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AN EXPERIMENT IN NUMERICAL INTEGRATION OF THE BAROTROPIC EQUATION BY A QUASI-LAGRANGIAN METHOD

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Summary. A method based on the graphical method proposed by Fjørtoft is described. A 24 and a 48 hour prognosis are compared with similar prognosis made by a more conventional procedure.

1. Introduction. The graphic procedure proposed by Fjørtoft (1952) for integrating the barotropic equation has proved to give results comparable with those obtained by other methods. The aim of the experiment which will be described in this paper, has been to find out if this method also is suited for high speed computers if properly arranged. The method may be characterized as a Lagrangian one in the sense that the movement of the individual particles is focused, whereas the Eularian method is concerned only with local changes. It is, however, not strictly Lagrangian, since no special substantial coordinates are introduced.

Numerically, the Lagrangian method differs widely from the Eulerian one. Therefore, a comparison between integrations carried out by both methods most likely will throw light upon the numerical errors. This problem has been touched upon in a previous investigation (Økland 1958), and will be studied more closely later. In this paper we shall restrict ourselves to present a comparison in a single case without any detailed discussion.

2. The differential equation. We use the barotropic equation in the following form:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\left(\nabla^2\psi + f\right) = \varkappa f^2\frac{\partial\psi}{\partial t}, \tag{1}$$

where ψ is a stream function, initially derived from the height of the 500 mb surface, f is the Coriolis parameter, $\varkappa$ is a constant, ∇ is the horizontal nabla operator, and

$$\boldsymbol{v} = \boldsymbol{k}\times\nabla\psi. \tag{2}$$

The right hand side of (1) is similar to the term introduced by CRESSMAN (1958) in order to control the very long waves.

At the boundary we assume

$$\frac{\partial \psi}{\partial t} = 0, \tag{3}$$

and further

$$\frac{\partial}{\partial t} \nabla^2 \psi = 0 \tag{4}$$

where there is inflow across the boundary, but $\nabla^2 \psi$ given by (1) where there is outflow.

Equation (1) can also be written

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)\left(\nabla^2 \psi + f - \varkappa f^2 \psi\right) = -\varkappa f^2 \boldsymbol{v} \cdot \nabla \psi - 2\varkappa f \psi \boldsymbol{v} \cdot \nabla f. \tag{5}$$

The first term on the right hand side vanishes owing to equation (2). The last term can be shown to be smaller than the Rossby term all over the hemisphere if an appropriate constant is added to ψ, and it can be neglected without introducing serious errors.

Following FJØRTOFT (1955) we now introduce a smoothed field $\psi^{(N)}$ defined by

$$\psi^{(N)} = \psi + \frac{\nabla^2 \psi}{a_N}, \tag{6}$$

where a_N is the eigenvalue corresponding to the spheric harmonic component which is entirely removed by the smoothing. Inserting from (6), equation (5) can be written as

$$\left(\frac{\partial}{\partial t} + \hat{\boldsymbol{v}} \cdot \nabla\right)\left(\nabla^2 \psi + f - \varkappa f^2 \psi\right) = 0 \tag{7}$$

where

$$\hat{\boldsymbol{v}} = \boldsymbol{k} \times \nabla \hat{\psi} \tag{8}$$

$$\hat{\psi} = \frac{1}{1 + \dfrac{\varkappa f^2}{a_N}} \left(\psi^{(N)} + \frac{f}{a_N}\right). \tag{9}$$

Equation (7) shows that the quantity

$$\nabla^2 \psi + f - \varkappa f^2 \psi \tag{10}$$

is individually conserved in a velocity field defined by (8) and (9).

9. Discussion of the method. A complete Lagrangian method would be this: At initial time the value of (10) in a regular square grid is calculated, and also the endpoints of the trajectories originating in the gridpoints and following the stream defined by $\hat{\psi}$ for a certain time $\triangle t$, which is so short that $\hat{\psi}$ can be assumed to be stationary. The value of ψ at time $\triangle t$ in the endpoints at the trajectories is then given by a rather complicated elliptic differential equation (WIIN-NIELSEN 1958).

