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# THE STABILITY OF COUETTE-FLOW IN NON-STRATIFIED AND STRATIFIED VISCOUS FLUIDS


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**Summary.** Throughout this paper, small perturbations of a plane Couette-flow in a viscous fluid are discussed. Section 4 deals with an incompressible homogeneous fluid with the result that the Couette-flow is stable for small perturbations. The amplitude of the perturbation velocities decreases with respect to time. In section 5–7, we find that in a stratified fluid, we can always have neutral wave-solutions for sufficiently large values of the Reynolds number, however small the value of the static stability. In section 6, a curve in a $(\varkappa, R)$-plane is found along which we have neutral wave-perturbations. Here $\varkappa$ is the wave-number and $R$ is the Reynolds number. The wave-solutions for values of $\varkappa$ and $R$ on both sides of the curve of neutral stability are, however, stable.


**1. Introduction.** The problem of a theoretical determination of the transition to turbulence has lead a number of workers to study the stability, with respect to infinitesimal disturbances, of laminar motion of viscous fluids. For a detailed list of authors see C. C. LIN [1].

In the case of a homogeneous fluid, the linearized differential equation determining the streamfunction is a regular equation having four independent solutions which are analytical functions of the vertical coordinate $z$. In the case of plane Couette-flow it is more convenient for discussion to use asymptotic expansions of the stream-function. This case has been discussed by L. HOPF [2], W. WASOW [3] and others. All investigations tend to show that the flow is stable. In the case of flows with curved velocity profiles it is more convenient for discussion to find the streamfunction by use of asymptotic series. In an important paper, W. HEISENBERG [4] has given two asymptotic methods for this purpose. C. C. LIN [5] has given a more detailed discussion of the validity of the asymptotic series obtained by these methods and of the "crossing substitution" in order to find the correct asymptotic series in different regions in the complex $z$-plane. Flows with velocity profiles of the symmetrical type or of the boundary-layer type are found to be unstable for sufficiently large values of the Reynolds number. Furthermore, a curve of neutral stability divides the ($\varkappa$, $R$)-plane in a region of instability and a region of stability; $R$ is the Reynolds number and $\varkappa$ is the wave-number.

In the case of a stratified fluid, however, the streamfunction has a logarithmic singularity at the point where $u_0 - c = 0$ ($u_0$ is the mean velocity and $c$ the velocity of propagation). H. SCHLICHTING [6] has discussed this case for flows of the boundary-layer type. He found that the effect of the static stability of a stably stratified fluid is to diminish the region of instability mentioned above in the case of a homogeneous fluid.

The present paper deals with the stability of plane Couette-flow of an incompressible viscous fluid. Sections 2 and 3 give the mathematical formulation of the problem. In section 4, the case of a homogeneous fluid is considered. HOPF found an infinite set of stable wave-solutions in this case for a given value of the wave-number and of the Reynolds number. The least stable wave-solutions found by HOPF are, however, different from the corresponding wave-solutions which will be found in the present paper; the approximations made by HOPF will be shown to be incorrect for these solutions. The case of a stably stratified fluid is dealt with in sections 5, 6 and 7. In sections 5 and 6 the static stability is assumed to be small, whereas section 7 gives a brief discussion in the case of finite values of the static stability. A comparison with the inviscid case is given in section 8. In the inviscid case the velocity and the vorticity become infinite at the singular point ($u_0 - c = 0$). The development of an arbitrary infinitesimal disturbance can, however, be found by a method given by A. ELIASSEN, E. HØILAND and E. RIIS [7].

**2. The perturbation equations.** The velocity of the mean flow is given by

$$u_0 = \frac{du_0}{dz} z, \tag{2.1}$$

where $\frac{du_0}{dz}$ is a constant (Fig. 1), $z$ is the vertical coordinate and the flow is confined between two rigid, horizontal planes in relative motion. The planes are situated at $z = 0$ and $z = h$. The density $\varrho_0$ of the mean flow, assumed to decrease exponentially with height, is given by

$$\frac{1}{\varrho_0}\frac{d\varrho_0}{dz} = -\gamma, \tag{2.2}$$

where $\gamma$ is a constant.

