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OCEAN CURRENT AS AN INITIAL PROBLEM

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Summary. A stationary current in heterogeneous water is accompanied by a definite density distribution in such a manner that the pressure gradient is balanced by the deflecting force of the Earth's rotation.

If this geostrophical equilibrium is not present initially, the current will generate internal oscillations, which will be propagated away from the area where the initial current is present.

The waves will be damped and at last the whole system will tend to a stationary state.

In the present paper we have chosen a special density distribution based on observations and we have considered a few different current systems.

In the first case we consider a rectilinear current system of finite width which decreases with depth and is zero at the bottom. The surfaces of equal density are initially horizontal.

The transition from the non stationary to the stationary state has been studied and the results are given in diagrams. (Fig. 2, 3, 4 and 5).

In the second case we have considered a non balanced vortex, the initial state and the resulting stationary vortex are given in diagrams. (Fig. 6 and 7).

At last we have considered a rectilinear current system in a sea which is bounded on both sides. In this case there will be no stationary state, because the waves are reflected from the sides of the canal.

The oscillations will at first be similar to those, found in the first case, but later the reflected waves will cause deviations.

1. Introduction and basic equations. The adjustment of a non balanced velocity field towards geostrophic equilibrium in a stratified fluid has been treated by B. Bolin. [1]

However, he has restricted himself to the treatment of a special case, when the density is a linear function of depth and, moreover, the limiting process from the non stationary solution to the corresponding stationary system is only an approximate one, and therefore it might be of interest to give the problem a treatment which in some respect is more general.

We assume that the equations of motion may be linearized and that the vertical acceleration may be neglected, and consequently the pressure may be regarded as static.

The Coriolis parameter is also regarded as constant.

The equations of motion are then:

$$(1,1) \qquad \begin{aligned} &\frac{\partial u}{\partial t} - \lambda v + \frac{1}{\varrho}\frac{\partial p}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} + \lambda u + \frac{1}{\varrho}\frac{\partial p}{\partial y} = 0, \\ &g + \frac{1}{\varrho}\frac{\partial p}{\partial z} = 0. \end{aligned}$$

We assume that the fluid may be treated as incompressible, and the equation of continuity is then:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The condition of incompressibility is expressed by the equation:

$$\frac{d\varrho}{dt} = \frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial x} + v\frac{\partial \varrho}{\partial y} + w\frac{\partial \varrho}{\partial z} = 0.$$

To simplify the equations, we put:

$$\varrho = \varrho_0(z) + \varrho_1(x,y,z,t)$$

$$p = p_0 + g\int_z^h \varrho_0\, dz + p_1$$

where ϱ_1 and p_1 are assumed to be small quantities. The condition of incompressibility may then be simplified, and takes the form:

$$\frac{\partial \varrho_1}{\partial t} + w\frac{\partial \varrho_0}{\partial z} = 0.$$

Furthermore we put:

$$w = \frac{d\zeta}{dt}.$$

ζ is then the elevation of a water particle from its equilibrium position. Introducing this in the equation above, it may be integrated with respect to t, giving:

$$\varrho_1 + \zeta\frac{d\varrho_0}{dz} = 0.$$

We put the arbitrary constant equal to zero, which means that the density variation is due to vertical displacements of the water particles only.

Wit these simplifications, the basic equations take the form:

$$(1,2)\qquad \begin{aligned} &\frac{\partial u}{\partial t} - \lambda v + \frac{1}{\varrho_0}\frac{\partial p_1}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} + \lambda u + \frac{1}{\varrho_0}\frac{\partial p_1}{\partial y} = 0, \\ &g\varrho_1 + \frac{\partial p_1}{\partial z} = 0, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial^2 \zeta}{\partial z \partial t} = 0, \\ &\varrho_1 + \zeta \frac{d\varrho_0}{dz} = 0. \end{aligned}$$

Let $w(z)$ be a solution of the differential equation:

$$(1,3)\qquad \frac{d}{dz}\left(\varrho_0 \frac{dw}{dz}\right) - \frac{g}{c^2} w \frac{d\varrho_0}{dz} = 0,$$

with the boundary conditions:

$$(1,4)\qquad \begin{aligned} &w = 0\ ;\ z = 0, \\ &c^2 \frac{dw}{dz} - gw = 0\ ;\ z = h. \end{aligned}$$

The differential equation has an infinite number of solutions corresponding to an infinite set of eigenvalues $c_0, c_1, c_2 \ldots$

It may be of interest to note that $\frac{dw}{dz}$ satisfies an integral equation of the form:

$$(1,5)\qquad \varrho_0 \frac{dw}{dz} = \frac{g}{c^2}\left[\varrho_0 \int_0^z \frac{dw}{dz}\, dz + \int_z^h \varrho_0 \frac{dw}{dz}\, dz\right].$$

This integral equation may also be used when $\varrho_0(z)$ is a discontinuous function of depth.

The integral equation may easily be transformed to an equation with a symmetric kernel.

In fact if we put:

$\varphi = \sqrt{\varrho_0}\,\frac{dw}{dz}$, the integral equation may be written in the form:

$$\varphi(z) = \lambda \int_0^h K(z,s)\,\varphi(s)\,ds,$$

where:

$$\lambda = \frac{g}{c^2},$$

$$K(z,s) = \sqrt{\frac{\varrho_0(z)}{\varrho_0(s)}}, \quad s \leq z$$

$$= \sqrt{\frac{\varrho_0(s)}{\varrho_0(z)}}, \quad s \geq z.$$

The bilinear formula for the kernel is:

$$K(z, s) = \sum_{n=0}^{\infty} \frac{\varphi_n(z)\,\varphi_n(s)}{\lambda_n},$$

or introducing again:

$$\varphi_n(z) = \sqrt{\varrho_0(z)}\,\frac{dw_n}{dz}, \quad \lambda_n = \frac{g}{c^2_n},$$

we find:

$$g\sqrt{\frac{\varrho_0(z)}{\varrho_0(s)}} = \Sigma\, c_n^2 \sqrt{\varrho_0(z)\,\varrho_0(s)}\;\frac{dw_n}{dz}\,\frac{dw_n}{ds}$$

or

$$g = \varrho_0(s)\,\Sigma\, c^2_n \frac{dw_n}{dz}\,\frac{dw_n}{ds} \qquad s \leq z$$

$$= \varrho_0(z)\,\Sigma\, c^2_n \frac{dw_n}{dz}\,\frac{dw_n}{ds} \qquad s \geq z.$$

If we put $s = z$ and integrate between o and h, we get:

(1,6) $$gh = \sum_{n=0}^{\infty} c^2_n.$$

In the application of the theory to a special case which will be given below we choose a density distribution which was observed at one of the "Snellius" anchor stations 135 a, because the necessary calculation of the eigenfunctions and eigenvalues had already been performed.

The density distribution (σ_t) and the first order eigenfunctions $w_1\,(z)$ and $u_1\,(z) = c_1 \frac{dw_1}{dz}$ are represented in Fig. 1.

The velocity of propagation of this internal wave was found to be 205,4 cm/sec, while the ordinary or zero order wave has a velocity of propagation of 106,2 m/sec or about 50 times that of the internal wave.

