GEOFYSISKE PUBLIKASJONER
GEOPHYSICA NORVEGICA
VOL. XXI NO. 8

# AN EXPERIMENT IN NUMERICAL PREDICTION OF THE 500 MB WIND FIELD

BY KAARE PEDERSEN

FREMLAGT I VIDENSKAPS-AKADEMIETS MØTE DEN 13DE NOVEMBER 1959 AV ELIASSEN

## LIST OF SYMBOLS:

| | |
|---|---|
| $f$ | Coriolis parameter |
| $g$ | acceleration of gravity |
| $h$ | height of tropopause |
| $J$ | Jacobian operator |
| $t$ | time |
| $x, y$ | Cartesian coordinates |
| $z$ | height of 500 mb level |
| $\eta$ | absolute vorticity |
| $\psi$ | geostrophic stream function obtained from the 500 mb heights |
| $\nabla$ | two dimentional del operator |
| $\nabla^2$ | two dimentional Laplacian operator |

**Summary.** Two 48 hour predictions of the geostrophic stream function at the 500 mb level are computed on one initial situation. One prediction uses the 4 point formula commonly substituted for the Jacobian operator, the other uses an 8 point formula for the Jacobian operator. The prognostic equation is the barotropic vorticity equation with divergence.

On the larger scales the two predictions are very similar. The displacements of six smaller troughs, which are observed to have a "barotropic displacement", are studied. It is found that their displacements in the prediction using the 4 point formula and in the prediction using the 8 point formula on an average are 66 and 78% of the observed displacements respectively.

**1. Introduction.** In numerical weather prediction it is necessary to make several approximations of mathematical character. The difficulty in testing these approximations arises from the fact that the solution to the set of differential equations which describes the chosen physical model of the atmosphere is known only to the extent the atmosphere behaves like the model. One may test the goodness of each approximation on theoretical types of flow. Another way to test the mathematical approximations would be to set up a numerical procedure in which the various approximations could be altered separately, and the effect on the different predictions studied. The differences between predictions using different degree of accuracy in the numerical procedure may indicate if a desired degree of accuracy is reached.

The mathematical approximation tested in this paper is the finite difference formula usually substituted for the Jacobian operator. Two 48 hour predictions are computed on one initial situation. One prediction uses the formula usually substituted for the Jacobian operator, the other uses a formula of higher degree of accuracy. The two formulae are also tested on harmonic functions.

**2. The physical model.** Our prognostic equation is the barotropic vorticity equation with divergence, CRESSMAN (1958). We have

$$(\nabla^2 - a^2)\frac{\partial \psi}{\partial t} = \mathcal{J}(\eta, \psi), \tag{2.1}$$

where $a^2 = f^2/(1 - \varepsilon)g\bar{h}$. $\varepsilon$ which represents the ratio of the density of the stratosphere to that of the troposphere, is given the value 7/8, corresponding to a value of the ratio $\frac{1}{h}\frac{\partial h}{\partial t}\Big/\frac{1}{z}\frac{\partial z}{\partial t}$ of about 4. $\bar{h}$ is the height of the tropopause averaged along latitude circles. The values of $\bar{h}$ representative for the winter months is taken from DEFANT and TABA (1958). The other quantities are defined in the list of symbols.

At the boundary of the forecast area we assume

$$\frac{\partial \psi}{\partial t} = 0 \tag{2.2}$$

and

$$\frac{\partial \nabla^2 \psi}{\partial t} = 0. \tag{2.3}$$

According to a recommendation by FJØRTOFT (personal communication), the stream function is obtained from the following simplified form of the balance equation

$$\nabla \cdot f \nabla \psi = g \nabla^2 z. \tag{2.4}$$

The boundary condition is

$$\frac{\partial \psi}{\partial s} = \frac{g}{f}\frac{\partial z}{\partial s}, \tag{2.5}$$

where $\partial s$ is an element of the boundary.

**3. The mathematical model.** The change in the quantity $(\nabla^2 - \alpha^2)\psi$ is for each time step computed by a finite difference formula for the Jacobian operator in each point of a square grid, and the Helmholtz equation for the tendency is solved by relaxation.

The forecast area covers the northern hemisphere north of a selected latitude circle, 19,6°N. It is found suitable to use a Mecator projection. Besides other advantages one may then easily maintain a correct zonal average of the stream function and thus reduce the scale of the residue field in the relaxation procedure. The mesh size, $d$, in the quadratic grid was chosen equal to 5° longitude. To avoid too dense a grid in the northern area, the mesh size north of 70,18°N was chosen to be $2d$, and the area north of 81,67°N was represented by the pole values only. Thus the total number of grid points was 1405.

