k > k_a$$

(1.11 b) $$\omega = c_1 \cos \sqrt{k^2_a - k^2}\,z + c_2 \sin \sqrt{k^2_a - k_2}\,z \qquad k < k_a$$

where

$$k^2_a = \frac{\beta g}{U^2} - \frac{\beta^2}{4}$$

and c_1 and c_2 are arbitrary constants. The solution satisfying the boundary condition on the ground is

(1.12 a) $$\omega = c_3 \sinh \sqrt{k^2 - k^2_a}\,(z - h_o) \qquad k > k_a$$

(1.12 b) $$\omega = c_3 \sin \sqrt{k^2_a - k^2}\,(z - h_o) \qquad k < k_a$$

where c_3 is an arbitrary constant. The solution (1.12 a) does not fulfil the boundary condition at infinity and must be rejected. The solution (1.12 b) however, fulfils the condition for every k less than k_a. We therefore obtain the following result: When $k > k_a$ no free waves exist. When $k < k_a$ a free wave exists for every k. Behind the mountain we will get the interaction of all possible waves with wave numbers less than k_a, and the result will be a non-periodical motion which decays downstream. If $\frac{\beta g}{U^2} < \frac{\beta^2}{4}$ no free waves exist for any k-value. For atmospherical values of the stability this inequality corresponds to U greater than about 200 m sec^{-1}.

On the mathematical solution in the two models discussed above. — As mentioned earlier a mathematical discussion of surface waves set up by a bottom corrugation was given by Kelvin and Rayleigh, and a similar discussion for the other model studied above, by Lyra and Queney. All of them assume that the motion is stationary, and they therefore meet with the same difficulty: the solution is not unique if a free wave exists. In the problem of surface waves the integrand in the Fourier integral will have a pole on the real axis for the wave number corresponding to the free wave (analogues to expression (1.7) above); in the Lyra-Queney model the singularity will be a branch point for $k = k_a$. This difficulty is due to the fact that in the stationary problem all possible kinds of initial values are introduced.

Kelvin solved this difficulty by interpreting the Fourier integral as meaning the principle value of the integral, and adding free waves so that he got only a wave downstream. Rayleigh avoided the singularity by introducing a small, artificial friction proportional to the velocity and, in the end result, taking the coefficient of friction as equal to zero. Lyra and Queney have in their work adopted both methods.

None of these methods is satisfying. It should be a result of the theory that the waves only exist downstream and not an assumption. And, on the other hand, why should it be necessary to introduce friction in this problem? It could seem as if friction were the effect giving lee waves. This is of course not true. The physical reasoning above gave lee waves (since the group velocity was less than the phase velocity) without introducing any kind of friction.

The difficulty above is due to the assumption that the motion is stationary. As pointed out by Høiland [6] the solution is unique if the problem is attacked as an initial value problem. When t increases the initial value solution approaches the stationary solution. Høiland proved this for surface waves. Wurtele [7] and Palm [8] independently proved that the same was true in the model discussed by Lyra and Queney. In both works an estimate was given on the time taken before the motion was practically stationary (without friction). The result was about a couple of hours.

In the Lyra-Queney model we also meet with the problem of the form of the boundary condition to be applied at infinity. The boundary condition used is that the perturbation energy must be finite (for one Fourier component). But is that the correct one? In the initial value problem this boundary condition is superfluous. The motion is at all times given uniquely by the initial velocity and acceleration field and the boundary condition at the ground.

The connection between the initial value solution and the stationary solution may be described as follows. We will discuss one Fourier component, i. e. the bottom corrugation is of sinusoidal form. As system of reference we choose that in which the mean flow is zero. At $t = 0$ the mountain is set in motion in such a fashion that it very quickly obtains its constant velocity $-U_0$. The disturbance, propagating vertically, reaches after a time $t = \frac{z}{s}$ (s the velocity of sound) the height z. What will then happen at that height will depend on whether k is greater or less than k_a. If $k < k_a$ a resonance wave exists, and only after the resonance group has passed, will the motion approximately have attained its permanent value. In the system of references where the mountain is not moving, the solution will then be approximately equal to the stationary solution obtained by Queney, the approximation being ever better when t increases. When $k > k_a$ no resonance wave exists. In this case it is impossible by pure reasoning to give a time beyond which the stationary solution may be said to be a good approximation. The case $k > k_a$ corresponds, for surface waves, to $U > \sqrt{gh}$, and $k < k_a$ corresponds to $U < \sqrt{gh}$.

It should also be mentioned that, as shown by Eliassen and Palm, the stationary solution is made unique by requiring the mountain to be the only energy source [9].

Comparison with observations. — Let us now see how the result obtained in the Lyra—Queney model fits the observations. The most striking feature in the observations made behind Sierra Nevada ([10], see also Fig. 4 in the present paper) is that the maximum wave amplitude is located just above the mountain with the amplitude decreasing upwards, and that the wave is repeated downstream. These features agree poorly with the result from the Lyra—Queney model. To get better correspondence between the theory and the observations, effective alterations must be made in the model to obtain resonance waves for distinct k values. As first pointed out by Scorer [11] such effective alterations are a variation in the stability and in the wind profile with height.

In the remainder of this paper we discuss resonance waves in models which approximate the conditions in the atmosphere. It will be assumed that the motion is stationary, and that the waves are located downstream.

2. A model with constant wind shear and stability in the troposphere and constant wind and stability in the stratosphere. A better approximation to the atmosphere than the Lyra—Queney model is a model with constant wind shear and stability below the tropopause and constant wind and stability above the tropopause. This model has been briefly discussed by Wurtele [12].

Assuming the fluid incompressible, the differential equation valid in both layers, may be written

$$\frac{d^2\omega}{dz^2} + \left(\frac{\beta g}{U^2} - \varkappa^2 + \frac{\beta U z}{U}\right)\omega = 0 \tag{2.1}$$

where

$$\varkappa^2 = \frac{\beta^2}{4} + k^2. \tag{2.2}$$

For an compressible atmosphere equation (2.1) is not more valid. Good approximation for the equation determining ω is, however, as usual obtained by replacing in equation (2.1) βg with the actual stability $S = \frac{g(\gamma_d - \gamma)}{\overline{T}}$ where γ_d is the dry adiabatic lapse rate, γ the actual lapse rate and $\overline{T}$ the mean temperature. The term $\frac{\beta U_z}{U}$ is for atmospheric conditions small compared with the stability term and will here be cancelled in the troposphere as well as in the stratosphere. By introducing this, equation (2.1) becomes

$$\frac{d^2\omega}{dz^2} + \left(\frac{S}{U^2} - \varkappa^2\right)\omega = 0. \tag{2.3}$$