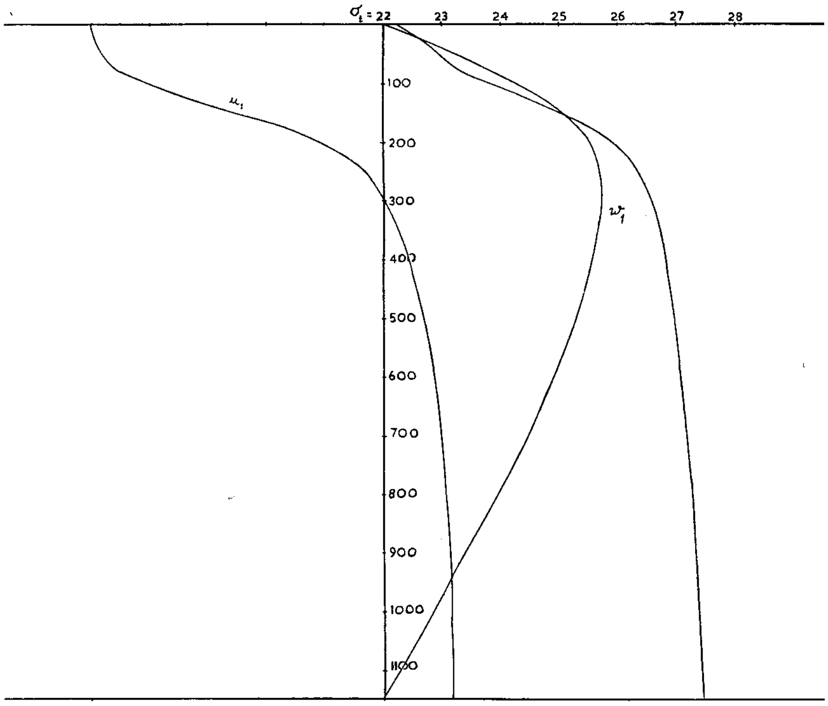

Fig. 1: Density distribution with depth and first order solution of internal wave equation and as functions of depth.

We assume that u, v, ζ, p, and ϱ_1 may be expanded in series of eigenfunctions $w_n(z)$ and $u_1(z)$

$$
\begin{aligned}
u &= \Sigma\, U_n(x,y,t)\, u_n(z),\\
v &= \Sigma\, V_n\, u_n(z),\\
p_1 &= \Sigma\, Z_n\, c_n\, u_n(z),\\
\zeta &= \Sigma\, Z_n\, w_n(z),\\
\varrho_1 &= -\frac{d\varrho_0}{dz}\sum Z_n\, w_n(z).
\end{aligned}
$$

Introducing these series in the basic equations (1,1), we get the following differential equations for U_n, V_n and Z_n:

$$
\begin{aligned}
&\frac{\partial U_n}{\partial t} - \lambda V_n + c_n \frac{dZ_n}{dx} = 0,\\
(1,7)\qquad &\frac{\partial V_n}{\partial t} + \lambda U_n + c_n \frac{\partial Z_n}{\partial y} = 0,\\
&\frac{\partial Z_n}{\partial t} + c_n\left(\frac{\partial U_n}{\partial x} + \frac{\partial V_n}{\partial y}\right) = 0.
\end{aligned}
$$

At the same time the boundary conditions:

$$\zeta = 0 \;;\; z = 0,$$

$$p = p_0 \;;\; z = h,$$

are satisfied.

We now formulate Cauchy's problem for this system of equations.

We want a set of solutions which for $t = o$ gives:

$$u = U_0(x,y,z)$$

$$v = V_0(x,y,z)$$

$$\zeta = Z_0(x,y,z)$$

We assume that it is possible to expand these functions in series of eigenfunctions $u_n(z)$ and $w_n(z)$.

The coefficients of these series may be determined in the Fourier manner, since the eigenfunctions have orthogonal properties:

$$\int_o^h \varrho_0 u_n(z)\, u_m(z)\, dz = 0\,,\; n \neq m$$

$$(\varrho_0 w_n w_m)_h - \int_o^h \frac{d\varrho_0}{dz} w_n w_m dz = 0,\; n \neq m$$

The eigenfunctions may be normalised by the condition:

$$\int_o^h \varrho u_n^2 dz = 1.$$

or a similar condition.

We consequently assume that:

$$U_0(x,y,z) = \Sigma\, U_{n,0}(x,y) u_n(z)$$

and similarly for the other initial values. The coefficient $U_{n,0}$ is determined by the formula:

$$\int_o^h U_0(x,y,z)\, \varrho_0 u_n(z)\, dz = U_{n,0}(z,y).$$

In this manner the problem is reduced to the following one. Determine the solution of the system of partial differential equations (1,7), which for $t = o$ gives:

$$U_n = U_{n,0}\,,\; V_n = V_{n,0}\,,\; Z_n = Z_{n,0}.$$

2. Solution for an infinite sea. To simplify the writing we drop the indices which may be replaced in the final solution.
Let φ be a solution of the partial differential equation:

$$\frac{\partial^2 \varphi}{\partial t^2} + \lambda^2 \varphi - c^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) = 0, \tag{2,1}$$

which satisfies the conditions of Cauchy:

$$\varphi = 0 \, , \frac{\partial \varphi}{\partial t} = \Phi \, (x,y) \qquad \text{when } t = 0.$$

Furthermore we put:

$$\psi = \int_0^t \varphi \, dt.$$

ψ is then a solution of the non homogeneous partial differential equation:

$$\frac{\partial^2 \psi}{\partial t^2} + \lambda^2 \psi - c^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = \Phi, \tag{2,2}$$

satisfying the initial conditions:

$$\psi = 0, \frac{\partial \psi}{\partial t} = 0, \frac{\partial^2 \psi}{\partial t^2} = \Phi \qquad \text{when } t = 0.$$

We may then write down a number of solutions of the system of equations (1,7).

$$\begin{aligned} U_1 &= \frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial y^2} \\ V_1 &= - \lambda \frac{\partial \psi}{\partial t} + c^2 \frac{\partial^2 \psi}{\partial x \partial y} \\ Z_1 &= c \lambda \frac{\partial \psi}{\partial y} - c \frac{\partial^2 \psi}{\partial x \partial t} \end{aligned} \tag{2,3}$$

This set of solutions satisfied the initial conditions:

$$U = \Phi \, (x,y) \, , \, V = 0 \, , \, Z = 0 \; ; \, t = 0.$$

II. If we interchange x and y and replace λ by $-\lambda$ we get the following set of solutions:

$$\begin{aligned} U_2 &= \lambda \frac{\partial \psi}{\partial t} + c^2 \frac{\partial^2 \psi}{\partial x \partial y}, \\ V_2 &= \frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2}, \\ Z_2 &= - c \lambda \frac{\partial \psi}{\partial x} - c \frac{\partial^2 \psi}{\partial y \partial t}, \end{aligned} \tag{2,4}$$

satisfying the initial conditions:

$$U_2 = 0\,,\ V_2 = \Phi\,(x,y)\,,\, Z = 0\ ;\ t = 0\,.$$

III.

$$U_3 = -c\,\frac{\partial^2 \psi}{\partial x\,\partial t} - c\,\lambda\,\frac{\partial \psi}{\partial y}\,,$$

$$V_3 = -c\,\frac{\partial^2 \psi}{\partial y\,\partial t} + c\,\lambda\,\frac{\partial \psi}{\partial x}\,,$$

$$Z_3 = \frac{\partial^2 \psi}{\partial t^2} + \lambda^2 \psi\,.$$

Here the initial conditions are:

$$U_3 = 0\,,\ V_3 = 0\,,\, Z_3 = \Phi\,(x,y)\ ;\ t = 0\,.$$

By partial differentiations and combinations of these solutions we get others which also may be of interest.