The procedure could be iterated in the following way: The values of $\hat{\psi}$ at time $\triangle t$ is computed from the values of ψ, and assuming as before this field to be stationary, the prolongation of the trajectories corresponding to a new time interval of length $\triangle t$

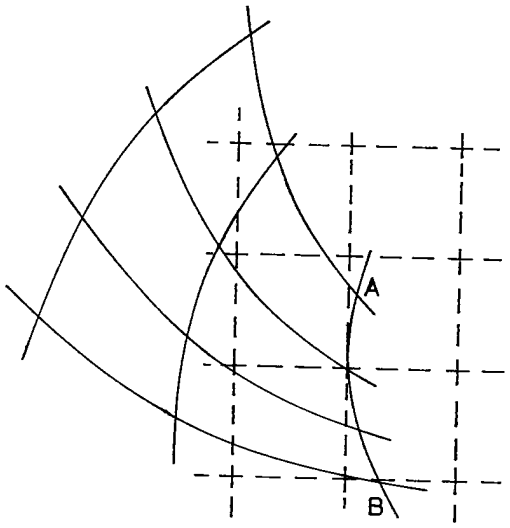

Fig. 1. A square grid (broken lines) is displaced and distorted (solid lines). In the direction AB the gridpoints have become more crowded, and consequently phenomena on smaller scales may have been created.

is calculated. The values of ψ at time $2 \triangle t$ in the new endpoints of the trajectories are then got from an elliptic equation similar to that mentioned before, but with other coefficients since they depend on the mutual position of the endpoints.

This procedure would give the value of ψ in points which are unevenly spaced over the integration area. In some places the points will have become more crowded along certain directions than they are in the initial regular grid. Here, correspondingly, the field given by (10) will have become more small-scaled as compared with the initial field. In other directions the gridpoints will have become less crowded, and (10) a more slowly varying field (Fig. 1). Hence it is seen that by considering gridpoints moving with the particles, a large density of points is obtained where this is needed for a detailed description of the field, and vice versa[1].

In this paper we shall not be concerned with this quite interesting numerical feature of the general Lagrangian method. In other words, we shall abstain from utilizing the possibility this method gives to follow the creation of phenomena on smaller scales

[1] It would, however, be necessary after some time steps to find the value of ψ in a regular grid by interpolation before the distortion had become too great. This interpolation would then necessarily involve a smoothing of the smaller scales if the number of gridpoints is kept constant.

without increasing the number of gridpoints. We can then as well describe the field at every time step by means of a regular grid, provided the grid is dense enough to describe the smallest scales which are of interest. This has further the advantage that the elliptic equation one has to solve, can be made much simpler.

One way of doing this would be to find the value of (10) in the regular grid by interpolation from the values in the endpoints of the trajectories, before solving the elliptic equation. We have, however, found it more convenient to displace the gridpoints backward in the field $\hat{\psi}$, that is to find the position of points which during the time $\triangle t$ move up to the gridpoints of the regular grid. The value of (10) at initial time must then be calculated by interpolation.

The method described in this paper differs therefore from the Eularian method only in the way by which the local changes of (10) are arrived at in a regular grid. In this connection the first point to be considered is the approximation one has to use for (10). For a reason which will be explained later, we shall proceed in the following way:

Let the value of ψ at time t be given in the gridpoints of a square grid. Drawing the parallels to the gridlines half way between them, we will get the integration area covered by squares which have the gridpoints in their middle. We denote the surface on the earth corresponding to one of this squares by σ. When displaced in the velocity field $-\hat{\boldsymbol{v}}$, this surface will be transformed into another, commonly distorted one, σ', with the same area. From (7) we then get

$$c - \int_{\sigma} \varkappa f^2 \psi d\sigma = c' - \int_{\sigma'} \varkappa f^2 \psi d\sigma + \int_{\sigma'} f d\sigma' - \int_{\sigma} f d\sigma \tag{11}$$

where c and c' are the circulations along the boundaries of the surfaces σ and σ' respectively, and where the right hand side is calculated from the value of ψ at time t. The left hand side, however, has to be expressed in terms of the unknown value of the stream function at time $t + \triangle t$.