The expression used for the Laplacian operator is the commonly used 5 point formula. It is shown elsewhere, PEDERSEN (1958), that for disturbances of wavelengths larger than $4d$ this formula has errors less than 5%. For the Jacobian operator two formulae are used, one is the commonly used 4 point formula, the other is an 8 point formula of a higher degree of accuracy. The two formulae are given and discussed in appendix I. The change in vorticity at the pole is assumed given by the transport of vorticity across the northernmost grid-line,

$$A \frac{\partial}{\partial t} \nabla^2 \psi_p = T, \tag{3.1}$$

where $\nabla^2\psi_p$ is the vorticity at the pole, $A$ is the area north of the northernmost grid-line and $T$ is the transport of vorticity across this grid-line.

We may compute the tendency in the stream function at the pole in the following way. We introduce a scalar function, $q$, given by

$$\nabla^2 q - \alpha^2 q = 0, \tag{3.2}$$

with $q = 1$ and $\partial q/\partial\varphi = 0$ at the pole ($\varphi$ is latitude). It may then be shown that (2.1) is identical with the following equation

$$\nabla \cdot q^2 \nabla \frac{1}{q} \frac{\partial \psi}{\partial t} = q\, \mathcal{J}(\eta, \psi). \tag{3.3}$$

The equations (2.2), (3.1) and (3.3) determine the tendency in the zonal mean value of the stream function at each latitude. The tendency at the pole may now be computed by assuming that the zonal mean value of the stream function at the two northernmost grid-lines and at the pole is located on a parabola with horizontal tangent at the pole. This pole value is used as a boundary condition in the relaxation.

The equation used for relaxation is (2.1). A solution of this equation is produced over the whole area by relaxation on the grid with mesh size 2d. The solution south of latitude 70.18°N is then improved by relaxation on the dense grid.

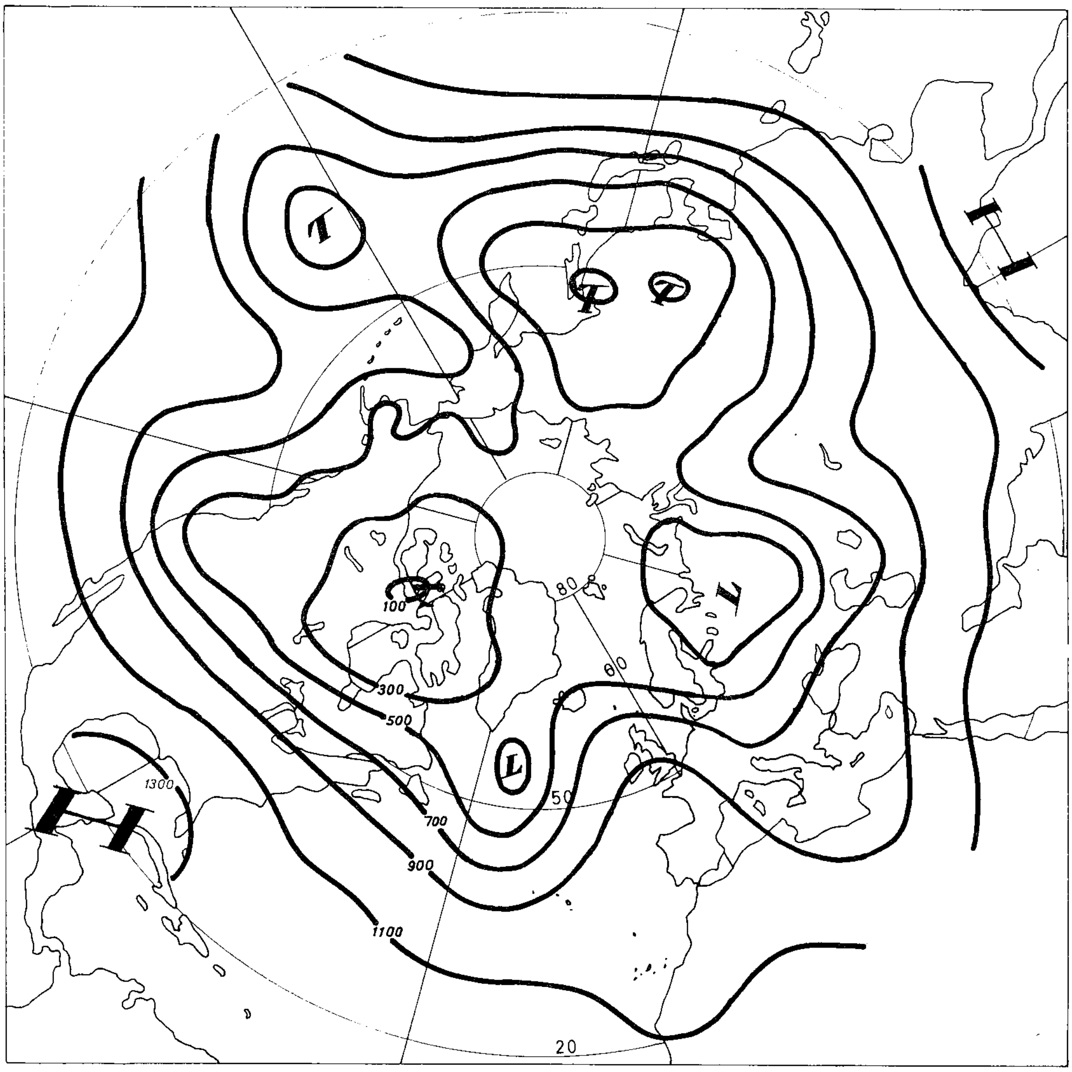


Fig. 1. Stream function obtained from the 500 mb height 6 December 1956, 1500 GMT.