If we take:

$$U_4 = \frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} = -\frac{1}{c}\,\frac{\partial U_3}{\partial t}\,,$$

$$V_4 = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = -\frac{1}{c}\,\frac{\varrho V_3}{\partial t}\,,$$

$$Z_4 = \frac{\partial Z_1}{\partial x} + \frac{\partial Z_2}{\partial y} = -\frac{1}{c}\,\frac{\partial Z_3}{\partial t}\,,$$

we find:

IV.

(2,6)

$$U_4 = \frac{\partial^2 \varphi}{\partial x\,\partial t} + \lambda\,\frac{\partial \varphi}{\partial y}\,,$$

$$V_4 = \frac{\partial^2 \varphi}{\partial y\,\partial t} - \lambda\,\frac{\partial \varphi}{\partial x}\,,$$

$$Z_4 = -\frac{1}{c}\left(\frac{\partial^2 \varphi}{\partial t^2} + \lambda^2 \varphi\right),$$

satisfying the initial conditions:

$$U_4 = \frac{\partial \Phi}{\partial x}\,,\ V_4 = \frac{\partial \Phi}{\partial y}\,,\, Z_4 = 0\,;\ t = 0\,.$$

If the initial disturbance Φ is restricted to a finite region, the whole solution will tend to zero when $t \to \infty$.

Other vanishing solutions may also be found.

V. At last we shall consider the solution given by:

$$
\begin{aligned}
U_5 &= \frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x} = -\lambda \frac{\partial^2 \psi}{\partial x \partial t} + \frac{\partial}{\partial y} (\Phi - \lambda^2 \psi), \\
V_5 &= \frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial x} = -\lambda \frac{\partial^2 \psi}{\partial y \partial t} - \frac{\partial}{\partial x} (\Phi - \lambda^2 \psi), \qquad (2,7) \\
\mathcal{Z}_5 &= \frac{\partial \mathcal{Z}_1}{\partial y} - \frac{\partial \mathcal{Z}_2}{\partial x} = \frac{\lambda}{c} \left(\frac{\partial^2 \psi}{\partial t^2} + \lambda^2 \psi - \Phi \right).
\end{aligned}
$$

This set of solutions satisfies the initial conditions:

$$U = \frac{\partial \Phi}{\partial y}$$

$$V = -\frac{\partial \Phi}{\partial x}$$

$$\mathcal{Z} = 0; \; t = 0.$$

and may be used to investigate the development of a non balanced vortex.

The solution of the partial differential equation (2,1) satisfying Cauchy's conditions is known and may be written in the form:

$$\varphi = \frac{1}{2\pi c} \iint_C \Phi(\alpha, \beta) \frac{\cos \frac{\lambda}{c} \sqrt{c^2 t^2 - \varrho^2}}{\sqrt{c^2 t^2 - \varrho^2}} \, d\alpha \, d\beta \qquad (2,8)$$

where:

$$\varrho^2 = (x - \alpha)^2 + (y - \beta)^2$$

and the domain C is the area bounded by the circle:

$$|\varrho| \lesseqgtr ct$$

If the initial function Φ satisfies the condition:

$$\iint \Phi(x, y) \, dx \, dy < \infty$$

φ will tend to zero with increasing value of time, t.

On the other hand:

$$\psi = \int_0^t \varphi \, dt$$

will tend to a limit when $t \to \infty$,

$$\psi_\infty = \frac{1}{2\pi c^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(\alpha, \beta) \, K_0 \left(\frac{\lambda \varrho}{c} \right) d\alpha \, d\beta \,. \qquad (2,9)$$

Here K_0 is the modified Bessel function of zero order and second kind:

$$K_0(x) = \int_0^\infty \frac{\cos x t \, dt}{\sqrt{1+t^2}} = \int_0^\infty \frac{t \, J_0(x t) \, dt}{1+t^2}$$

$$= \int_1^\infty \frac{e^{-xt} \, dt}{\sqrt{t^2-1}}.$$

If Φ is a function of $r = \sqrt{x^2+y^2}$ only, one of the integrations may be performed and we find:

$$\psi_\infty = \frac{1}{c^2} K_0\left(\frac{\lambda r}{c}\right) \int_0^r \Phi(\varrho) \, I_0\left(\frac{\lambda \varrho}{c}\right) \varrho \, d\varrho + \frac{1}{c^2} I_0\left(\frac{\lambda r}{c}\right) \int_r^\infty \Phi(\varrho) \, K_0\left(\frac{\lambda \varrho}{c}\right) \varrho \, d\varrho. \tag{2,10}$$

If Φ is a function of y only we have the integral:

$$\varphi = \frac{1}{2} \int_0^t \left[\Phi(y+c\tau) + \Phi(y-c\tau)\right] J_0\left(\lambda \sqrt{t^2-\tau^2}\right) d\tau, \tag{2,11}$$

and:

$$\psi_\infty = \frac{1}{2\lambda} \int_0^\infty \left[\Phi(y+c\tau) + \Phi(y-c\tau)\right] e^{-\lambda\tau} \, d\tau. \tag{2,12}$$

The limiting value of the solution is then:

$$U = \Phi(y) - \frac{\lambda}{2} \int_0^\infty \left[\Phi(y+c\tau) + \Phi(c\tau)\right] e^{-\lambda\tau} \, d\tau \tag{2,13}$$

$$V = 0$$

and:

$$Z = \frac{\lambda}{2} \int_0^\infty \left[\Phi(y+c\tau) - \Phi(y-c\tau)\right] e^{-\lambda\tau} \, d\tau.$$

For the calculation of the non stationary values it is more convenient to start with a Fourier integral:

$$\varphi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} e^{i\mu(y-\beta)} \, \Phi(\beta) \, \frac{\sin t \sqrt{\lambda^2 + c^2\mu^2}}{\sqrt{\lambda^2 + c^2\mu^2}} \, d\mu \, d\beta.$$

This is an integral of the equation:

$$\frac{\partial^2 \varphi}{\partial t^2} + \lambda^2 \varphi - c^2 \frac{\partial^2 \varphi}{\partial y^2} = 0.$$

which satisfies the conditions of Cauchy:

$$\varphi = 0, \frac{\partial \varphi}{\partial t} = \Phi(y); \, t = 0.$$

In this form the integration with respect to t can easily be performed, giving:

formed

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mu(y-\beta)} \frac{1 - \cos t\sqrt{\lambda^2 + c^2\mu^2}}{\lambda^2 + c^2\mu^2} d\mu \, d\beta.$$

If the integral:

$$\int_{-\infty}^{\infty} e^{-i\mu\beta} \Phi(\beta) \, d\beta = G(\mu)$$

can be calculated, we have:

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\mu) e^{i\mu y} \frac{1 - \cos t\sqrt{\lambda^2 + c^2\mu^2}}{\lambda^2 + c^2\mu^2} d\mu.$$

which can be computed by a numerical process.