According to the discussion above the condition for getting a wave in the lee of the mountain is the existence of a stationary free wave. The free wave must satisfy the following boundary conditions: The vertical velocity must vanish on the ground, the kinetic energy must be finite at infinity and the pressure and the vertical velocity must be the same on both sides of the tropopause. Since the boundary conditions are homogeneous we have only three constants at our disposal and we should expect to obtain an equation in k which determines whether or not any resonance wave exists. However, when $k < \sqrt{SU_s^{-2} - \frac{\beta^2}{4}} = k_a$ (U_s the velocity in the stratosphere) the solution of equation (2.3) in the stratosphere is a sine and a cosine function so that both solutions satisfy the boundary condition at infinity. Thus we see that when $k < k_a$, a resonance wave will exist for all wave numbers, in accordance with our previous result. All these waves will create a resultant motion which decays quite rapidly downstream. If discrete resonance waves exist, k must be greater than k_a. The solution of equation (2.3) in the stratosphere may then be written

$$\omega = e^{-\lambda z} \tag{2.4}$$

where

$$\lambda = \sqrt{k^2 - k_a^2}. \tag{2.5}$$

If the factor $\frac{S}{U^2} - \varkappa^2$ in equation (2.3) is negative also in the troposphere, the curvature will have the same sign for all z and ω cannot be zero on the ground. In order to get a free wave $\frac{d^2\omega}{dz^2}$ must change sign and therefore $\frac{S}{U^2}$ must be greater below the tropopause than above. A possible way of satisfying this requirement is that the stability is greater in the troposphere than in the stratosphere. However, this is not in agreement with observations. The other possibility is that U is sufficient small below the tropopause, i. e. increases with height in the troposphere, and this is also what is usually observed. We arrive at the following conclusion: Since the stability is greater in the stratosphere than in the troposphere, a positive wind shear is necessary in the troposphere in order to get lee waves. If the stability and wind shear in the troposphere are small, no lee waves will exist. If these two parameters are large enough, at least one lee wave will exist with a wave-length less than

(2.6) $$L_a = \frac{2\pi}{k_a} = 2\pi \left(SU_s^{-2} - \varkappa^2\right)^{-\frac{1}{2}}.$$

We notice that the smaller the wind shear and the stability, the closer will the actual wave length be to L_a.

In order to obtain more quantitative results we will discuss this model mathematically. Equation (2.3) may in the troposphere be written.

(2.7) $$\frac{d^2\omega}{dz^2} + \left(\frac{S}{a^2z^2} - \varkappa^2\right)\omega = 0,$$

choosing origin at the level where $U = 0$ and with a denoting the wind shear. Introducing

(2.8) $$\bar{\omega} = z^{-1/2}\omega,$$

and

(2.9) $$\nu^2 = \frac{S}{a^2} - \frac{1}{4},$$

we obtain

(2.10) $$\frac{d^2\bar{\omega}}{dz^2} + \frac{1}{z}\frac{d\bar{\omega}}{dz} + \left[\frac{\nu^2}{z^2} - \varkappa^2\right]\bar{\omega} = 0.$$

Equation (2.10) is a Bessel equation having as solutions Bessel functions of imaginary order and imaginary argument. The two real solutions may be called $F_\nu(\varkappa z)$ and $G_\nu(\varkappa z)$ (discussed and tabulated in [13]), so that in the troposphere

(2.11) $$\omega = c_1 z^{\frac{1}{2}} F_\nu(\varkappa z) + c_2 z^{\frac{1}{2}} G_\nu(\varkappa z)$$

with c_1 and c_2 denoting arbitrary constants. In the stratosphere the solution is

(2.12) $$\omega = c_3 e^{-\lambda z} \qquad k > k_a$$

where c_3 is an arbitrary constant and λ is defined by equation (2.5).

At the tropopause the pressure and the vertical velocity must be the same on both sides. This leads to the conditions

$$\Delta \omega = 0$$

(2.13) $\qquad z = h$

$$U_s \, \Delta \frac{d\omega}{dz} - \omega \alpha = 0$$

where Δ denotes the difference between the value just below and just above the tropopause, and $z = h$ is the level of the tropopause. These equations together with the boundary condition at the ground ($z = h_o$) give the following equation for the wave number of the resonance wave

(2.14)

$$G_\nu (\varkappa h_o) \left[\left(\lambda h - \frac{1}{2} \right) F_\nu (\varkappa h) + h F_\nu{}' (\varkappa h) \right] = F_\nu (\varkappa h_o) \left[\left(\lambda h - \frac{1}{2} \right) G_\nu (\varkappa h) + h G_\nu{}' (\varkappa h) \right].$$

Here $F_\nu{}' (\varkappa h)$ and $G_\nu{}' (\varkappa h)$ are the derivatives of $F_\nu (\varkappa h)$ and $G_\nu (\varkappa h)$ with respect to the argument. Equation (2.14) may be solved graphically. This has been done for a selection of values of the parameters γ, U_s and U_M, where U_M denotes the velocity just above the mountain top level. It is assumed that the stratosphere is isothermal, that the distance from the top of the mountain to the tropopause is 7 km and that the top of the mountain is 1 km above the ground ($z = h_o$). The following results are then obtained:

	Values of the parameters:	Results:
(1)	$\gamma = 0.5 \; 10^{-2} \, °C \, m^{-1}$ $U_s = 50$ m sec^{-1} $U_M = 15$ m sec^{-1}	$L_a = 14.7$ km $L_r = 13.3$ km
(2)	$\gamma = 0.5 \; 10^{-2} \, °C$ m^{-1} $U_s = 40$ m sec^{-1} $U_M = 15$ m sec^{-1}	$L_a = 11.8$ km $L_r = 11.5$ km
(3)	$\gamma = 0.5 \; 10^{-2} \, °C$ m^{-1} $U_s = 35$ m sec^{-1} $U_M = 15$ m sec^{-1}	$L_a = 10.3$ km L_r does not exist
(4)	$\gamma = 0.7 \; 10^{-2} \, °C$ m^{-1} $U_s = 50$ m sec^{-1} $U_M = 15$ m sec^{-1}	$L_a = 14.7$ km $L_r = 14.7$ km
(5)	$\gamma = 0.7 \; 10^{-2} \, °C$ m^{-1} $U_s = 35$ m sec^{-1} $U_M = 10$ m sec^{-1}	$L_a = 10.3$ km $L_r = 10.0$ km

(6) $\gamma = 0.7\ 10^{-2}\ °C\,\text{m}^{-1}$ $\quad L = 8.8$ km
$U_M = 10$ m sec^{-1} $\quad L_r = 8.8$ km.
$U_s = 30$ m sec^{-1}

Here L_r denotes the wave length of the resonance wave, (the lee wave). These results are in agreement with the qualitative results obtained above. We notice that in case 3 no resonance wave exists because the shear is too small. In case 4 and 6, $L_a \approx L_r$, indicating that a slight decrease in the stability or the wind shear will result in no lee wave.

The result may be compared with the observations made behind Sierra Nevada on (a) December 18 1951 [10] and (b) January 30 1952 (see Fig. 6). In the first case, (a), the observed wind and stability were the same as those chosen in case (4) above, and the observed wave length was about 15 km. In the second case, (b), the observed wind and stability were those chosen in case (6), and the observed wave length was about 8 km.

In both cases 4 and 6, for which comparison with observations are available, the agreement is very good — so good that it must be accidental. It should be pointed out that in both cases a slight decrease in the stability will result in disappearance of the lee wave. For example, a choice of $\gamma = 0{,}68\ 10^{-2}\ °C\,\text{m.}^{-1}$ gives no resonance wave in either. The observations show that when lee waves occur, the stability usually cannot be considered as constant in the troposphere. The typical situation is characterized by a thin layer of strong stability in the middle, bounded by layers of weak stability above and below. We therefore are led to consider a model with variations in the stability below the tropopause. It should perhaps be pointed out, however, that a layer with strong stability is by no means necessary for lee waves to occur.