The time step used was one hour. This proved to be too large near the northern boundary of the grid. It was necessary to introduce a smoothing function to avoid computational instability. A linear smoothing operator was applied 12 times after 12, 24 and 36 time steps. The smoothing operator and the effect of 36 smoothings are given in appendix II.

**4. Results.** The initial situation is the 6 December 1956, 1500 GMT. The data are read from the Daily Series of Synoptic Weather Maps, published by U.S. Weather Bureau. Figs. 1 and 2 show the stream functions for the 6 and 8 December. Fig.

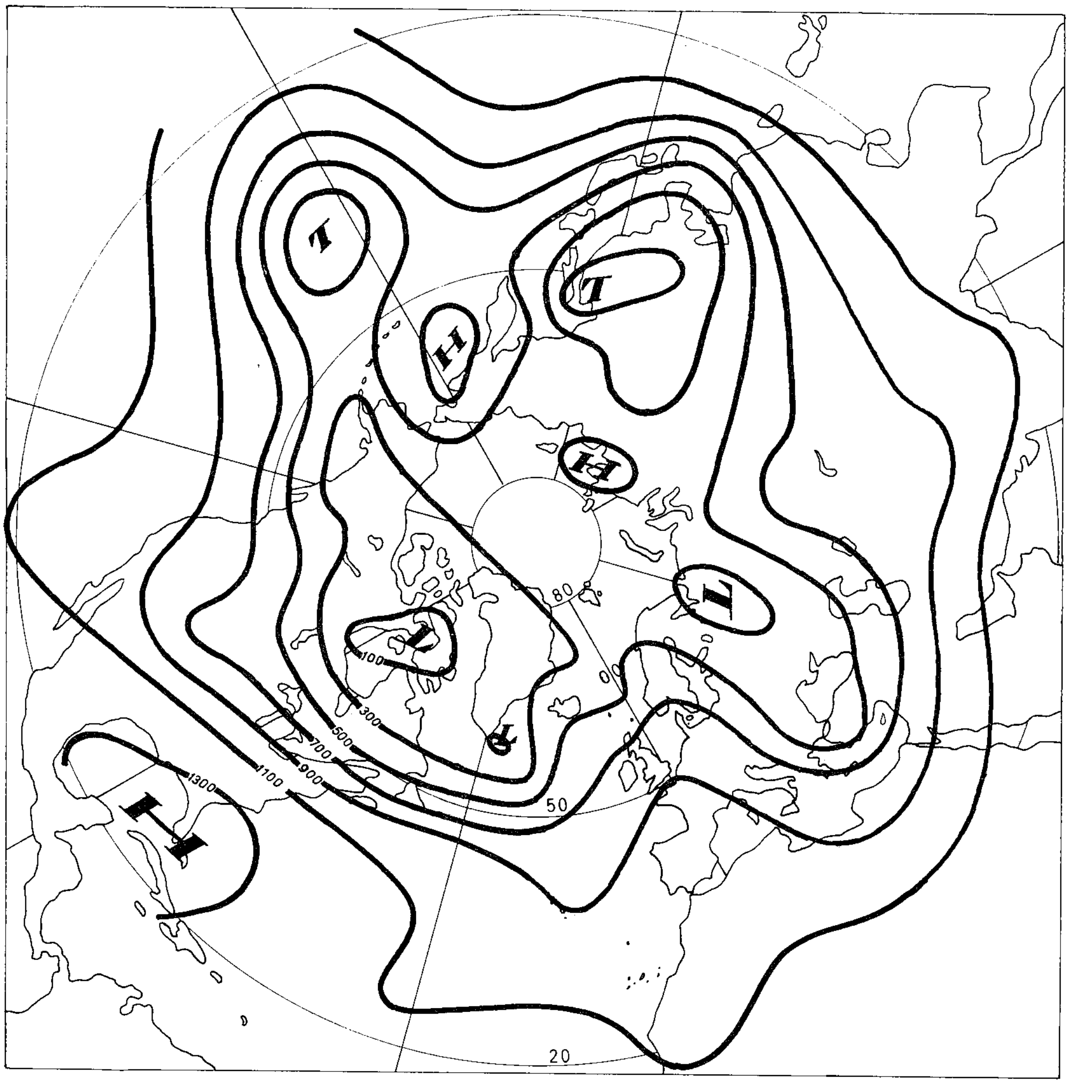


Fig. 2. Stream function obtained from the 500 mb height 8 December 1956, 1500 GMT.