In the following we have chosen:

$$\Phi(y) = e^{-ky^2}$$

giving:

$$G(\mu) = \sqrt{\frac{\pi}{k}} \, e^{-\frac{\mu^2}{4k}}$$

and:

(2,15)
$$\psi = \frac{1}{2\sqrt{k\pi}} \int_{-\infty}^{+\infty} e^{-\frac{\mu^2}{4k}} \cos \mu y \frac{1 - \cos t\sqrt{\lambda^2 + c^2\mu^2}}{\lambda^2 + c^2\mu^2} d\mu$$

art with

3. Current in an infinite ocean. In the following numerical application we assume that the initial current is given by:

$$u = e^{-ky^2}\left(1 - a u_1(z)\right), v = 0, \zeta = 0$$

where u_1 is the eigenfunction corresponding to the internal wave of the first order represented in Fig. 1. The corresponding solution is given by the formulae:

$$u = U_0(y,t) - a\, U_1(y,t)\, u_1(z)$$
$$v = V_0(y,t) - a\, V_1(y,t)\, u_1(z)$$
$$\zeta = Z_0(y,t)\, w_0(z) - a Z_1(y,t)\, w_1(z)$$

The functions U_0, U_1 etc. are expressed by:

$$U = \frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial y^2} = \Phi - \lambda^2 \psi$$

$$V = -\lambda \frac{\partial \psi}{\partial t}$$

$$Z = c\lambda \frac{\partial \psi}{\partial y}$$

where ψ is defined by (2,15).

It will be convenient to introduce non-dimensional variables by:

$$c\mu = \beta z\,,\ k = \varkappa b^{-2},\ y = b\eta\,,\ \frac{\lambda b}{c} = \beta\,.$$

The integral (2,15) may then be written in the form:

$$\lambda^2 \psi = \frac{\beta}{\sqrt{\varkappa \pi}} \int_0^\infty e^{-\frac{\beta^2 z^2}{4\varkappa}} \cos \beta z \eta\, \frac{1 - \cos \lambda t \sqrt{1+z^2}}{1+z^2}\, dz.$$

For the zero order wave we have $c_0 = \sqrt{gh} = 106.2$ m/sec. While the internal wave of the first order give:

$$c_1 = 2.054 \text{ m/sec.}$$

If we then take $\lambda = 10^{-4}$ and $b = 40$ km we get:

$$\beta_o = 0.038 \text{ and } \beta_1 = 2.0.$$

When t tends to infinity the stationary value of the integral is:

$$\lambda^2 \psi = \frac{\beta}{\sqrt{\varkappa \pi}} \int_0^\infty e^{-\frac{\beta^2 z^2}{4\varkappa}} \frac{\cos \beta z \eta}{1+z^2}\, dz.$$

An equivalent form of the integral is:

$$\frac{\beta}{2} \int_0^\infty \left[e^{-\varkappa (z+\eta)^2} + e^{-\varkappa (z-\eta)^2} \right] e^{-\beta z}\, dz,$$

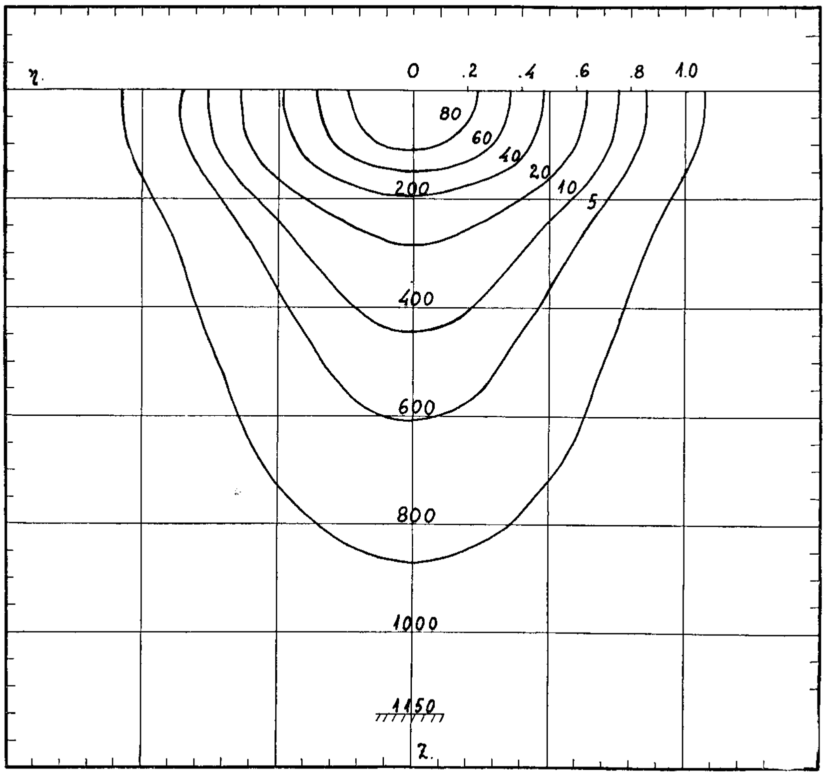

Fig. 2: Rectilinear current system. Assumed initial current distribution.

and this may be expressed by the error function:

$$\mathrm{Erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx\,.$$

The non stationary values of the integral $\lambda^2\psi$ may be calculated by numerical integration.

In Fig. 2 we give a representation of the initial current. The coefficient a is chosen such that the velocity is zero at the bottom.

The resulting stationary current is given by the formulae:

$$u = \Phi - \lambda^2\psi_0 - 4,2\,(\Phi - \lambda^2\psi_1)\, u_1(z)\,,$$

$$v = 0\,,$$

$$\zeta = \frac{1}{\beta_0}\frac{\partial}{\partial\eta}(\lambda^2\psi_0)\, w_0(z) - 4.2\frac{1}{\beta_1}\frac{\partial}{\partial\eta}(\lambda^2\psi_1)\, w_1(z)\,.$$

A graphical representation of the current and the corresponding deformation of the density surfaces are given in Fig. 3.

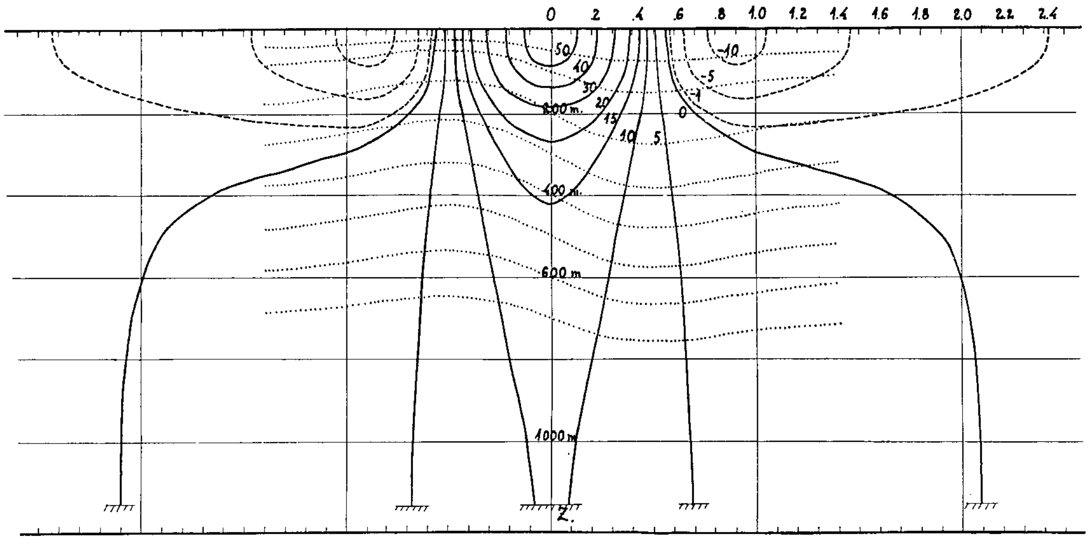

Fig. 3: Rectilinear current system. Final current distribution and density surfaces.