Small perturbations are superimposed on the mean flow. Assuming the fluid to be incompressible and the motion to be two-dimensional, the linearized equations determining the motion are

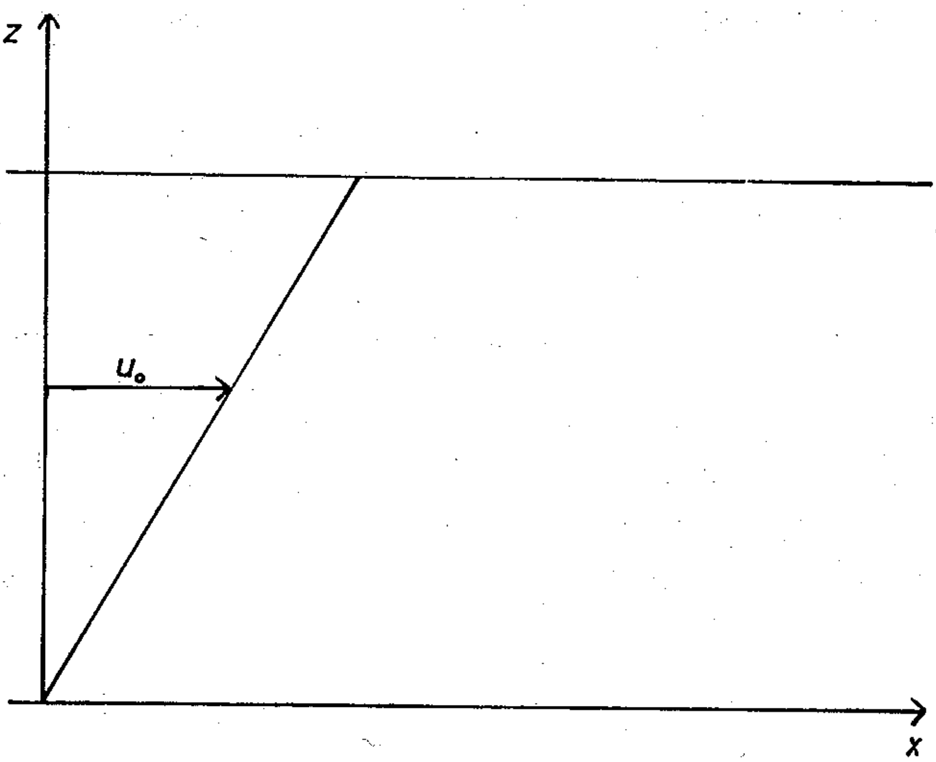


*Fig. 1.* The mean flow.

$$\varrho_0\left[\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\boldsymbol{v}+w\frac{du_0}{dz}\boldsymbol{i}\right]=-\nabla p-g\varrho\,\boldsymbol{k}+\nu\varrho_0\nabla^2\boldsymbol{v}, \tag{2.3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\,, \tag{2.4}$$

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\varrho+w\frac{d\varrho_0}{dz}=0\,, \tag{2.5}$$

where $\varrho$, $p$ and $\boldsymbol{v}=u\,\boldsymbol{i}+w\,\boldsymbol{k}$ are perturbation quantities and $\nabla=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{k}\frac{\partial}{\partial z}$. Elimination of $p$ from equation (2.3) gives

$$\begin{aligned}&\nabla\varrho_0\times\left[\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\boldsymbol{v}+w\frac{du_0}{dz}\boldsymbol{i}-\nu\nabla^2\boldsymbol{v}\right]\\&+\varrho_0\nabla\times\left[\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\boldsymbol{v}+w\frac{du_0}{dz}\boldsymbol{i}-\nu\nabla^2\boldsymbol{v}\right]=-g\nabla\varrho\times\boldsymbol{k}\,.\end{aligned} \tag{2.6}$$