The circulations must be approximated by some finite difference expressions. Using conformal mapping, c can be expressed by the same formula all over the map, apart from a factor which depends on the scale of the map. The simplest way is to use the common five-point formula for the Laplacian.

As to c', one must take into consideration the location of the surface σ', and the calculation will be much more complicated.

Generally the finite difference expression which one has to use in the approximation to the circulation, fails to describe all the different harmonic components of the field correctly. It is necessary, however, that the expressions for c and c' behave in the same way in this respect. If, for instance, the approximation to c' should underestimate a certain component more than the approximation to c does, the result would be an underestimation of this component in the solution.

As already mentioned, energy will in course of the integration be transferred to smaller scales. When too small, these scales will not be properly described by the values

in the gridpoints in a regular grid, and may falsely be interpreted as waves on greater scales. This implies that the smallest scales ought to be smoothed. Primarely the use of integrated values of the vorticity as introduced above, should contribute to such a smoothing. The calculation of c' also implies an interpolation in order to get the velocity at the boundary of the surface σ', and this can be done in a way that suppresses the smallest wave components.

Another advantage of using the circulation, c', instead of the vorticity, should also be stressed, namely that the circulation round a surface composed of an arbitrary number of the surface σ', is individually conserved during the time $\triangle t$.

4. The calculation. In order to simplify the calculation, we have used the five-point formula for the approximation to c. If the symbol $\nabla^2\psi$ is used for this formula, one can approximate equation (11) by

$$\nabla^2\psi - \frac{\varkappa d^2 f^2}{m^2}\psi = c' - \frac{\varkappa d^2 f'^2}{m^2}\psi' + (f' - f)\frac{d^2}{m^2}, \tag{12}$$

where d is the mesh size, and m is the scale factor of the map. Primed symbols for the stream function and the Coriolis parameter refer to the midpoint of σ', and unprimed to the gridpoint in the middle of σ.

The stream function which serves as initial data for the integration, has been calculated from the equation

$$\nabla \cdot f \nabla \psi = \nabla^2 \Phi,$$

where Φ is the geopotential of the 500 mb surface. The boundary condition for this equation is assumed to be

$$\frac{\partial \psi}{\partial s} = \frac{1}{f}\frac{\partial \Phi}{\partial s}$$

where s measures the distance along the boundary.

The first time step $\frac{1}{2}\triangle t$, has to be an uncentered one, in the sense that the given stream function is used for the computation of the trajectories. Later on the trajectories are derived from the value of ψ at time $n\frac{1}{2}\triangle t$. The time step is then made equal to $\triangle t$, and ψ at time $(n-1)\frac{1}{2}\triangle t$ is used for the computation of the right hand side of (12).

The trajectories are computed in the following way: First the velocity $-\hat{\boldsymbol{v}}$ is calculated in all gridpoints, and stored. When the velocity in an arbitrary point is needed, it is calculated from the values in the gridpoints by linear interpolation. The trajectories are computed in steps corresponding to a certain time interval δt. In order to compute one step, starting in a point with position vector $\boldsymbol{r}$, the velocity $-\hat{\boldsymbol{v}}$ in this point is first found, and then the velocity $-\hat{\boldsymbol{v}}'$ in the point $\boldsymbol{r}' = \boldsymbol{r} - \hat{\boldsymbol{v}}\,\delta t$. Then $\boldsymbol{r}_1 = \boldsymbol{r} - \frac{1}{2}(\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}')\,\delta t$ is an approximate value of the position after the time δt.

The procedure can obviously be iterated, using now instead of $-\hat{\boldsymbol{v}}'$, the velocity in the point $\boldsymbol{r}_1$. It has shown up, however, that with δt equal to or less than 4 hours, $\boldsymbol{r}_1$ is a sufficiantly accurate value.