3. A model with constant wind shear and layers with constant stability in the troposphere, and constant wind and stability in the stratosphere. — The model we are going to discuss in this section has the following features: The wind and the temperature are constant in the stratosphere, respectively 50 m sec^{-1} and 220° C. The height of the tropopause above the ground is 8 km. The wind shear is constant in the troposphere so that $U_M = 15$ m sec^{-1}. The troposphere is divided into three layers with constant stability. The lowest layer (1) is assumed to have a depth of 1.5 km and a lapse rate of $\gamma = 0.8\ 10^{-2}\ °C\,\text{m}^{-1}$. The second layer (2) has a depth of 0.5 km and a temperature increase of $0.5\ 10^{-2}\ °C\,\text{m}^{-1}$, being an inversion layer. The third layer (3) has a depth of 6 km and $\gamma = 0.78\ 10^{-2}\ °C\,\text{m}^{-1}$. The values of the temperature gradients give an average of $\gamma = 0.7\ 10^{-2}\ °C\,\text{m}^{-1}$.

In each layer equation (2.10) is valid when ν is given the proper value. In the stratosphere will the solutions have the form (1.11). Applying the boundary conditions at the ground, at the tropopause and at infinity, we end up with the following equation determining the wave number of the resonance wave:

$$\text{(3.1)}\qquad \left[\delta G_3(\varkappa h_2) + \frac{\psi}{\varphi} F_3(\varkappa h_2)\right]\left[G'_2(\varkappa h_2) - \frac{M}{N} F'_2(\varkappa h_2)\right] =$$
$$\left[G_2(\varkappa h_2) - \frac{M}{N} F_2(\varkappa h_2)\right]\left[\delta G'_3(\varkappa h_2) + \frac{\psi}{\varphi} F'_3(\varkappa h_2)\right].$$

Here G_n and F_n are the two independent solutions of equation (2.10) in layer n. Furthermore,

$$\delta = \frac{h_3^{-\frac{1}{2}} e^{-\lambda h_3}}{G_3(\varkappa h_3)} - \frac{\psi F_3(\varkappa h_3)}{\varphi G_3(\varkappa h_3)},$$

$$\psi = h_3^{-\frac{1}{2}}\left[0.5 - h_3\lambda - \frac{h_3 \varkappa G'_3(\varkappa h_3)}{G_3(\varkappa h_3)}\right] e^{-\lambda h_3},$$

$$\varphi = \varkappa h_3\left[F'_3(\varkappa h_3) - \frac{F_3(\varkappa h_3)}{G_3(\varkappa h_3)} G'_3(\varkappa h_3)\right],$$

$$M = \frac{\varkappa h_1 G_2(\varkappa h_1)}{\mu}\left[G'_1(\varkappa h_1) - \frac{G_1(\varkappa h_o)}{F_1(\varkappa h_o)} F'_1(\varkappa h_1)\right] - \varkappa h_1 G'_2(\varkappa h_1),$$

$$N = \frac{\varkappa h_1 F_2(\varkappa h_1)}{\mu}\left[G'_1(\varkappa h_1) - \frac{G_1(\varkappa h_o)}{F_1(\varkappa h_o)} F'_1(\varkappa h_1)\right] - \varkappa h_1 F'_2(\varkappa h_1),$$

$$\mu = G_1(\varkappa h_1) - \frac{G_1(\varkappa h_o)}{F_1(\varkappa h_o)} F_1(\varkappa h_1),$$

and

$$h_0 = \frac{U_o}{a}$$

$$h_1 = h_o + 1.5 \text{ km}$$
$$h_2 = h_o + 2 \text{ km}$$
$$h_3 = h_o + 8 \text{ km}$$

This equation which can be solved graphically leads to a wave length $L_r = 13.3$ km. By comparing this result with the result obtained in Section 2, we notice that the introduction of the stability layers has had the same effect as increasing the stability in the troposphere. As mentioned earlier, the observed wave length corresponding to the model under discussion is about 15 km. The difference between the observed and computed wave length is not greater than what should be expected due to the difference between the actual atmosphere and the model and due to the looseness in the observations.

If in this model $U_s = 35$ m sec^{-1} and $U_M = 15$ m sec^{-1}, no lee wave will appear.

4. A model with constant wind shear and layers with constant stability in the troposphere, and constant wind shears and stability in the statosphere. In the preceeding we have assumed that the wind is constant in the stratosphere. Observations, however, show that often a better approximation can be made by assuming that the wind decreases linearly above the tropopause and finally becomes easterly. In this section we will discuss a model having this property. Aside from the wind gradient in the stratosphere, which is chosen as $\alpha^* = -2{,}46\ 10^{-3}\ \mathrm{sec}^{-1}$ the model is the same as that discussed in Section 4. Equation (2.10) will now be the equation for ω in all four layers if z for the stratosphere is interpreted as the vertical coordinate with origin at a height $h_4 = \dfrac{U_T}{\alpha^*} = 20$ km above the tropopause and U_0 denotes the wind at the tropopause. Equation (2.10) will have a singularity at origin, giving «Cats eyes» at that level. This phenomenon will not be discussed here.*

By applying the boundary conditions we end up with the following equation to determine the wave number for the resonance wave

$$\left[\delta^* G_3(\varkappa h_2) + \frac{\psi^*}{\varphi} F_3(\varkappa h_2)\right]\left[G'_2(\varkappa h_2) - \frac{M}{N} F_2'(\varkappa h_2)\right] = \left[G_2(\varkappa h_2) - \frac{M}{N} F_2(\varkappa h_2)\right]\left[\delta^* G'_3(\varkappa h_2) + \frac{\psi^*}{\varphi} F'_3(\varkappa h_2)\right]. \tag{4.1}$$

Here

$$\psi^* = 0.5\, G_4(\varkappa h_4)\left[3\left(\frac{h_3}{h_4}\right)^{\frac{1}{2}} + \left(\frac{h_3}{h_4}\right)^{\frac{1}{2}} - \frac{2\varkappa h_3 G'_3(\varkappa h_3)}{G_3(\varkappa h_3)}\left(\frac{h_4}{h_3}\right)^{\frac{1}{2}} + \left(\frac{h_3}{h_4}\right)^{\frac{1}{2}} \varkappa h_4 G_4(\varkappa h_4)\right]$$

and

$$\delta^* = \frac{G_4(\varkappa h_4)}{G_3(\varkappa h_3)}\left(\frac{h_4}{h_3}\right)^{\frac{1}{2}} - \frac{F_3(\varkappa h_3)\psi^*}{G_3(\varkappa h_3)\,\varphi}.$$

The other quantities are defined in connection with equation (3.1). Equation (4.1) is solved graphically, and two solutions are found giving (1) $k_r = 0.475\ 10^{-3}\mathrm{m}^{-1}$ and (2) $k_r = 0.315\ 10^{-3}\mathrm{m}^{-1}$. The corresponding wave lengths are (1) $L_r = 13.2$ km and (2) $L_r = 20$ km. We notice that the first wave length is the same as the wave length found in Section 3. The second wave is due to the shear in the stratosphere.

It would have been of interest to discuss the effect to a change in the wind shear in the stratosphere. This requires a considerable amount of work and is not done here. It seems likely that the long wave will have a small amplitude in the lower part of the atmosphere when the shear in the stratosphere is small and the tropopause is found rather high up. When the wind shear is great in the stratosphere and the tropopause low it may be possible that the longest wave is also dominant in the lower part of the atmosphere.