Normally the first term will be small compared to the second term for sufficiently small values of $\gamma$. Considering periodic disturbances, we introduce the streamfunction $\Psi(z)e^{ik(x-ct)}$ defined by

$$u = \frac{d\Psi}{dz} e^{ik(x-ct)} , \qquad w = - ik\Psi e^{ik(x-ct)} . \tag{2.7}$$

Furthermore we eliminate $\varrho$ from equations (2.5) and (2.6). The equation for the streamfunction can then be written as

$$(u_0 - c)^2 (\Psi'' - k^2\Psi) - \frac{\nu}{ik} (u_0 - c) (\Psi'''' - 2k^2\Psi'' + k^4\Psi) - g\gamma\Psi = \\ - \gamma (u_0 - c)\left[(u_0 - c) \Psi' - \frac{du_0}{dz}\Psi - \frac{\nu}{ik} (\Psi''' - k^2\Psi')\right] . \tag{2.8}$$

The kinematical coefficient of viscosity $\nu$, has been treated as a constant. The term on the right hand side of equation (2.8) is due to the first term in equation (2.6). Introducing non-dimensional quantities, equation (2.8) can be written as

$$(z_1 - c_1)^2 (\psi'' - \varkappa^2\psi) + \frac{i (z_1 - c_1)}{\varkappa R} (\psi'''' - 2\varkappa^2\psi'' + \varkappa^4\psi) + s\psi = \\ s \frac{h}{g}\left(\frac{du_0}{dz}\right)^2 (z_1 - c_1)\left[(z_1 - c_1) \psi' - \psi + \frac{i}{\varkappa R} (\psi''' - \varkappa^2\psi')\right] , \tag{2.9}$$

where

$$s = - \frac{g\gamma}{\left(\frac{du_0}{dz}\right)^2} , \; z_1 = \frac{z}{h} , \; \varkappa = kh, \; c_1 = \frac{c}{h\frac{du_0}{dz}} , \tag{2.10}$$

$$R = \frac{h^2 \frac{du_0}{dz}}{\nu} , \; \psi (z_1) = \Psi (z) ,$$

$R$ is the Reynolds number and $s$ is the Richardson number. Equation (2.9) is the equation found by H. Schlichting [6], if we put $u'' = 0$ in his equation. In a homogeneous fluid $s = 0$, and equation (2.9) reduces to the Orr-Sommerfeld equation

$$(z_1 - c_1) (\psi'' - \varkappa^2\psi) + \frac{i}{\varkappa R} (\psi'''' - 2\varkappa^2\psi'' + \varkappa^4\psi) = 0 . \tag{2.11}$$

We introduce a new independent variable $\zeta$ given by

$$\zeta = (\varkappa R)^{1/3} (z_1 - c_1) - i\delta^2 , \tag{2.12}$$

where

$$\delta = \left(\frac{\varkappa^2}{R}\right)^{\frac{1}{3}} . \tag{2.13}$$

Equation (2.9) obtains

$$i(\zeta + i\delta^2)\,\Phi'''' + (\zeta + i\delta^2)(\zeta - i\delta^2)\,\Phi'' + [s - \delta^2\zeta(\zeta + i\delta^2)]\,\Phi = \tag{2.14}$$
$$(\varkappa R)^{-1/3}\, s\,\frac{h}{g}\left(\frac{du_0}{dz}\right)^2 (\zeta + i\delta^2)\,[(\zeta + i\delta^2)\,\Phi' - \Phi + i\,(\Phi''' - \delta^2\Phi')]\,,$$

where

$$\Phi(\zeta) = \psi(z_1). \tag{2.15}$$

If $\Phi$ is expanded in power series in $(\varkappa R)^{-1/3}$, we have as a first approximation to equation (2.14)

$$i(\zeta + i\delta^2)\,\Phi'''' + (\zeta + i\delta^2)(\zeta - i\delta^2)\,\Phi'' + [s - \delta^2\zeta(\zeta + i\delta^2)]\,\Phi = 0\,, \tag{2.16}$$

which is equivalent to neglecting the first term in equation (2.6). In the following discussion we shall assume that equation (2.16) can be utilized to find the stream-function with a sufficient degree of accuracy.