3 shows the 48 hours prediction of the stream function using the 8 point formula for the Jacobian operator. Streamlines are drawn for each 200 meter, at 60°N the gradient equals the gradient in geopotential meter. The prediction using the 4 point formula for the Jacobian operator is very similar to this prediction. Fig. 4 shows the difference between the two predictions. The differences are seen to appear on a rather small scale, the amplitudes vary between 40 and 80 meter. Due to the smoothing performed these amplitudes are strongly reduced. The 24 hour unsmoothed differences (the computational instability did not occur until after 24 hours) were even larger than the

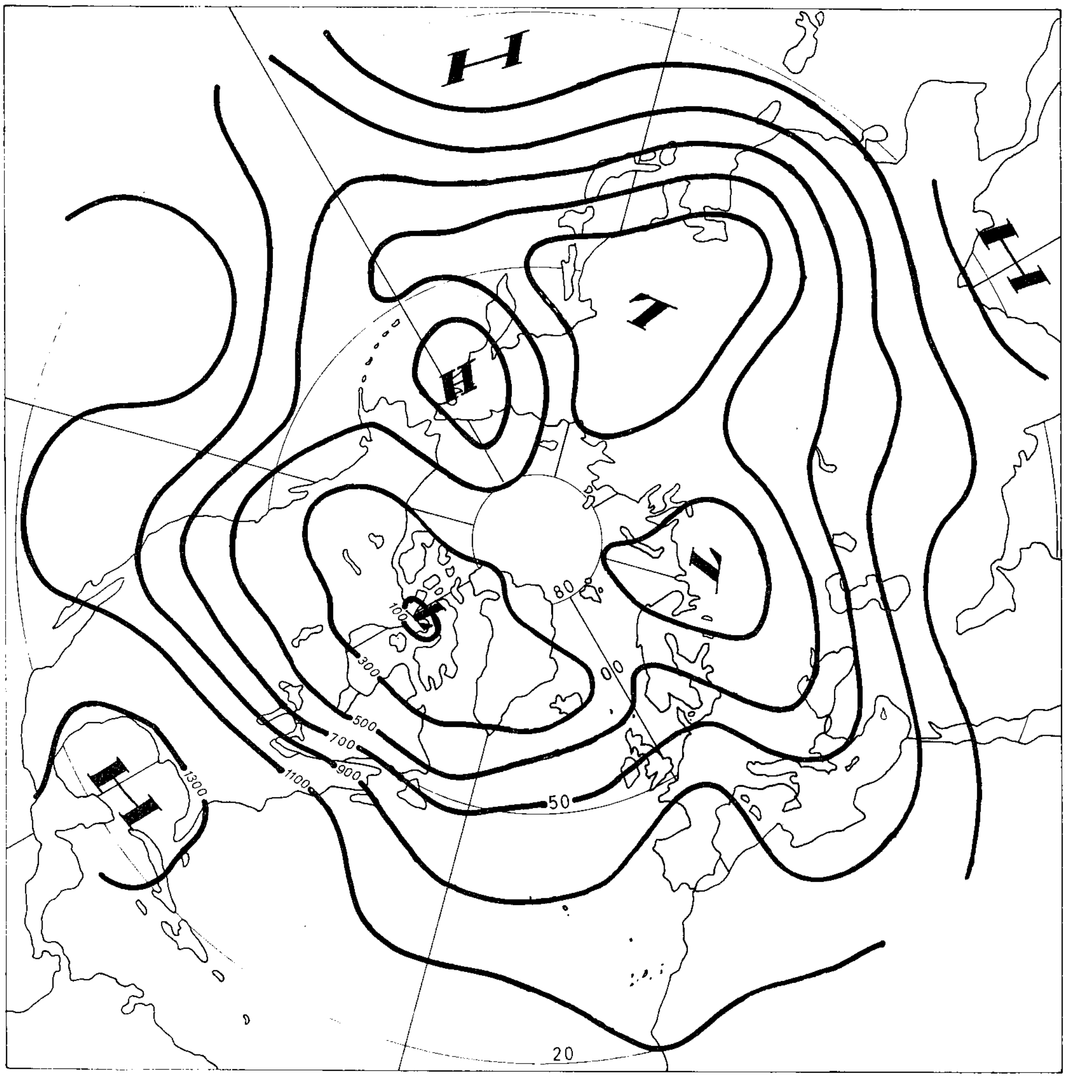


Fig. 3. The 48 hour prediction for the stream function using the 8 point formula for the Jacobian operator.

corresponding smoothed 48 hours differences. But these computations do to some extent demonstrate the effect of using uncorrect formulae for the Jacobian operator. An inspection of the maps shows that the differences between the two predictions result mainly from a different displacement of the small scale troughs and ridges. The displacements of six small scale troughs are studied. The position of a trough is defined by the point of maximum cyclonic vorticity in the trough. These six troughs are observed to move very nearly with the observed geostrophic mean wind along the dis-