: *Added during the proof reading.* Stictly speaking, the solution in layer 4 is a combination of G_4 and F_4. The formulas for the resonance waves and the amplitude in layers 1, 2, 3, are, however, approximately independent of the form of the solution in layer 4.

The velocity and pressure field. In the rest of the paper we are concerned with the velocity and pressure fields behind the mountain in the model under discussion. Since we want to compare the result obtained by this model with the observations in the Sierra Nevada region, we have chosen a moutanin profile which is an approximation of the profile of Sierra Nevada. It is assumed that the mountain profile given by

$$\zeta = -\zeta_0 \frac{2}{\pi} \operatorname{arctg} \frac{x}{a}\,, \tag{4.2}$$

where a is chosen so as to fit the real mountain reasonably, should have approximately the right effect since, usually, a layer of very stable air exists on the windward side (see Fig. 1). a is chosen as 1 km which corresponds to $\zeta = -0.9\zeta_0$ when $x = 6.3$ km.

The mountain will give rise to a vertical velocity w_0 at the mountain, given by

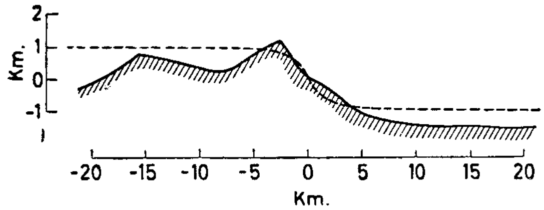

Fig. 1. Hyperbolic tangent approximation.

$$w_o = U_o \frac{d\zeta}{dx} = -\frac{2U_o}{\pi}\,\frac{a\zeta_o}{a^2 + x^2} = -\frac{2}{\pi}\,U_o\zeta_o \int_0^\infty e^{-ka}\cos kx\, dk. \tag{4.3}$$

Due to the linearization we have

$$w = w_o \qquad \text{at } z = h_o. \tag{4.4}$$

If

$$w_o = \gamma \cos kx, \tag{4.5}$$

the w field will be of the form

$$w = \frac{\gamma z^{\frac{1}{2}}}{\Delta\,(k)}\, e^{\frac{1}{2}\beta z} \cos kx\, [f_n(k)F_n(\varkappa z) + g_n(k)G_n(\varkappa z)], \tag{4,6}$$

where index n refers to layer n, and $\Delta\,(k)$ is the same in all layers. A mountain defined by equation (4.2) will then give

$$w = -\frac{2}{\pi}\,U_o\zeta_o e^{\frac{1}{2}\beta z}\, z^{\frac{1}{2}} \int_0^\infty \frac{e^{-ka}}{\Delta\,(k)}\,[f_n(k)F_n(\varkappa z) + g_n(k)G_n(\varkappa z)]\,\cos\, kx\, dk \tag{4.7}$$

or

$$w = -\frac{U_o\,\zeta_o}{\pi}\, e^{\frac{1}{2}\beta z} z^{\frac{1}{2}} \int_0^\infty \frac{e^{-ka}}{\Delta\,(k)}\,[f_n(k)F_n(\varkappa z) + g_n(k)\,G_n(\varkappa z)]\, e^{ikx}dk\, - \tag{4.8}$$

$$\frac{U_o\,\zeta_o}{\pi}\, e^{\frac{1}{2}\beta z} z^{\frac{1}{2}} \int_0^\infty \frac{e^{-ka}}{\Delta\,(k)}\,[f_n(k)\,F_n(\varkappa z) + g_n(k)\,G_n(\varkappa z)]\, e^{-ikx}dk.$$

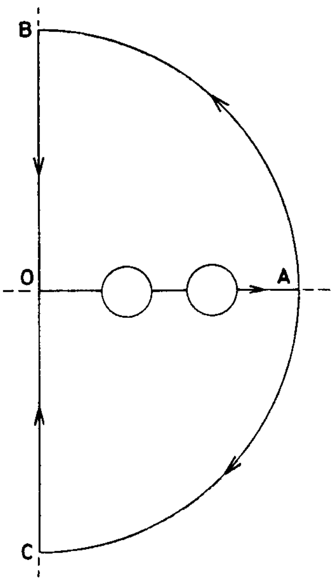

Fig. 2. Path of integration in the complex k-plane.

If a resonance wave exists, $\triangle(k)$ must be zero for a real value of k. In the actual case two resonance waves exist so that $\triangle(k)$ has two zero points on the real axis. As discussed in the introduction these singularities are introduced by the assumption of stationary motion. The solution giving waves on the lee side only is obtained if in the first integral the singularities are by-passed along the half circle so that the singular points are on the left hand side, and, in the second integral, along the half circle having the singular points on the right hand side (see Fig. 2). To evaluate the integrals we transform them by means of Cauchy's theorem. For $x > 0$ the first integral in equation (4.8) is equal to the integral along OB plus the residue terms when $R \to \infty$ (R is the radius in the half circle) since the contribution along the arc AB vanishes. For $x > 0$ the first integral is equal to the integral along OC when $R \to \infty$. The second integral is for $x < 0$ equal to the integral along OC plus the residue terms and for $x < 0$ equal to the integral along OB, again when $R \to \infty$.

Writing

$$w = w_r + w_p \tag{4.9}$$

where w_r is the contribution due to the residue terms (resonance waves), we get

$$w_r = -4U_o\,\zeta_o e^{\frac{1}{2}\beta z} z^{1/2} \sum_{k=k_1}^{k_2} e^{-ka} \sin kx \left[\frac{f_n(k)}{\triangle'(k)} F_n(\varkappa z) + \frac{g_n(k)}{\triangle'(k)} G_n(\varkappa z) \right] \tag{4.10}$$

when $x > 0$, and

$$w_r = 0 \tag{4.11}$$

when $x < 0$. Here k_1 and k_2 denote the wave numbers for the resonance waves. w_p is symmetrical with respect to x and is, after a transformation, for positive x given by

$$w_p = \frac{2U_o\zeta_o}{\pi} e^{\frac{1}{2}\beta z} z^{\frac{1}{2}} \int_0^\infty \frac{1}{\triangle(ik)} [f_n(ik)F_n(\varkappa^* z) + g_n(ik)G_n(\varkappa^* z)] \sin ka\, e^{-kx}\, dk. \tag{4.12}$$

Here $\varkappa^*$ is defined by

$$\varkappa^* = \sqrt{\frac{\beta^2}{4} - k^2}.$$

To compute in the actual case the effect of the resonance waves, w_r, we have to find $f_n(k)$, $g_n(k)$ and $\triangle'(k)$ ($k = k_1, k_2$). This is done by solving equation (2.10) in the four layers and applying the proper boundary conditions. It is found that

(4.13) $$\triangle(k) = H_o' K_o - K_o' H_o,$$

where

$$H_o = \delta^* G_3(\varkappa h_2) + \frac{\psi^*}{\varphi} F_3(\varkappa h_2)$$

$$K_o = G_2(\varkappa h_2) - \frac{M}{N} F_2(\varkappa h_2)$$

$$H_o' = \left[\delta^* G_3'(\varkappa h_2) + \frac{\psi^*}{\varphi} F'_3(\varkappa h_2)\right]$$

$$K_o' = \left[G_2'(\varkappa h_2) - \frac{M}{N} F_2'(\varkappa h_2)\right].$$

Furthermore, $f_n(k)$ and $g_n(k)$ are given by:

(4.14) $$f_4(k) = 0$$

(4.15) $$g_4(k) = [K_o F'_2(\varkappa h_2) - F_2(\varkappa h_2) K_o'] \frac{E}{N}$$

(4.16) $$f_3(k) = \frac{\psi^*}{\varphi} g_4(k)$$

(4.17) $$g_3(k) = \delta^* g_4(k)$$

(4.18) $$f_2(k) = -\frac{M}{N} g_2(k)$$

(4.19) $$g_2(k) = \frac{H_o}{K_o} g_4(k)$$

(4.20) $$f_1(k) = -\frac{G_1(\varkappa h_o)}{F_1(\varkappa h_o)} g_1(k)$$

(4.21) $$g_1(k) = \frac{F_2(\varkappa h_1)}{\mu} f_2(k) + \frac{G_2(\varkappa h_1)}{\mu} g_2(k).$$

Here

$$E = \varepsilon \varkappa h_1 \left[G_1'(\varkappa h_1) - \frac{G_1(\varkappa h_o)}{F_1(\varkappa h_o)} F_1'(\varkappa h_1)\right] - \frac{\gamma F'_1(\varkappa h_1) \varkappa h_1}{h_0^{\frac{1}{2}} F_1(\varkappa h_o)}$$

with

$$\varepsilon = \frac{\gamma F_1(\varkappa h_1)}{h_o^{\frac{1}{2}} \mu F_1(\varkappa h_o)}.$$

The other quantities are defined earlier. k is the wave number of the resonance wave (k_1, k_2).

The amplitudes of the two resonance waves in question are found by putting $k = 0{,}315 \; 10^{-3}$ m $^{-1}$ and $k = 0{,}475 \; 10^{-3}$m $^{-1}$. $f_n(k)$ and $g_n(k)$ are found directly

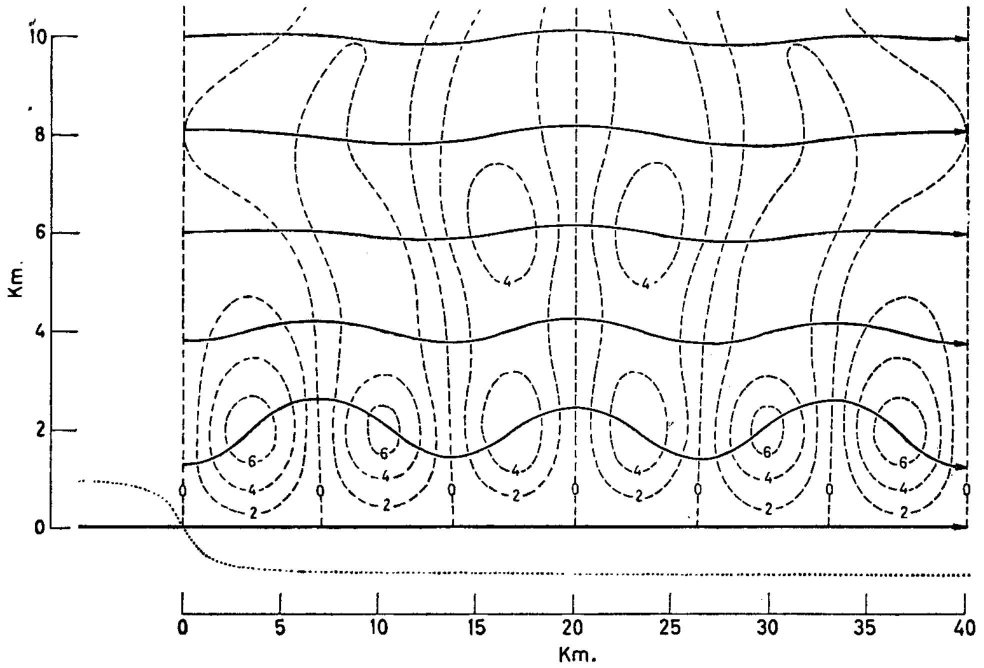

Fig. 3. The vertical velocity field due to resonance waves and the resultant streamline field (Units of w: m sec^{-1}).

from the formulae above. $\triangle' (k_r)$ is taken from the curve for $\triangle (k)$. The field of w_r together with the streamlines are shown in Fig. 3.

To get the complete field of w we have to add the field of w_p. This is not done here since no simple way of getting this field has been found. However, some information of w_p may be obtained without any calculation. In the first place, formula (4.12) shows that w_p decreases rapidly with increasing x. Furthermore, since w_r is zero when $z = h_0$, one of the streamlines due to the field of w_p will coincide with the profile of the mountain. The field of w_p will have a discontinuity at $x = 0$ so that the complete field shall be continuous. The main effect of the field of w_p therefore is to take away the discontinuity at $x = 0$ and to give a down-draft region behind the mountain.

The values for the stability and wind distribution introduced in this model are chosen in agreement with the observations in the Sierra Nevada region December 18 1951. On that day a pronounced lee wave was observed. It was thought that it would be of interest to ascertain if models like those discussed above would give a wave solution, and what the velocity and pressure field of this wave would look like. The observations indicate a wave length of about 15 km in the lower troposphere and also indicate that the maximum wave amplitude is just above the mountain and that the amplitude decreases upwards. The observed data are, however, too few to be used in

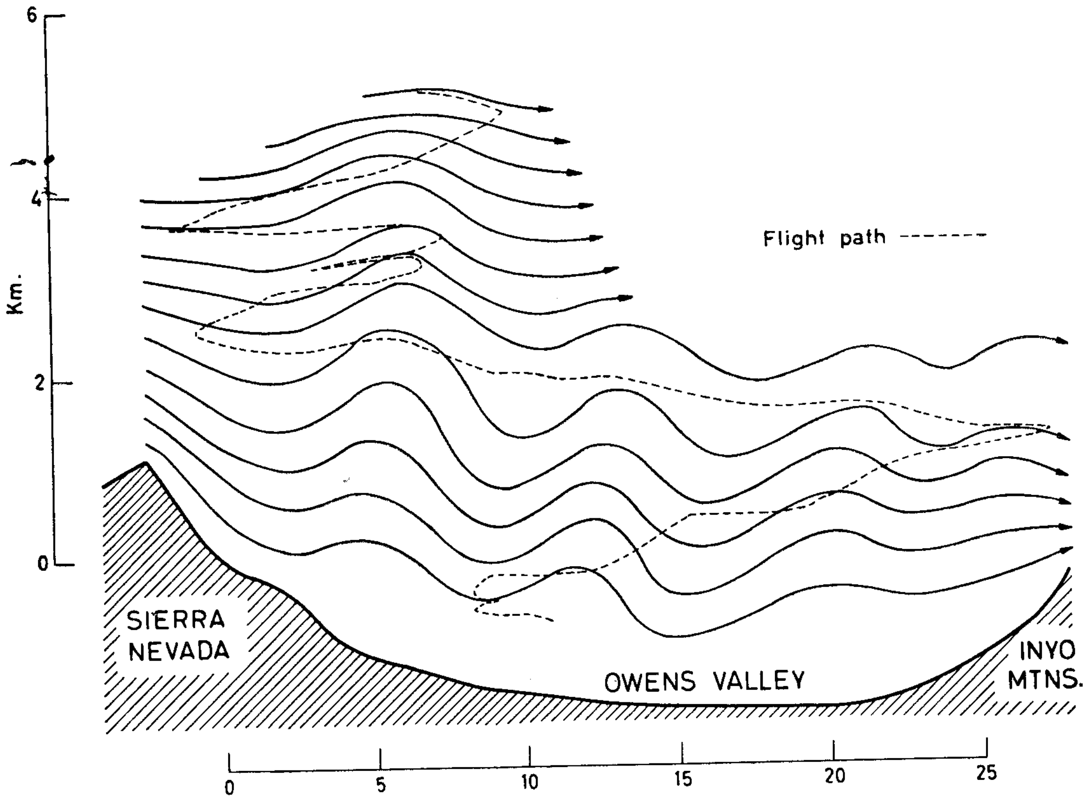

Fig. 4. Observed streamlines behind Sierra Nevada, January 30 1952. Mountain Wave Project, Department of Meteorology, U.C.L.A., Los Angeles.