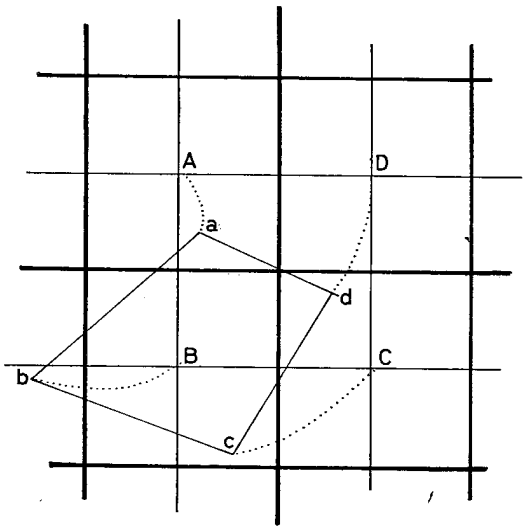

Fig. 2. Heavy lines are the regular grid. The square ABCD is displaced and transformed into a new surface which is approximated by abcd.

In this way the endpoints of the trajectories are calculated, starting in the corners of the surfaces σ, and the surfaces σ' are taken to be the quadrangles determined by these endpoints (Fig. 2). The circulation along the boundaries of σ' is then approximated by

$$\sum_{i=1}^{4} \boldsymbol{l}_i \cdot \boldsymbol{v}_i$$

where $\boldsymbol{l}_i$ is one of the sides, and $\boldsymbol{v}_i$ the velocity according to (2) in the midpoint of this side, interpolated from the values of ψ in the gridpoints.

The interpolation is done by means of polynomials, and it has been decided that this interpolation should reduce phenomena with wavelength smaller than *3d*, but maintain the greater ones. In order to attain this, one has to use polynomials of rather high degree, and the calculation of the coefficient will be very laborious. The procedure has therefore been simplified in the way that only 4 fixed sets of coefficients are used corresponding to 4 certain locations in the grid (Fig. 3). Then for an arbitrary point that set of coefficients is used which corresponds to the nearest of the 4 fixed locations. The error which is introduced in this way has no systematic dependence on the different scales of the field.

In order to get close correspondence with the five-point operator for the Laplacian, the veolocity components are to be calculated as the difference in ψ over a distance equal to d.

A fifth degree polynominal has been used when the interpolation is needed only in one direction. When interpolation in two directions is needed, the calculation is done in two steps by means of the same fifth degree polynominal.

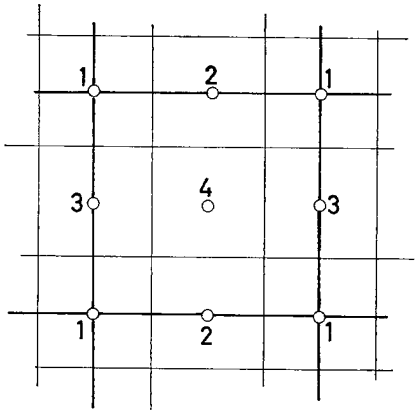

Fig. 3. This figure illustrates the 4 different types of interpolation. The thin lines divide the map in a number of squares. The set of coefficients used in the interpolation is that which corresponds to the midpoit of the square in which the arbitrary point is situated.

The Helmholtz equation has been solved by relaxation.

The boundary condition (3) is fulfilled in the usual way by holding the value of ψ unchanged in all the gridpoints along the boundary.

Condition (4) is formulated in the following way:

For the inner gridpoints we calculate c' in the described way when all the corners of the corresponding quadrangle are situated inside the boundary. In order to calculate $\boldsymbol{v}$ from the ψ field in points situated near the boundary, the integration area is assumed to be extended outside the boundary with a sufficient number of gridpoints, and the value of ψ in these points is supposed to be the same as in the nearest gridpoint at the boundary.

When one or more of the corners of a quadrangel σ' are situated outside the boundary, the right hand side of (12) is replaced by the Laplacian in the corresponding gridpoint, calculated by means of the usual five-point formula.

A polar stereographic map is used, and the grid size is taken to be 600 km at 60°N. The grid consists of 781 points, forming an octagon which covers the area north of about 15°N.

5. Results. A 24 and a 48 hour prognosis have been compared with those computed by PEDERSEN (1959) by means of an Eulerian method. The same stream function is used as initial data. Fig. 4 shows computed 48 hours changes of the stream function and Fig. 5 gives the difference between the Lagrangian and the Eulerian prediction. In the table below are given some statistical parametres concerning the results:

Statistical parametres for observed changes in the stream function (index R), changes computed by the Lagrangian method (index L), and changes computed by the Eurelian method (index E). σ is the standard deviation, and r the correlation coefficient. $\sigma_{(L-E)}$ is the standard deviation of the difference between tendensies computed by the two methods. Values in the 163 gridpoints situated north of 40° N have been used.