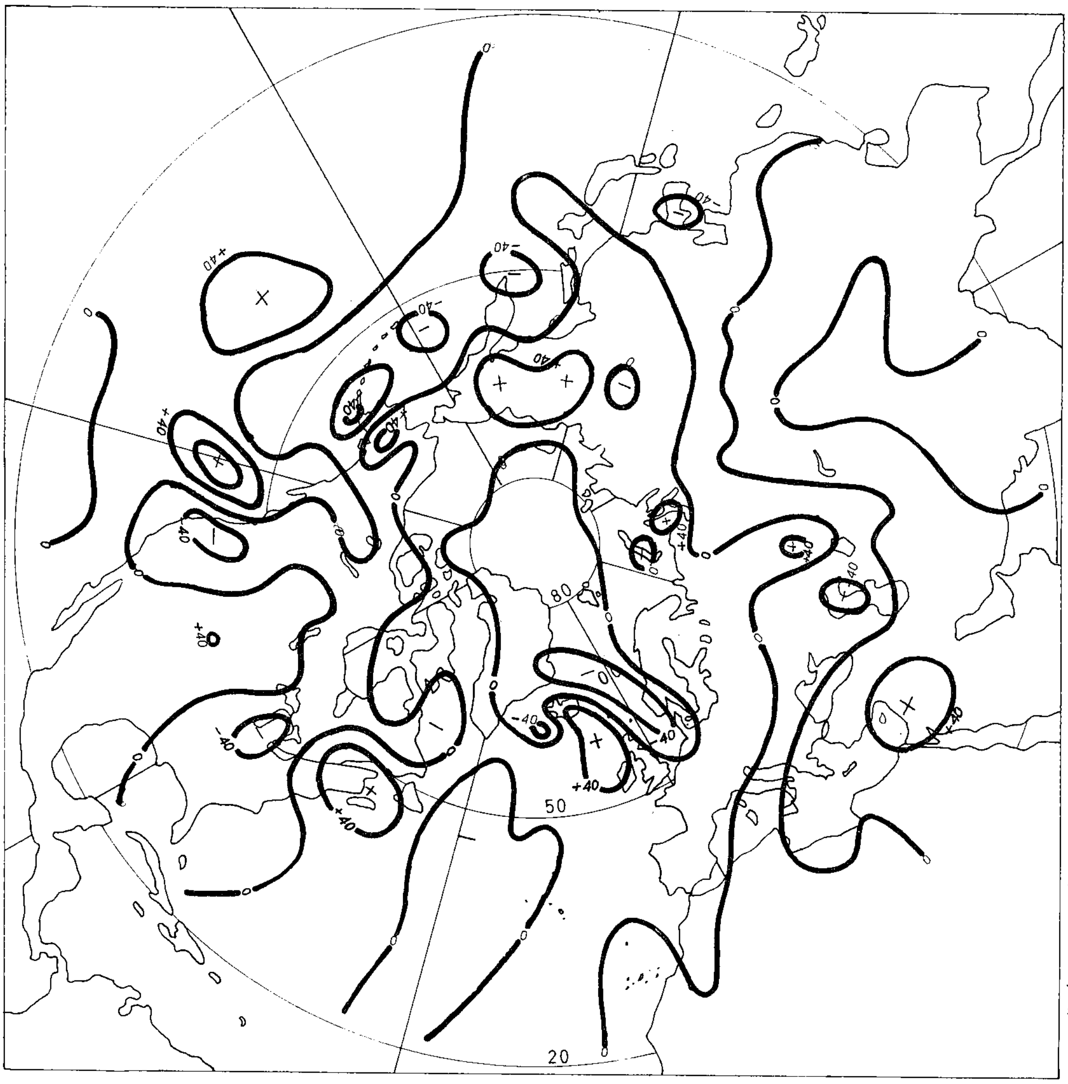


Fig. 4. Stream function difference between 48 hour predictions using the 8 point formula and the 4 point formula for the Jacobian operator.

placement path, so they should be expected to be well predicted by the barotropic equation. The Figs. 5a to 5f give an idea of the errors in the predicted positions of the troughs. Each figure gives the stream function profiles — along a certain latitude or longitude circle — of the two 48 hour predictions and of the stream function observed the 8 December. In Table 1 the predicted displacements of the troughs are given in % of the observed displacements.