an objective drawing of the velocity field and a comparison with Fig. 3. Recently, the observations made on January 30 1952 at Sierra Nevada have been analyzed. As mentioned earlier, a wave length of about 8 km was observed that day. A number of data was obtained so that it has been possible to construct a reliable streamline field (Fig. 4). The stability distribution in the troposphere is similar to that observed December 18 1951. The mean temperature gradient is the same, and the observed stability distribution can be approximated by a 3-layer model. The wind velocity in the troposphere can be approximated by a linear function of height with $U_M = 10\ \mathrm{m\,sec}^{-1}$ and $U_s = 30\ \mathrm{m\ sec}^{-1}$. Comparing Figs. 3 and 4 the calculated wave length should therefore be replaced by a shorter one (see case 6 Section 2). It should also be remembered that the field of w_p should be added to the computed field of w_r, giving a downdraft behind the mountain in the lower part of the troposphere.

The pressure field is found from the equation of motion. The result is shown in Fig. 5. It is noticed that the maximum pressure deviation is about 1.2 mb.

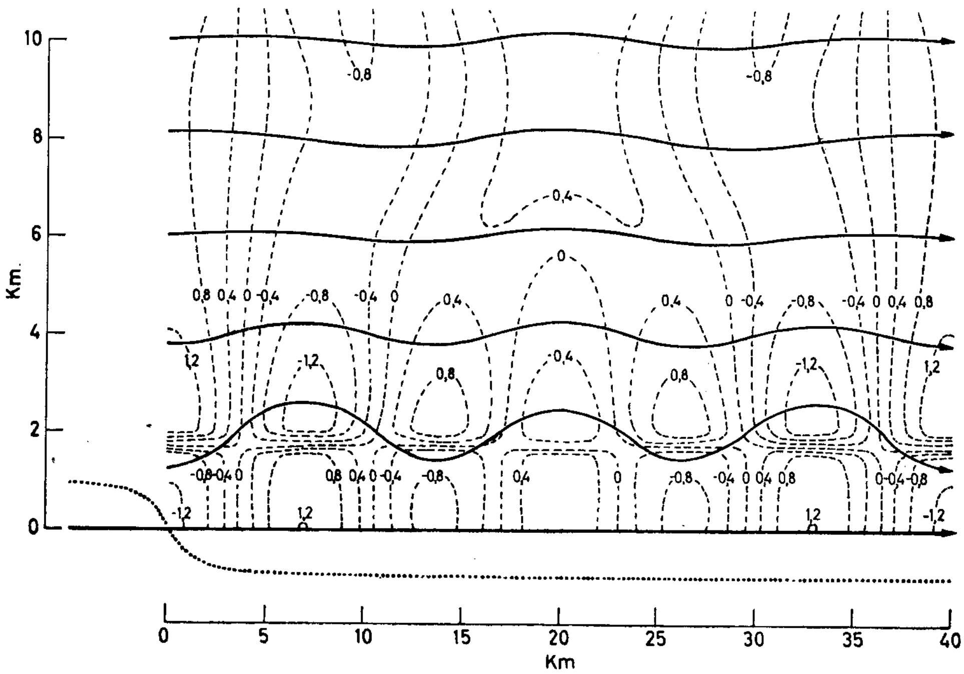

Fig. 5. Pressure field due to resonance waves (in mbs).

Remarks. The results obtained above have been obtained with equations which have been linearized. It is assumed that this approximation, as is generally believed, is good in a stationary model, although at some points near the mountain it breaks down.

Often a cloud is located at the first positive maximum of the vertical displacement. The cloud is sometimes in the form termed a roll cloud (see [10]). The existence of the cloud is in agreement with the lee wave theory. The turbulent character of the cloud may be due to a kind of instability or a non-linear effect.

THREE-DIMENSIONAL MOUNTAIN WAVES

5. The model atmosphere. — In this chapter we are concerned with three-dimensional mountain waves generated by a mountain peak. Waves created by a three-dimensional obstacle have been discussed by the author of this paper in the case of surface waves [14] and recently by Scorer and Wilkinson [15] and Wurtele (unpublished). To bring into account the effect of the stability and wind shear we choose here a model similar to the one studied in Section 2 of Chapter I.

It will be assumed that the stability and basic velocity is constant in the stratosphere, and that in the troposphere the stability is constant and the basic velocity increases linearly with height. In the example given later the following values of the parameters are chosen:

Stratosphere.

$\gamma = 0$
$\bar{T} = 220^0 K$
$U_s = 45 \text{ m sec}^{-1}$

$$S = \frac{g(\gamma_d - \gamma)}{\bar{T}} = 4.5\ 10^{-4} \text{ sec}^{-2}$$

Troposphere.

$\gamma = 0.5\ 10^{-2}\ {}^{\circ}\text{C m}^{-1}$
$\bar{T} = 250^{\circ} K$

$$\alpha = \frac{45-15}{9}\ 10^{-3} \text{ sec}^{-1} = 3.33\ 10^{-3} \text{ sec}^{-1}$$

$$S = \frac{g(\gamma_d - \gamma)}{\bar{T}} = 2.0\ 10^{-4} \text{ sec}^{-2}.$$

Here the symbols have the same meaning as in Chapter I, i. e. γ denotes the actual lapse rate, $\bar{T}$ the mean temperature, U_s the basic velocity in the stratosphere, α the wind shear, S the stability and γ_d the dry adiabatic lapse rate. As in Chapter I the equations are linearized and assumed to be independent of time. The motion will be described in a system of references with the z-axis positive upwards and the x-axis in the direction of the basic velocity.

The mountain profile. For simplicity the mountain is assumed to have rotational symmetry about the z-axis. The equation which describes the profile is chosen as

(5.1)
$$\zeta = \frac{\zeta_o}{\left(1 + \frac{r^2}{a^2}\right)^{\frac{1}{2}}}$$

where a and ζ_0 are given constants, and

(5.2)
$$r^2 = x^2 + y^2.$$

Formula (1) may also be written

(5.3)
$$\zeta = \frac{\zeta_o}{\left(1 + \frac{r^2}{a^2}\right)^{\frac{1}{2}}} = a\zeta_o \int_0^{\infty} e^{-ka} J_0(kr)\, dk = \frac{a\zeta_0}{2\pi} \int_0^{\infty} e^{-ka}\, dk \int_{-\pi}^{+\pi} e^{ik(x\cos\varphi + y\sin\varphi)}\, d\varphi,$$

where $J_0(kr)$ is a Bessel function of order zero. In an example given later ζ_0 and a are chosen so that $\zeta_0 = 2a = 0.5$ km ($r \approx \zeta_0$ gives $\zeta \approx \frac{1}{2}\zeta_0$).