	σ_R	σ_L	σ_E	$\sigma_{(L-E)}$	r_{LR}	r_{ER}	r_{LE}
24 hours	140	111	105	47	0.666	0.646	0.908
48 hours	187	153	134	73	0.546	0.518	0.878

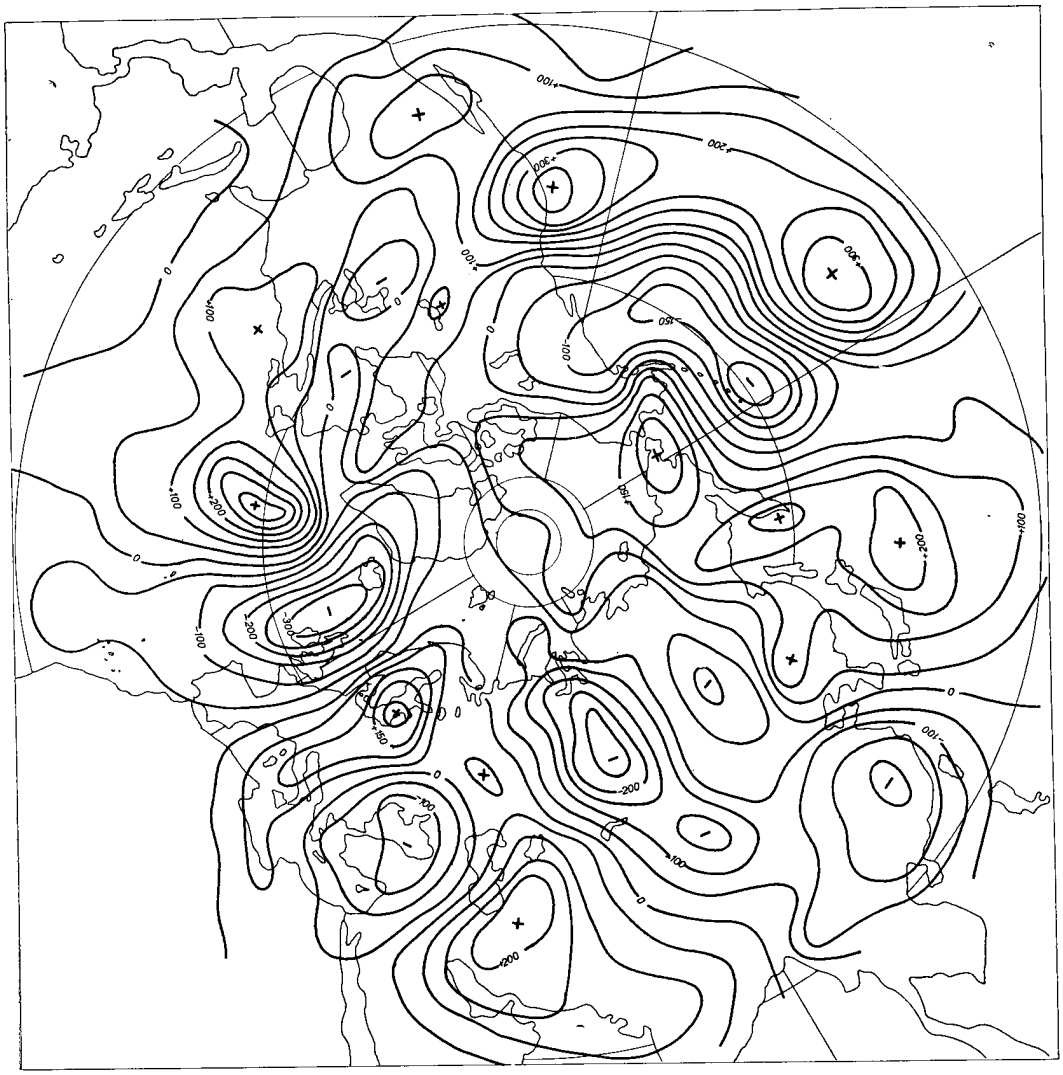

Fig. 4. Computed 48 hours change of the stream function.