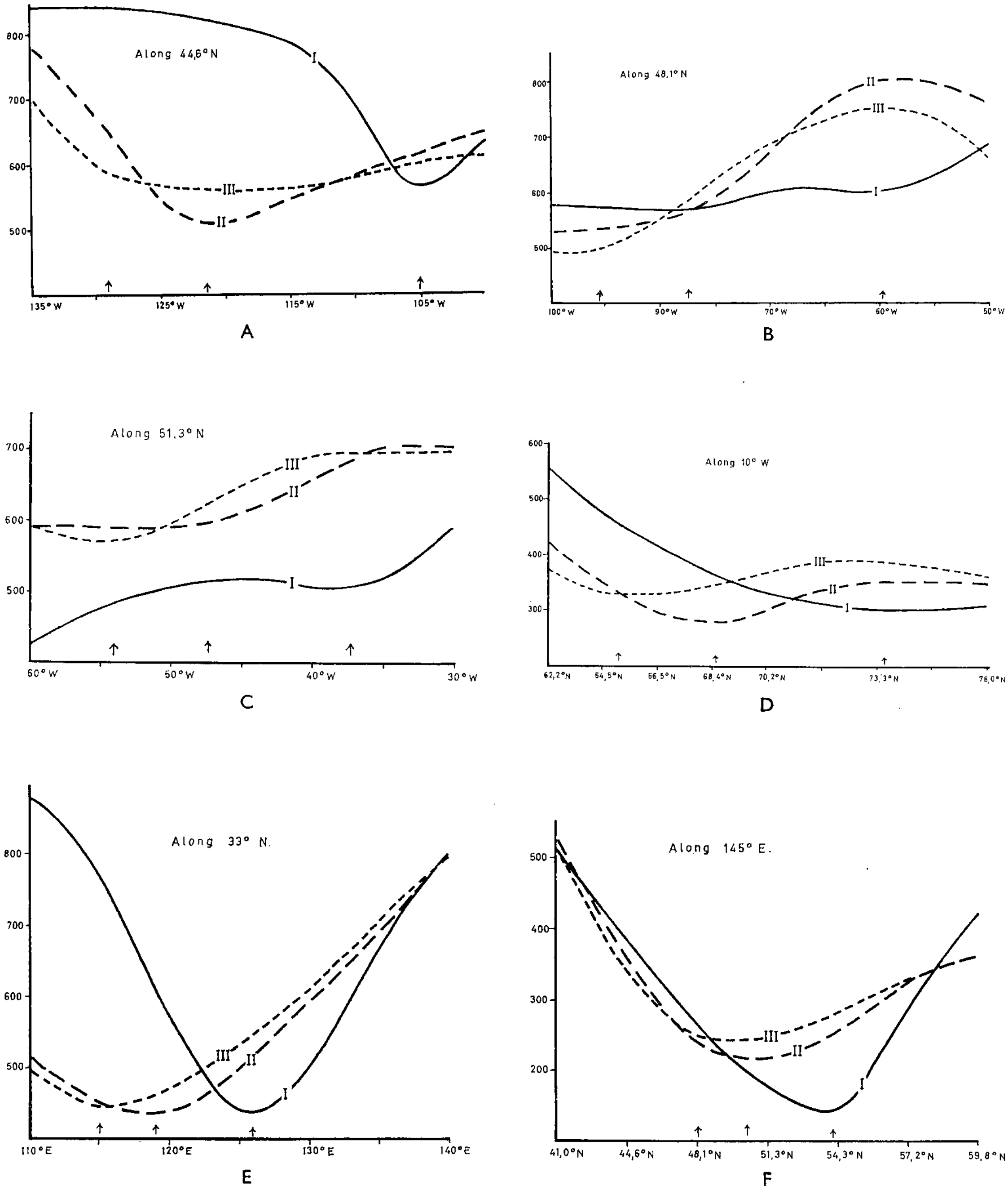


Fig. 5a to 5f. Curve I gives the observed stream function the 8 December. Curve II and III give the predicted stream function using the 8 point formula and the 4 point formula for the Jacobian operator respectively. The arrows indicate the position of maximum cyclonic vorticity.

Table 1. *The predicted displacements of the troughs in Figs. 5a to 5f in % of the observed displacements.*

| | Prediction using the 4 point formula | Prediction using the 8 point formula |
|---|---|---|
| Fig. 5a ............ | 44% | 68% |
| Fig. 5b ............ | 63% | 77% |
| Fig. 5c ........... | 67% | 72% |
| Fig. 5d ........... | 79% | 92% |
| Fig. 5e ............ | 67% | 78% |
| Fig. 5f ............. | 78% | 83% |
| Mean ............. | 66% | 78% |

Some of the errors are larger than expected from the discussion in appendix I. This may partly be due to the fact that the smoothing performed decreases the mean wind speed in the jet current in which the troughs are imbedded.