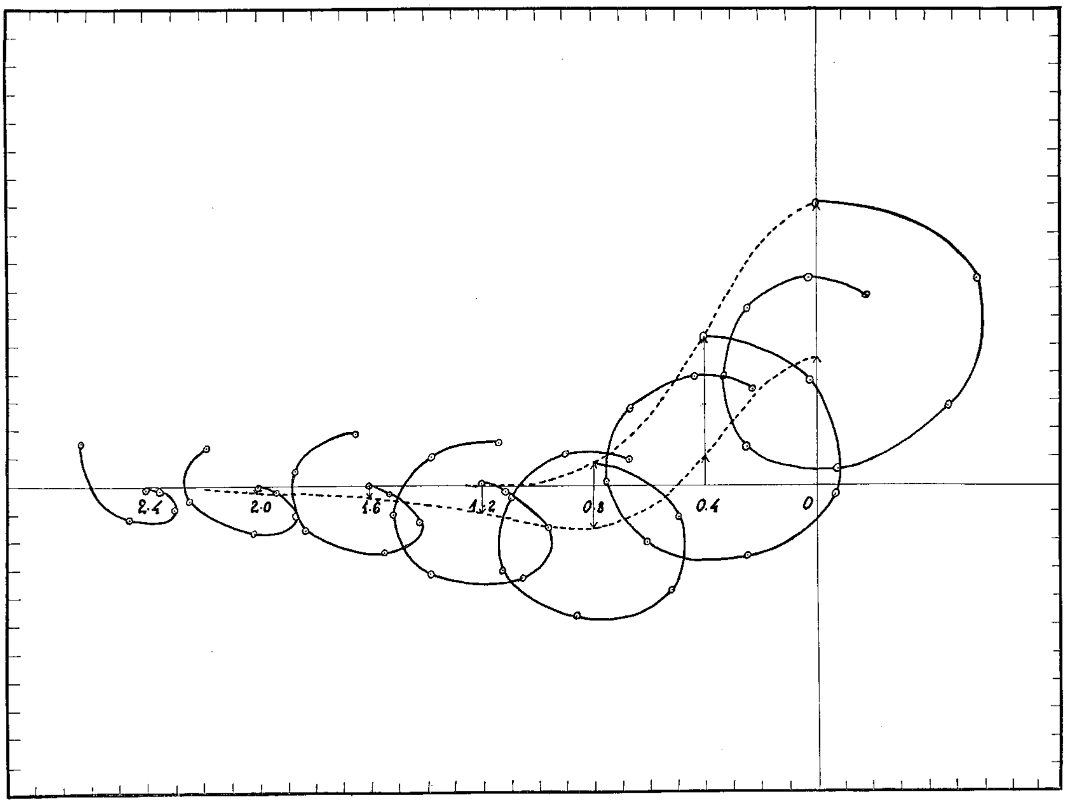

Fig. 4: Central vector diagrams of oscillating current.

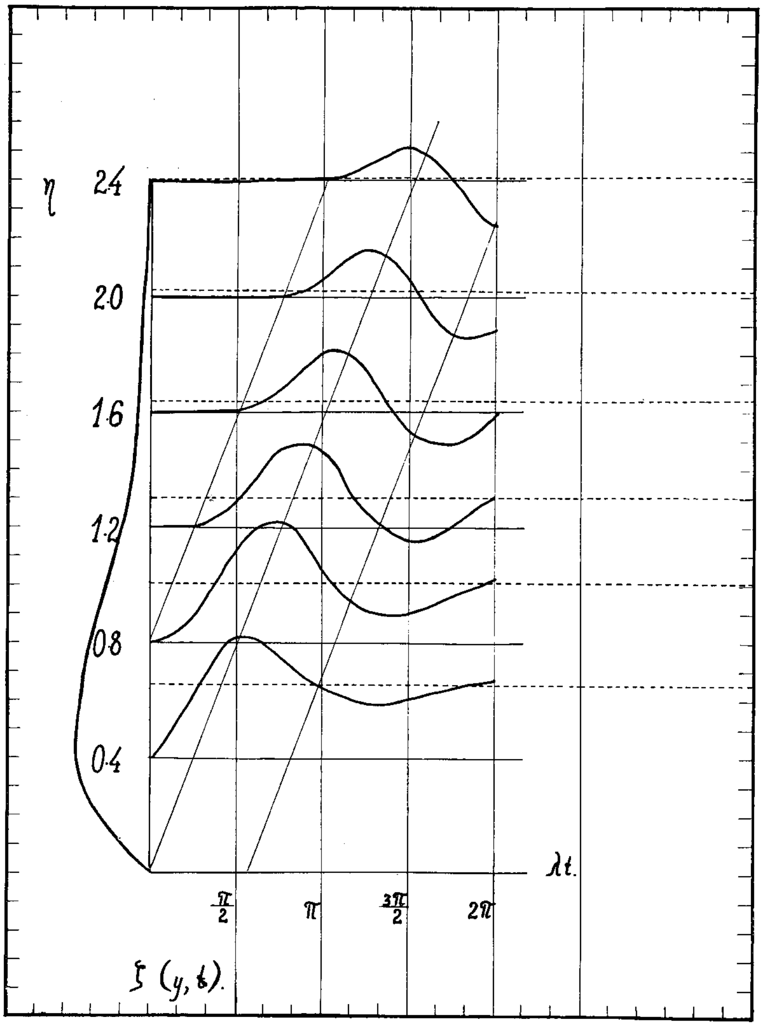

Fig. 5: Oscillations of density surfaces.

The velocity field in the stationary state is similar to that found by Bolin. In the upper layers the current system is more narrow and the velocities are smaller than the velocities of the initial current. In the deeper layers the velocities have increased and the current system is also much broader than at the surface. On both sides of the initial current system we have a weak counter current. The formation of the counter current is a consequence of the deformation of the density field. Since the ocean is regarded as infinite, the density surfaces must flatten out at a distance from the initial current field and approach the original position asymptotically.

A current going in one direction only, would demand that the density surfaces would be sloping from left to right and consequently the density surfaces would be higher on the left side. In an infinite sea this is impossible and we find that the density surfaces when going from left to right must rise from its initial level and then sink below the initial level and finally rise again. This is seen on Fig. 3, where a representation of ζ for different levels is given.

We have also made some calculations relating to the non stationary state of the current. The zero order wave will rapidly tend to the stationary state because the velocity of propagation is very large, and this wave will have no influence on the density field, and we have therefore only considered the first order internal wave. In Fig. 4 we give a number of central vector diagrams, showing the variation of the current during the first 12 pendulum hours for

$$\eta = 0,\ 0.4,\ 0.8,\ 1.2,\ 1.6,\ 2.0 \text{ and } 2.4.$$

The points marked on the curves represent the endpoint of the current vector for $\lambda t = 0°, 45°, 90° \ldots 360°$. The dashed curves give the initial and final distribution of the current as a function of η, the distance from the center of the current system.

The current vector oscillates in magnitude and direction. Near the center of the current system the oscillations have some resemblance with inertia oscillations, but it is to be remarked that the initial direction is attained in less than 12 pendulum hours.