6. The differential equation. The density in the undisturbed state, Q, may with good approximation be assumed to be

$$Q = Q_o e^{-\beta z} \tag{6.1}$$

where Q_0 and β are constants. The basic velocity in the troposphere, U, takes by a suitable choice of origin the form

$$U = az. \tag{6.2}$$

Intending to apply the Fourier method, the vertical velocity, w, is written

$$w = e^{\frac{1}{2}\beta z}\omega(z)e^{ik(x\cos\varphi + y\sin\varphi)}. \tag{6.3}$$

Applying (6.2) and making the same assumptions as we did to derive (2.3), we find

$$\frac{d^2\omega}{d^2z} + \left(\frac{S}{z^2a^2\cos^2\varphi} - \varkappa^2\right)\omega = 0. \tag{6.4}$$

As mentioned in Chapter I the solution of (6.4) is

$$\omega = A\, z^{\frac{1}{2}} F_\nu(\varkappa z) + B\, z^{\frac{1}{2}} G_\nu(\varkappa z). \tag{6.5}$$

Here A and B are arbitrary constants,

$$\nu = \frac{S}{a^2\cos^2\varphi} - \frac{1}{4}, \tag{6.6}$$

and $F_\nu(\varkappa z)$ and $G_\nu(\varkappa z)$ Bessel functions of imaginary argument and order.

In the stratosphere the differential equation is

$$\frac{d^2\omega}{dz^2} + \left(\frac{S}{U_s^2\cos^2\varphi} - \varkappa^2\right)\omega = 0. \tag{6.7}$$

The solution of this equation will be of exponential or trigonometric character according as

$$k^2 \lessgtr \frac{S}{U_s^2\cos^2\varphi} - \frac{\beta^2}{4} = k_a^2.$$

As will be soon demonstrated (compare Section 1 in Chapter I) the k domain corresponding to trigonometric solutions of (6.7) does not make any essential contribution to the complete Fourier integral and will therefore be cancelled. The solution of (6.7) is then

$$\omega = Ce^{-\lambda z} + De^{\lambda z} \tag{6.8}$$

where λ is real and determined by

$$\lambda^2 = \varkappa^2 - \frac{S}{U_s^2\cos^2\varphi},$$

and C and D are arbitrary constants.

The solution of the problem. The boundary conditions are:

1. At the ground ($z = h_o$): $w = U_o \zeta_x$

where

$$U_o = U(h_o)$$

2. At the tropopause ($z = h$): The vertical velocity and the pressure are continuous functions.

3. At the upper boundaıy ($z \to \infty$): The energy due to the wave motion must disappear.

From these three conditions we are able to find A and B. They are found to be

$$A = \frac{\eta h_o^{-\frac{1}{2}} \left[\left(\lambda h - \frac{1}{2} \right) G_\nu \ (\varkappa h) + \varkappa h G'_\nu \ (\varkappa h) \right]}{F_\nu \ (\varkappa h_o) \left[\left(\lambda h - \frac{1}{2} \right) G_\nu \ (\varkappa h) + \varkappa h G'_\nu \ (\varkappa h) \right] - G_\nu \ (\varkappa h_o) \left[\left(\lambda h - \frac{1}{2} \right) F_\nu \ (\varkappa h) + \varkappa h F_\nu' (\varkappa h) \right]}$$

(6.9)

$$B = - \frac{\eta h_o^{-\frac{1}{2}} \left[\left(\lambda h - \frac{1}{2} \right) F_\nu \ (\varkappa h) + \varkappa h F_\nu' \ (\varkappa h)] \right]}{F_\nu \ (\varkappa h_o) \left[\left(\lambda h - \frac{1}{2} \right) G_\nu (\varkappa h) + \varkappa h G'_\nu \ (\varkappa h) \right] - G_\nu \ (\varkappa h_o) \left[\left(\lambda h - \frac{1}{2} \right) F_\nu \ (\varkappa h) + \varkappa h F'_\nu (\varkappa h) \right.}$$

where

$$\eta = U_o e^{-ka} ik \frac{\zeta_o a}{2\pi} \cos\varphi \ e^{-\frac{1}{2}\beta h_o} .$$

A mountain of form (5.1) will then set up a vertical velocity

$$w = z^{\frac{1}{2}} e^{\frac{1}{2}\beta z} \int_{k_a}^{\infty} dk \int_{-\pi}^{\pi} [A \ (k) \ F_\nu \ (\varkappa h) + B \ (k) \ G_\nu \ (\varkappa z)] e^{ik(x \cos\varphi + y \sin\varphi)} \ d\varphi \tag{6.10}$$

plus a contribution from k values less than k_a. Here A (k) and B (k) are defined by (6.9). The evalution of the integrals of equation (6.10) will be different according to whether A (k) and B (k) have poles or not on the path of integıation. If the function have poles on this path, it follows from Cauchys theorem that the most important contributions to the integral will be due to the poles for moderate values of x $\cos \varphi + y \sin \varphi$. It is easily proved that no poles can exist for k values less than k_a. We therefore cancel the contributions for $k < k_a$, this approximation being better for increasing values of $x \cos \varphi + y \sin \varphi$. Changing the oıder of integration we obtain

$$w = z^{\frac{1}{2}} e^{\frac{1}{2}\beta z} \int_{-\pi}^{\pi} d\varphi \int_{k_a}^{\infty} [A \ (k) \ F_\nu \ (\varkappa z) + B \ (k) \ G_\nu \ (\varkappa z)] \ e^{ik(x \cos\varphi + y \sin\varphi)} \ dk. \tag{6.11}$$

From equation (6.9) it is seen that the k values corresponding to poles on the path of integration are the real solutions of the equation

$$(6.12)\quad F_\nu(\varkappa h_o)\left[\left(\lambda h - \frac{1}{2}\right)G_\nu(\varkappa h) + \varkappa h G_\nu'(\varkappa h)\right] = G_\nu(\varkappa h_o)\left[\left(\lambda h - \frac{1}{2}\right)F_\nu(\varkappa h) + \varkappa h F_\nu'(\varkappa h)\right]$$

This equation is solved numerically with the values of the parameters specified above. It is found that equation (6.12) has for all values of φ at least one real solution, and that the solutions may with good approximation be obtained from

$$(6.13)\quad G_\nu(\varkappa h_o) = 0,$$

and

$$(6.14)\quad k > k_a.$$

Equation (6.13) corresponds to a basic velocity which increases linearly with height throughout the atmosphere.