In order to test to what extent the meridional eddy transport of angular momentum is correctly predicted, the observed and the two predicted changes in the zonal mean values of the stream function is computed. The results are given in Fig. 6. The observed

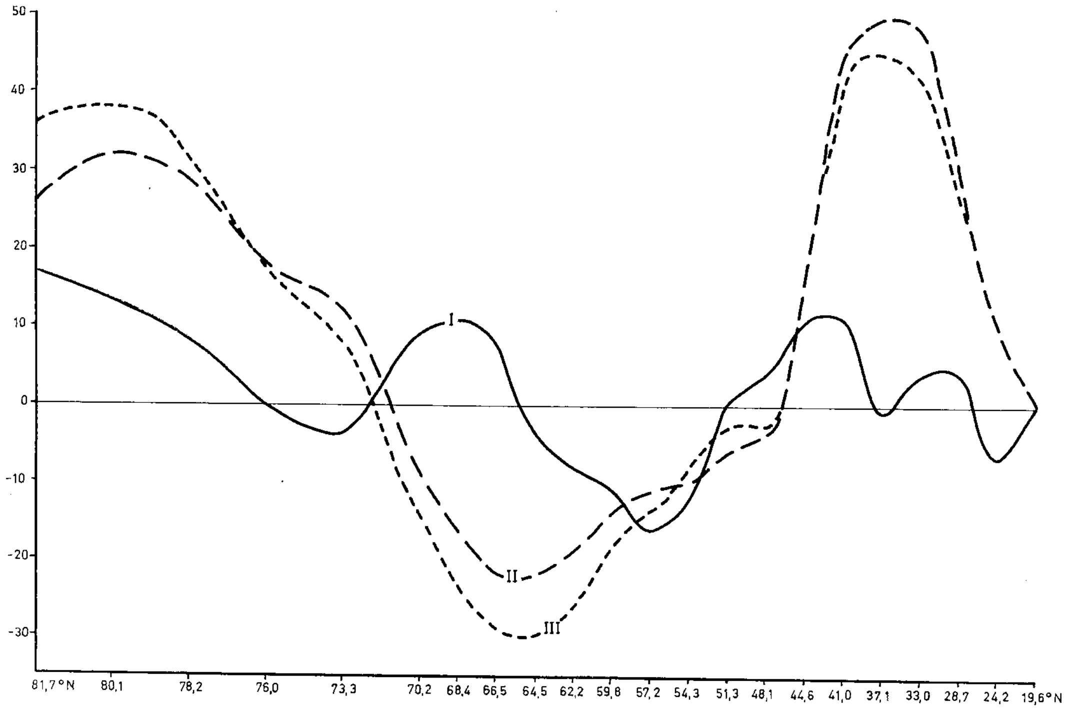


Fig. 6. Curve I gives the observed 48 hour changes in the zonal mean value of the stream function. Curve II and III give the predicted 48 hour changes in the zonal mean value of the stream function using the 8 point formula and the 4 point formula for the Jacobian operator respectively.

changes are small. There has been a slight increase in the westerlies in the zone between 44°N and 57°N and a slight decrease in the zone between 57°N and 68°N. The two predicted changes are very similar. There is a decrease in the westerlies south of 37°N and an increase in the zone between 37°N and 64°N. This is in good agreement with computations of the geostrophic poleward flux of angular momentum during a winter month (i.e. MINTZ 1955). The fact that the predicted increase and decrease in the westerlies is not observed, is probably due to meridional circulations and vertical eddy flux of momentum, which are not incorporated in this physical model. Since the two predictions are very similar and since the errors in the predictions are of a reasonable character, the predicted changes are believed to be mathematically fairly correctly computed.

**Acknowledgement.** The author is indebted to Professor ELIASSEN, University of Oslo, for valuable suggestions and discussions.

The computations are programmed for the Ferranti Mercury computer at the Norwegian Defence Research Establishment, the author wishes to express his thanks for the permission to use this computer free of charge.

The work has been sponsored by the Norwegian Research Council for Science and the Humanities through the Institute for Weather and Climate Research.

**Appendix I.** For the Jacobian operator the two following formulae have been used

$$\text{(I,1)} \qquad J_4(\eta, \psi) = \frac{1}{4d^2}\Big\{[\eta(x+d,y) - \eta(x-d,y)]\,[\psi(x,y+d) - \psi(x,y-d)] - [\eta(x,y+d) - \eta(x,y-d)]\,[\psi(x+d,y) - \psi(x-d,y)]\Big\}$$

and

$$\text{(I,2)} \qquad J_8(\eta, \psi) = \frac{4}{3} J_4(\eta, \psi) - \frac{1}{48d^2}\Big\{[\eta(x+2d,y) - \eta(x-2d,y)]\,[\psi(x,y+2d) - \psi(x,y-2d)] - [\eta(x,y+2d) - \eta(x,y-2d)]\,[\psi(x+2d,y) - \psi(x-2d,y)]\Big\}$$

The two formulae have errors of second and forth order in $d$ respectively. The following discussion may give an idea of the goodness of the formulae (I,1) and (I,2). We want to investigate the case when waves of a rather small scale are imbedded in a flow of larger scale. Suppose the stream function, $\psi$, is given by $\psi_1 + \psi_2$ where

$$\text{(I,3)} \qquad \psi_1 = \sin px \sin qy, \text{ and } \psi_2 = \sin Px \sin Qy.$$

$P$, $Q$, $p$ and $q$ are the wave-numbers. For simplicity we choose $P$ equal to $Q$ and $p$ equal to $q$. Suppose $p > Q$, we would then have

$$\text{(I,4)} \qquad J[\psi_1 + \psi_2, \nabla^2(\psi_1 + \psi_2)] \approx J(\psi_2, \nabla^2\psi_1).$$

By substituting the equations (I,3) in the exact and the two approximate formulae for the Jacobian, one finds the respective errors in the amplitudes. Table 2 gives the amplitude errors for $P$ equal to $2\pi/12d$ for some values of $p$.