Unavoidable mathematical errors will be associated with both methods, but since the methods are so different, one must believe that the errors will influence the result in quite different ways.

However, since nevertheless the two methods in the described case give practically the same results, one can draw the conclusion that the mathematical errors associated with both methods are small in this case.

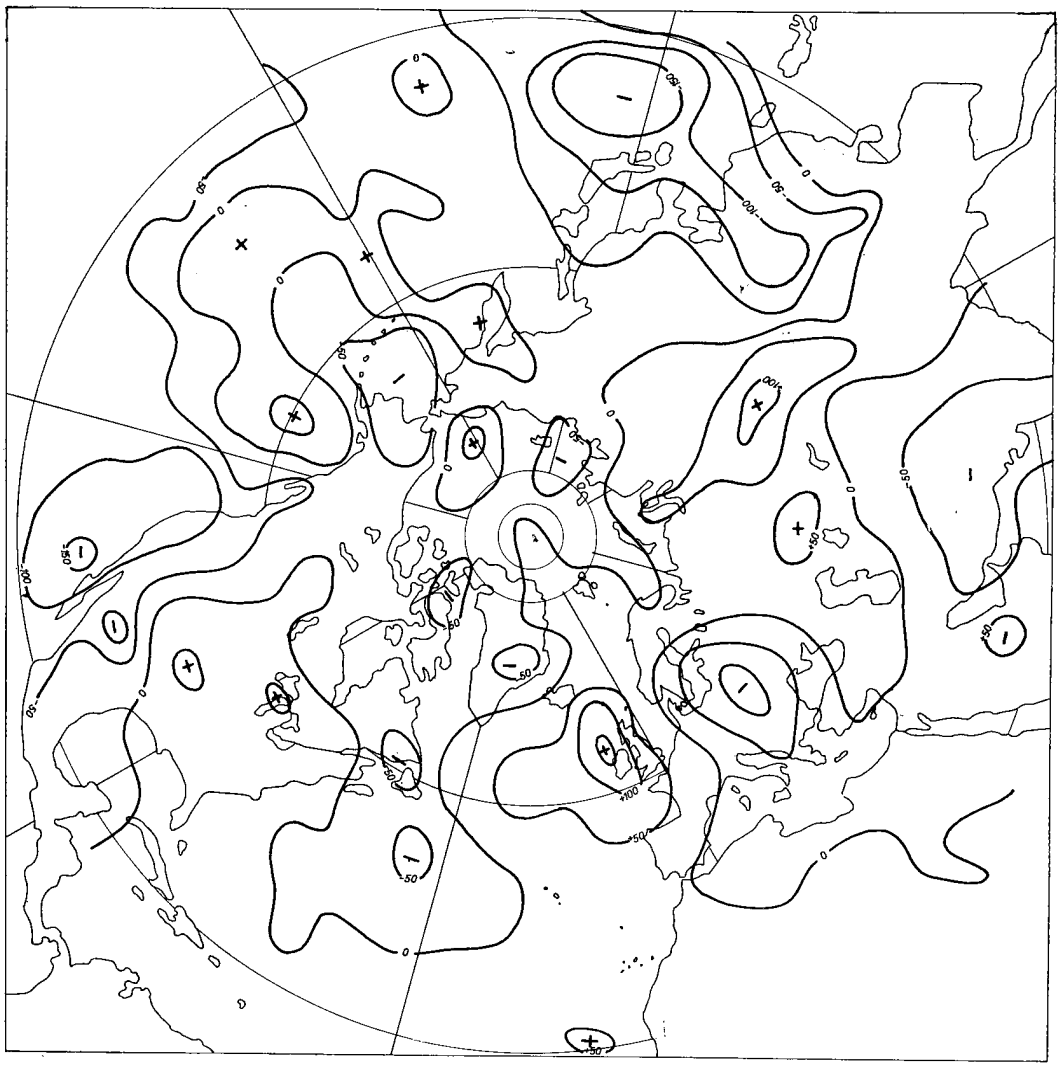

Fig. 5. Stream function difference between the Lagrangian and the Eularian prediction.

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