At some distance from the center of the current system the current is initially zero and it takes some time before the oscillations start.

In Fig. 5 we give the corresponding graph of ζ as a function of time. The initial value is everywhere zero, and we see that the density surfaces at first rise above the stationary value and perform decaying oscillations about the stationary state which is reached asymptotically.

4. Unbalanced Vortex. Consider an initial vortex given by:

$$u_0 = \frac{\partial \Phi}{\partial y},\ v_0 = -\frac{\partial \Phi}{\partial x},$$

$$\Phi = C e^{-k(x^2+y^2)} \left((1 - 4.2\, u_1(z)\right)$$

$$\zeta_0 = 0$$

The solution is here represented by the formulae:

$$U = -\lambda \frac{\partial^2 \psi}{\partial x \partial t} + \frac{\partial}{\partial y}(\Phi - \lambda^2 \psi)$$

$$V = -\lambda \frac{\partial^2 \psi}{\partial y \partial t} - \frac{\partial}{\partial x}(\Phi - \lambda^2 \psi)$$

$$Z = \frac{\lambda}{c}\left(\frac{\partial^2 \psi}{\partial t^2} + \lambda^2 \psi - \Phi\right)$$

and in the stationary state we get:

$$U = \frac{\partial}{\partial y}(\Phi - \lambda^2 \psi),$$

$$V = -\frac{\partial}{\partial x}(\Phi - \lambda^2 \psi),$$

$$\zeta = -\frac{\lambda}{c}(\Phi - \lambda^2 \psi).$$

In this case it will be convenient to introduce circular coordinates by:

$$x = r \cos \theta,$$

$$y = r \sin \theta,$$

$$u = \dot{r} \cos \theta - r\dot{\theta} \sin \theta,$$

$$v = \dot{r} \sin \vartheta + r\dot{\theta} \cos \theta.$$

If we put $\dot{r} = \dfrac{dr}{dt} = u_1$, and $r\dot{\theta} = v_1$, u_1 and v_1 are represented by the formulae:

$$U_1 = -\lambda \frac{\partial^2 \psi}{\partial r \partial t} + \frac{1}{\gamma}\frac{\partial}{\partial \theta}(\Phi - \lambda^2 \psi),$$

$$V_1 = -\lambda \frac{\partial^2 \psi}{r \partial \theta \partial t} - \frac{\partial}{\partial r}(\Phi - \lambda^2 \psi),$$

$$\zeta = \frac{\lambda}{b}\left(\frac{\partial^2 \psi}{\partial t^2} + \lambda^2 \psi - \Phi\right).$$

If the initial system is independent of θ, the stationary vortex is given by the formulae:

$$U_1 = 0,$$

$$V_1 = -\frac{\partial}{\partial r}(\Phi - \lambda^2 \psi),$$

$$\zeta = -\frac{\lambda}{c}(\Phi - \lambda^2 \psi),$$

and ψ is a solution of the equation:

$$\lambda^2 \psi - c^2 \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}\right) = \Phi,$$

$$\Phi(r) = b e^{-kr^2}.$$

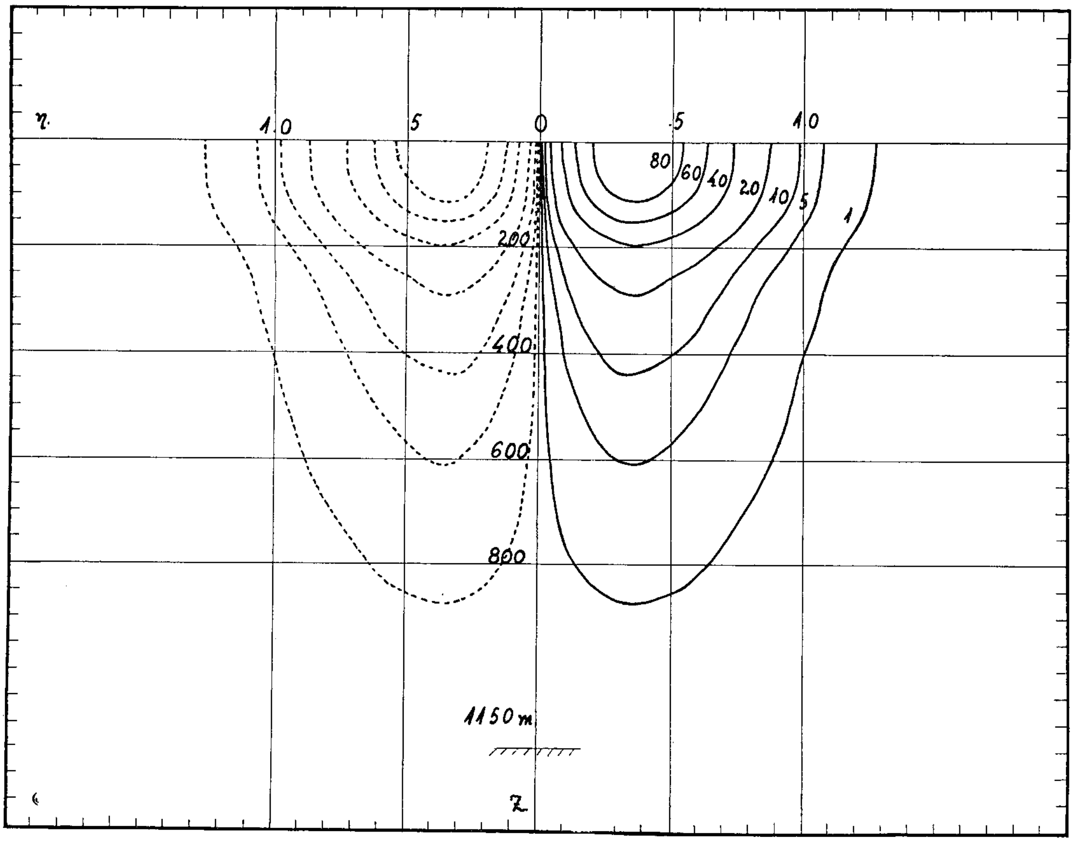

Fig. 6: Vortex. Initial current distribution.

The stationary value is given by the formula

$$\psi = \frac{1}{c^2} K_0\left(\frac{\lambda r}{c}\right)\int_0^r \Phi(\varrho) I_0\left(\frac{\lambda \varrho}{c}\right)\varrho\, d\varrho + \frac{1}{c^2} I_0\left(\frac{\lambda r}{c}\right)\int_r^\infty \Phi(\varrho) K_0\left(\frac{\lambda \varrho}{c}\right)\varrho\, d\varrho,$$

which can be computed by numerical integration. We shall give only the initial and the final state of the velocities and the vertical deformations of the density surfaces.