Table 2. *The amplitude errors in the formulae (I,1) and (I,2) for P equal to $2\pi/12d$ for three different values of p.*

| | The amplitude error in formula (I,1) | The amplitude error in formula (I,2) |
|---|---|---|
| $\frac{2\pi}{6d}$ | 21% | 6% |
| $\frac{2\pi}{5d}$ | 28% | 13% |
| $\frac{2\pi}{4d}$ | 39% | 19% |

The accuracy of the formulae (I.1) and (I.2) has been investigated by ØKLAND (1958) in the one-dimensional case.

**Appendix II.** The smoothing operator used was

$$\text{(II,1)} \qquad \psi(x,y) = 0{,}945\psi(x,y) + 0{,}015[\psi(x+d,y) + \psi(x,y+d) + \psi(x-d,y) + \psi(x,y-d)] + 0{,}006[\psi(x+d,y+d) + \psi(x-d,y+d) + \psi(x-d,y-d) + \psi(x+d,y-d)] - 0{,}0055[\psi(x+2d,y) + \psi(x,y+2d) + \psi(x-2d,y) + \psi(x,y-2d)] - 0{,}0015[\psi(x+2d,y+d) + \psi(x+d,y+2d) + \psi(x-d,y+2d) + \psi(x-2d,y+\mathrm{d}) + \psi(x-2d,y-d) + \psi(x-d,y-2d) + \psi(x+d,y-2d) + \psi(x+2d,y-\mathrm{d})] + 0{,}00125[\psi(x+2d,y+2d) + \psi(x-2d,y+2d) + \psi(x-2d,y-2d) + \psi(x+2d,y-2d)].$$

This smoothing was performed 36 times. The effect of the smoothing on functions of the type (I,3) is shown in Fig. 7. $L_x$ and $L_y$ are the wavelengths in the $x$- and $y$- direction respectively. The diagram is symmetrical with respect to the line $L_x$ equal to $L_y$.

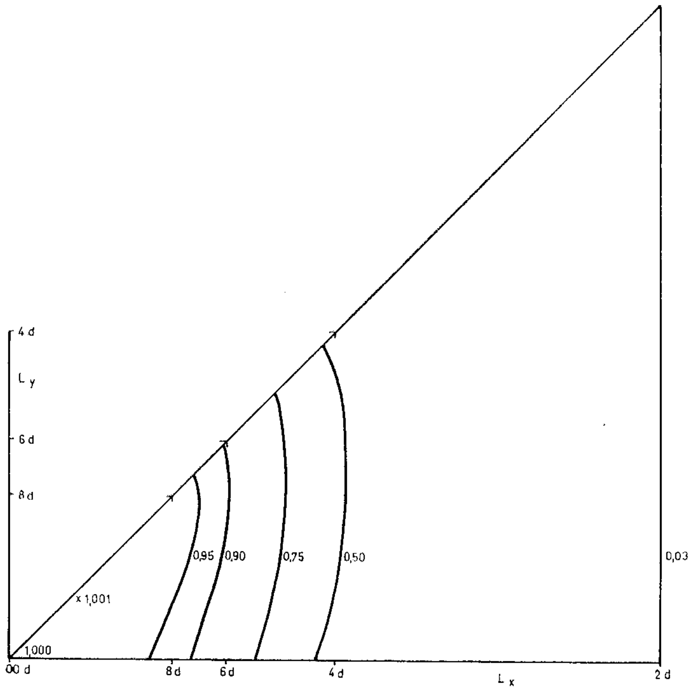


Fig. 7. The effect of 36 smoothings using the operator (II.1). 1.001 is the maximum value.

## REFERENCES

CRESSMAN, G. P., 1958: Barotropic divergence and very long atmospheric waves. *Monthly weather review.* Vol. **86,** No. 7.

DEFANT, FR., and H. TABA, 1958: The strong indexchange period from January 1 to January 7, 1956. *Tellus*, **10,** p. 225.

MINTZ, Y., 1955: Final computation of the mean geostrophic poleward flux of angular momentum and of sensible heat in the winter and summer of 1949. *General circulation project*, Final Report, Dept. of Met., Univ. of California.

PEDERSEN, K., 1958: A discussion of errors in finite difference operators substituted for the Laplace operator and the Jacobian. Report No. 2, Institute for Weather and Climate Research, Oslo.

ØKLAND, H., 1958: False dispersion as a source of integration errors. Scientific Report No. 1. Det Norske Meteorologiske Institutt.