Strictly speaking, the innermost integral in equation (6.11) does not exist when a pole occurs on the real axis. Physically, this is due to the fact that the problem has been treated as a stationary one without friction. To get a proper integral we can either discuss the problem as an initial value problem or as a stationary one with friction. Let us for simplicity introduce here a small Rayleigh friction [16]. (In the corresponding problem in Chapter I we postulated instead that the waves should be located downstream). It may then be shown that the poles are complex numbers approximately given by

$$(6.15)\quad \varkappa = \frac{\zeta}{h_o} + \frac{\mu i}{h_o\, a \cos\varphi}$$

where $\frac{\zeta}{h_o}$ denotes the corresponding pole without friction, and μ is here the coefficient of viscosity. By now letting μ tend to zero and applying Cauchy's theorem, the integral will approach a limit which is a proper integral. Thus we conclude, owing to (6.15) that the integral

$$\int_{k_a}^{\infty} [A(k)\, F_\nu(\varkappa z) + B(k)\, G_\nu(\varkappa z)]\, e^{ik(x\cos\varphi + y\sin\varphi)}\, dk$$

is to be interpreted as

$$\int_L [A(k)\, F_\nu(\varkappa z) + B(k)\, G_\nu(\varkappa z)]\, e^{ik(x\cos\varphi + y\sin\varphi)}\, dk$$

when $\cos\varphi > 0$, and as

$$\int_{L'} [A(k)\, F_\nu(\varkappa z) + B(k)\, G_\nu(\varkappa z)]\, e^{ik(x\cos\varphi + y\sin\varphi)}\, dk$$

when $\cos\varphi < 0$. Here L and L' denote the paths of integration consisting of the real axis from k_a to ∞ and semicircles around the poles so that these are on the left side

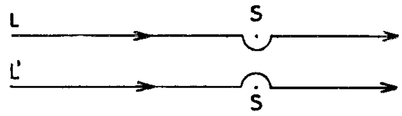

Fig. 6. S denotes singular point.

or on the right side, respectively (see Fig. 6). Formula (6.11) is now rewritten in the form

$$w = z^{\frac{1}{2}} e^{\frac{1}{2}\beta z} \int_{\sigma-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{L} [A\,(k)\,F_{\nu}\,(\varkappa z) + B\,(k)\,G_{\nu}\,(\varkappa z)]\, e^{ik(x\cos\varphi + y\sin\varphi)}\, dk \tag{6.16}$$

$$- z^{\frac{1}{2}} e^{\frac{1}{2}\beta z} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{L'} [A\,(k)\,F_{\nu}\,(\varkappa z) + B\,(k)\,G_{\nu}\,(\varkappa z)]\, e^{-ik(x\cos\varphi + y\sin\varphi)}\, dk.$$

The innermost integrals in this formula are evaluated by applying Cauchy's theorem. For moderate values of $x\cos\varphi + y\sin\varphi$ an approximation is obtained, as mentioned earlier, by taking into account only the contributions from the poles. With this approximation formula (6.16) is (by means of (6.9) and (6.13) found to be

$$w = X \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-ka} k \cos\varphi \left[\left(\lambda h - \frac{1}{2}\right) G_{\nu}\,(\varkappa h) + \varkappa h G_{\nu}'\,(\varkappa h)\right] F_{\nu}\,(\varkappa z) \cos k\,(x\cos\varphi + y\sin\varphi)}{G'_{\nu}(\varkappa h) \left[\left(\lambda h - \frac{1}{2}\right) F_{\nu}\,(\varkappa h) + h F_{\nu}'\,(\varkappa h)\right] \frac{d\varkappa}{dk}}\, d\varphi$$

(6.17)

$$- X \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-ka} k \cos\varphi\, G_{\nu}\,(\varkappa z) \cos k\,(x\cos\varphi + y\sin\varphi)}{G_{\nu}'\,(\varkappa h) \frac{d\varkappa}{dk}}\, d\varphi$$

when

$$x\cos\varphi + y\sin\varphi > 0,$$

and

(6.18) $$w = 0$$

when

$$x\cos\varphi + y\sin\varphi < 0.$$

Here

$$X = 2U_o a z^{\frac{1}{2}} h_o^{-\frac{3}{2}} \zeta_o e^{\frac{\beta}{2}(z-h_o)}$$

and k is a function of φ defined by equation (6.13).

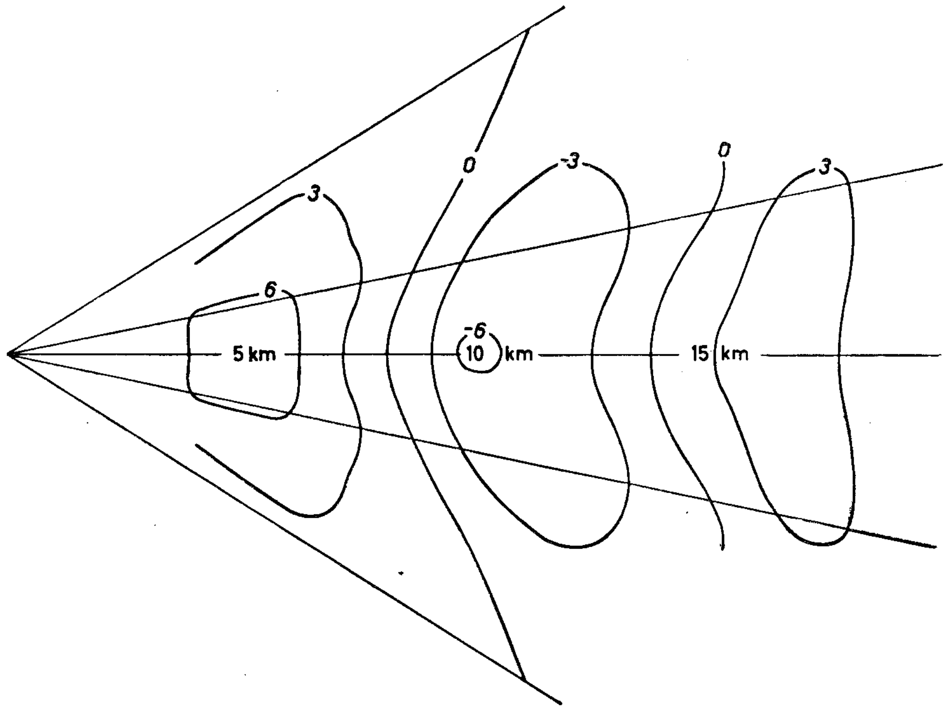

Fig. 7. Isolines for the vertical velocity at a height of 3 km (Units 10^{-1} m sec^{-1}).

The integrals in formula (6.17) may for large values of x and y be evaluated by means of the method of stationary phase. It seems, however, that in the present case this method is only applicable for values of x and y which are considerable greater than those interesting us. The integrals have therefore been computed by a straight forward method, and only for $z = 3$ km. The result of the calculations are shown in Fig. 7 where isolines for the vertical velocity are drawn.

By changing the parameters in the model the vertical velocity may be increased considerably. The vertical velocity is directly proportional to ζ_0 and U_0, and increases also with the steepness of the mountain. It should also be noted that a concentration of the stability acts in the same way.

Acknowledgement. The author is indebted to Professor Holmboe, University of California at Los Angeles, and to Professor Høiland, University of Oslo, for valuable suggestions and discussions. The research reported in Chapter I of this paper has been made possible through support and sponsorship extended by the Geophysical Research Division of the U. S. Air Force Cambridge Research Center. The work was essentielly carried out at the Department of Meteorology, University of California at Los Angeles, during the fall and spring of 1954—1955 and has been distributed as Scientific Report No. 3 under contract No. *A F* 19(604)—728. Chapter II of this investigation has been sponsored by Norwegian Research Council for Science and the Humanities, and has in part been distributed as Report No. 1, 1956, Institute for Weather and Climate Research, Oslo.

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