Fig. 6 represents a section of the initial vortex represented by the formulae:

$$u_1 = \dot{r} = 0$$

$$v_1 = r\dot{\theta} = 8\,\frac{r}{b}\,e^{-4\left(\frac{r}{b}\right)^2}\left(1 - 4.2\,u_1(z)\right)$$

$$\zeta = 0$$

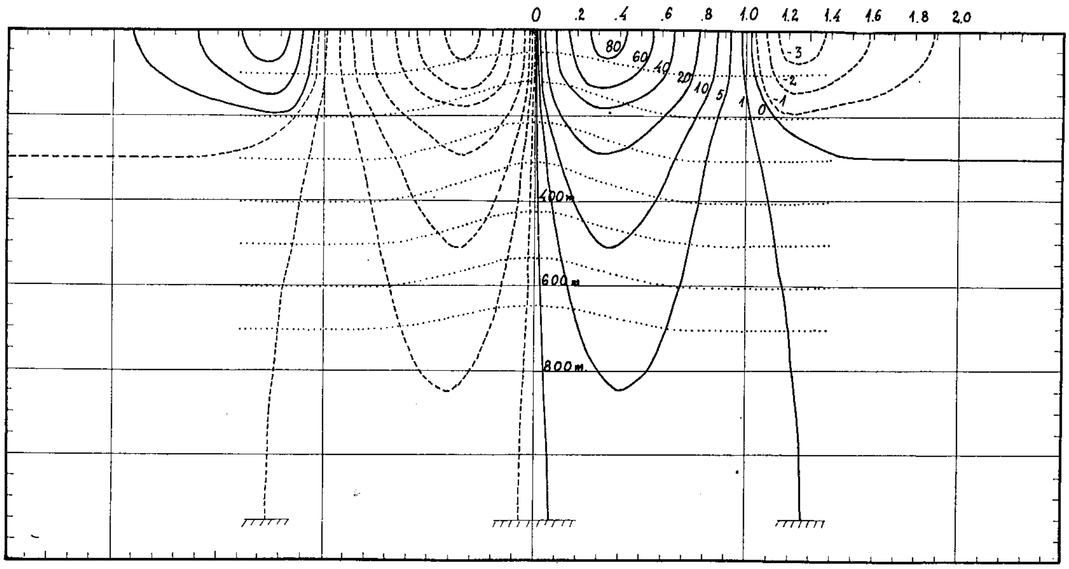

Fig. 7: Vortex. Final current distribution.

Fig. 7 gives the resulting stationary current and the corresponding deformation of the density surfaces. We also here find some of the characteristics which we had in the solution relating to the rectilinear current.

At the surface the vortex has shrinked and the velocities have diminished, while in the deeper layers the velocities have increased and spred out to a larger area. Around the vortex we find a circulation in the opposite direction. This is again a result of the deformation of the density field, which is also represented in Fig. 7.

5. Solutions when the sea is bounded. If the ocean is bounded in one or more directions the solution may be expressed by similar formulae, but the integral ψ has to satisfy boundary conditions besides the initial conditions.

In the first place we may take the case where we have an initial current outside a straight coast given by the line $y = o$.

ψ is then a solution of the same partial diferential equation subject to the initial conditions:

$$\psi = 0, \frac{\partial \psi}{\partial t} = 0, \frac{\partial^2 \psi}{\partial t^2} = \Phi(y). ; t = 0,$$

and $\psi = o$ when $y = o$.

To get the corresponding solution we put:

$$\psi = \int_o^\infty \sin \mu y \, A(\mu, t) \, d\mu ,$$

giving:

$$\int_0^\infty \sin \mu y \left[\frac{d^2 A}{dt^2} + (\lambda^2 + c^2\mu^2) A\right] d\mu = \Phi(y)$$

or:

$$A = \frac{2}{\pi} \int_0^\infty \sin \mu a \, \Phi(a) \, \frac{1 - \cos t \sqrt{\lambda^2 + c^2\mu^2}}{\lambda^2 + c^2\mu^2} \, da,$$

giving:

$$\psi = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin \mu y \sin \mu a \, \Phi(a) \, \frac{1 - \cos t \sqrt{\lambda^2 + c^2\mu^2}}{\lambda^2 + c^2\mu^2} \, d\mu \, da.$$

The limiting value of this solution when $t \to \infty$ is:

$$\psi = \frac{2}{\pi} \int_0^\infty \Phi(a) \int_0^\infty \frac{\sin \mu y \sin \mu a}{\lambda^2 + c\mu^2} \, d\mu \, da.$$

Here the integration with respect to μ may be performed, giving:

$$\int_0^\infty \frac{\sin \mu y \sin \mu a}{\lambda^2 + c^2\mu^2} \, d\mu = \frac{\pi}{4c\lambda} \left[e^{-\frac{\lambda}{c}\left|y - a\right|} - e^{-\frac{\lambda}{c}\left|y + a\right|} \right]$$

and:

$$\psi_\infty = \frac{1}{2c\lambda} \int_0^\infty \Phi(a) \left[e^{-\frac{\lambda}{c}\left|y - a\right|} - e^{-\frac{\lambda}{c}\left|y + a\right|} \right] da.$$

We shall not go any further with the discussion of this solution, but instead treat the case when the sea is bounded by two straight coasts:

$$y = \pm b.$$

In this case it will be nescessary to express the solution by means of Fourier series instead of Fourier integrals.

Initial current in a straight canal. If the current system is started in a sea bounded by straight coasts the current will not tend to any definite limit as long as no frictional resistance is present.

Suppose that the sea is bounded by the straight coast lines: $y = \pm b$

The solution is again given by the formulae:

$$U = \Phi - \lambda^2 \psi$$

$$V = -\lambda \frac{\partial \psi}{\partial t}$$

$$Z = c\lambda \frac{\partial \psi}{\partial y}$$

where ψ is a solution of the equation:

$$\frac{\partial^2 \psi}{\partial t^2} + \lambda^2 \psi - c^2 \frac{\partial^2 \psi}{\partial y^2} = \Phi(y),$$

but now subject to the conditions:

$$\psi = 0 \; ; \; y = \pm b.$$

The initial conditions are again:

$$\psi = 0 \; , \; \frac{\partial \psi}{\partial t} = 0 \, , \; \frac{\partial^2 \psi}{\partial t^2} = \Phi(y) \; ; \; t = 0 .$$

If we assume that the initial current system is symmetrical about the line $y = o$, the solution may be expressed by a Fourier series.
Let:

$$\Phi(y) = \Sigma c_n \cos (2n+1) \frac{\pi y}{2b} .$$

Then:

$$\psi(y) = \Sigma c_n \cos (2n+1) \frac{\pi y}{2b} \frac{1 - \cos t \sqrt{\lambda^2 + c^2 \beta_n{}^2}}{\lambda^2 + c^2 \beta_n{}^2}$$

where

$$\beta_n = (2n+1) \frac{\pi}{2b} .$$

To get a solution which can be compared with the solution in the case of an unlimited sea we take:

$$\Phi(y) = \frac{1}{b} \sqrt{\frac{\pi}{k}} \sum e^{-\frac{\beta_n{}^2}{4k}} \cos \beta_n y ,$$

and get:

$$\psi = \frac{1}{b} \sqrt{\frac{\pi}{k}} \sum e^{-\frac{\beta_n{}^2}{4k}} \cos \beta_n y \frac{1 - \cos t \sqrt{\lambda^2 + c^2 \beta_n{}^2}}{\lambda^2 + c^2 \beta_n{}^2} .$$

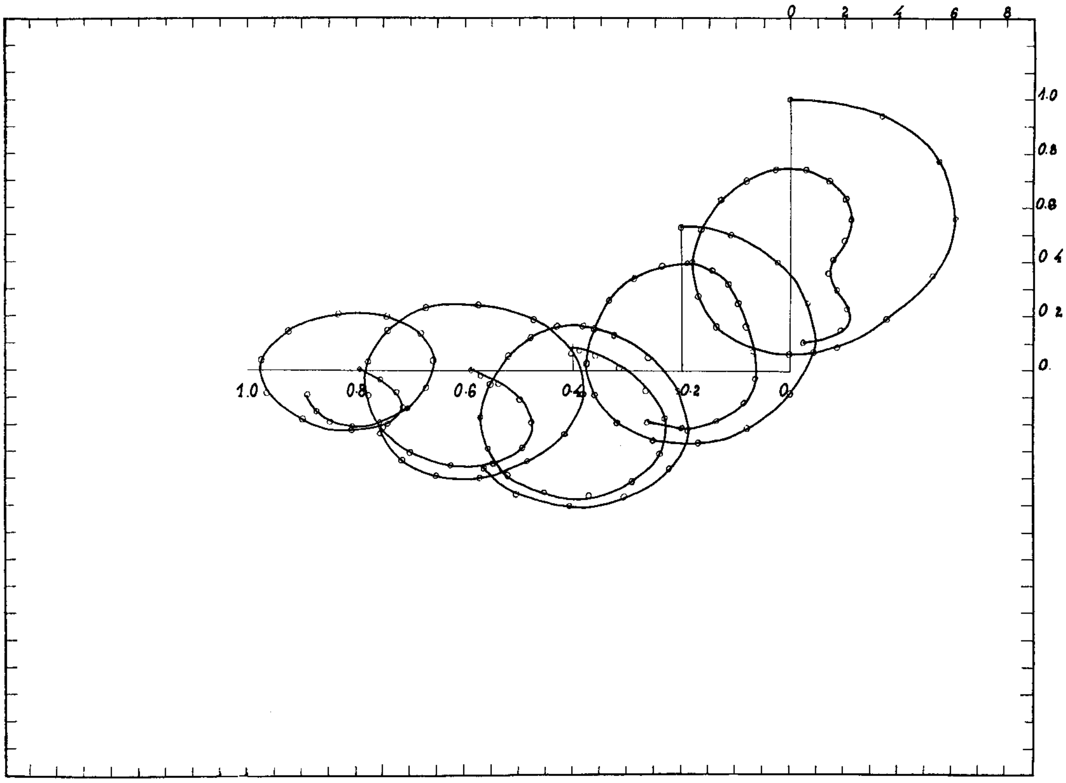

Fig. 8: Current in a straight canal.
Central vector diagrams of oscillating current.

When: $b \rightarrow \infty$ the limit is:

$$\frac{1}{\sqrt{k\pi}}\int_0^\infty e^{-\frac{\beta^2}{4k}}\cos\beta y\,\frac{1-\cos t\sqrt{\lambda^2+c^2\beta^2}}{\lambda^2+c^2\beta^2}\,d\beta,$$

which is the solution in the case of the unlimited sea.

The initial function $\Phi\ (y)$ is a Jacobian ϑ-function, and we have the formula:

$$\frac{1}{b}\sqrt{\frac{\pi}{k}}\sum e^{-\frac{\beta_n^2}{4k}}\cos\beta_n y = e^{-ky^2}\left(1+2\sum(-1)^n e^{-4n^2kb^2}\cosh 4nkby\right),$$

and for large values of kb^2, the series on the right side will practically reduce to the first term.

The solution is then given by the formulae:

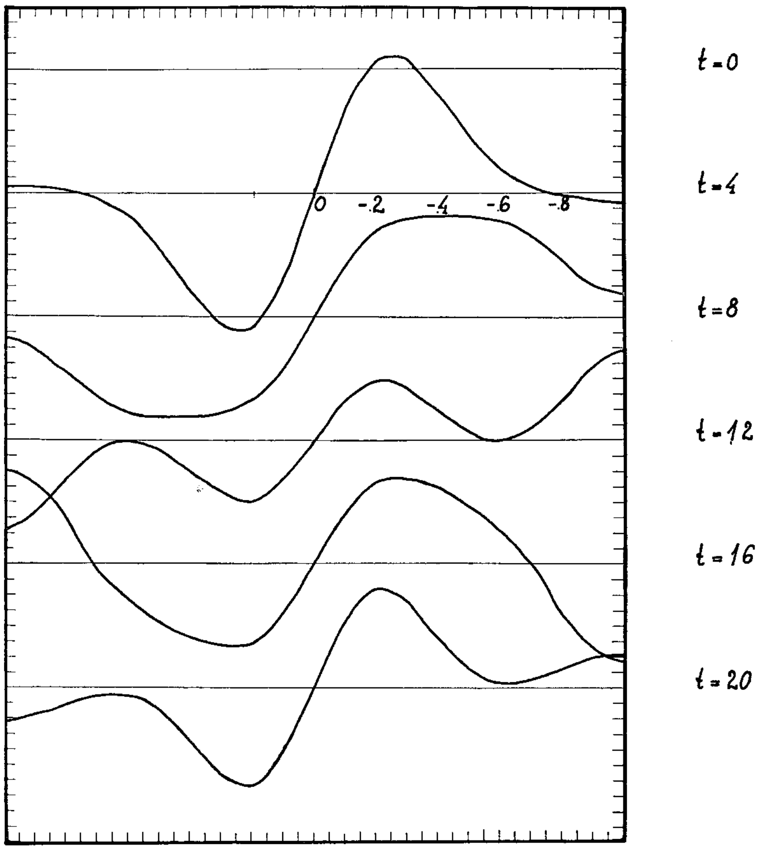

Fig. 9: Current in a straight canal. Oscillations of a density surfaces across the canal.

$$U=\frac{1}{b}\sqrt{\frac{\pi}{k}}\sum e^{-\frac{\beta_n^2}{4k}}\cos\beta_n y\,\frac{c^2\beta_n^2+\lambda^2\cos t\sqrt{\lambda^2+c^2\beta^2}}{\lambda^2+c^2\beta_n^2},$$

$$V=-\frac{1}{b}\sqrt{\frac{\pi}{k}}\sum e^{-\frac{\beta_n^2}{4k}}\cos\beta_n y\,\frac{\lambda\sin t\sqrt{\lambda^2+c^2\beta_n^2}}{\sqrt{\lambda^2+c^2\beta_n^2}},$$

$$Z=-\frac{1}{b}\sqrt{\frac{\pi}{k}}\sum e^{-\frac{\beta_n^2}{4k}}\sin\beta_n y\,\frac{\lambda c\beta_n(1-\cos t\sqrt{\lambda^2+c^2\beta_n^2})}{\lambda^2+c^2\beta_n^2}.$$

These functions will be quasi periodic and do not tend to limiting values for increasing values of time t.

We have calculated a numerical example using the values:

$$kb^2 = 16\,, \frac{\lambda b}{c} = 4\,.$$

The zero order wave will perform rapid oscillations, but will have practically no influence on the density field, and we have therefore only considered the first order internal wave.

In Fig. 8 we have given central vector diagrams for the velocities at the points η = 0, 0.2, 0.4, 0.6 and 0.8. The endpoint of the current vector has been indicated for the first 26 pendulum hours. A comparison of this diagram with the corresponding diagram for the unlimited sea shows some resemblance for the first 12 pendulum hours, but then the effect of the reflection from the sides of the canal will be more marked.

In Fig. 9 is represented a section of a density surface across the canal. The deviation of this surface from its equilibrium position is given for

$$t = 0,\ 4,\ 8,\ 12,\ 16 \text{ and } 20 \text{ pendulum hours.}$$

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