In section 4, long wave-solutions ($\varkappa \ll 1$) will be found. Assuming the Reynolds number $R$ so large that $\varkappa R$ is finite or even large compared to unity, we put $\varkappa = 0$ in equation (2.9) except in those terms where $\varkappa$ is multiplied by $R$. This shows that the first approximations of equation (2.16) and (2.12) are

$$i\zeta\Phi'''' + \zeta^2\Phi'' + s\Phi = 0\,, \tag{2.17}$$

$$\zeta = (\varkappa R)^{1/3}\,(z_1 - c_1). \tag{2.18}$$

**3. The frequency equation.** The boundary conditions are

$$\Psi(z) = \Psi'(z) = 0 \text{ for } z = 0 \text{ and } z = h.$$

These conditions are equivalent to

$$\Phi(\zeta) = \Phi'(\zeta) = 0 \text{ for } \zeta = \zeta_0 \text{ and } \zeta = \zeta_1\,, \tag{3.1}$$

where

$$\zeta_0 = -(\varkappa R)^{1/3}\,c_1 - i\delta^2,\ \zeta_1 = (\varkappa R)^{1/3}\,(1 - c_1) - i\delta^2\,. \tag{3.2}$$

If $\Phi_j(\zeta)$, $j = 1, 2, 3$ or $4$, are four independent solutions of equation (2.16), the frequency equation determining $c_1$ can be written as

$$\begin{vmatrix} \Phi_{10} & \Phi_{20} & \Phi_{30} & \Phi_{40} \\ \Phi'_{10} & \Phi'_{20} & \Phi'_{30} & \Phi'_{40} \\ \Phi_{11} & \Phi_{21} & \Phi_{31} & \Phi_{41} \\ \Phi'_{11} & \Phi'_{21} & \Phi'_{31} & \Phi'_{41} \end{vmatrix} = 0\,, \tag{3.3}$$

where $\Phi_{jk} = \Phi_j(\zeta_k)$ and $\Phi'_{jk} = \Phi'_j(\zeta_k)$.

The wave-solutions are

$$\left.\begin{array}{l}\text{stable}\\ \text{neutral}\\ \text{unstable}\end{array}\right\} \text{ as } Im(c_1) \lesseqgtr 0 \, .$$

Since

$$\zeta_1 - \zeta_0 = (\varkappa R)^{1/3} \, , \tag{3.4}$$

$\zeta_0$ and $\zeta_1$ must lie on a straight line parallel to the real axis in the complex $\zeta$-plane. When $z$ increases from 0 to $h$, $\zeta$ will vary from $\zeta_0$ to $\zeta_1$ along this straight line. Acccordingly we have

$$\begin{cases}\text{stability waves when} & 0 < \arg(\zeta_1 + i\delta^2) < \arg(\zeta_0 + i\delta^2) < \pi \, ,\\ \text{neutral waves when} & \arg \zeta_0 = \pm \pi \text{ or } 0 \, ,\\ \text{instability waves when} & -\pi < \arg(\zeta_0 + i\delta^2) < \arg(\zeta_1 + i\delta^2) < 0 \, .\end{cases} \tag{3.5}$$

In a homogeneous fluid, $s = 0$, the equation for the stream-function (2.16) has no singularity for $\zeta = -i\delta^2$. In this case the condition for instability (3.5) can be replaced by

(3.6) instability waves for $s = 0$ when $\arg(\zeta_0 + i\delta^2) > \pi$ and $\arg(\zeta_1 + i\delta^2) < 0$ .

Putting

$$f_{jk}(\zeta) = \begin{vmatrix} \Phi_j(\zeta) & \Phi_k(\zeta) \\ \Phi_j'(\zeta) & \Phi_k'(\zeta) \end{vmatrix} \, , \tag{3.7}$$

the frequency equation (3.3) can be written as

$$\begin{aligned} & f_{12}(\zeta_0) f_{34}(\zeta_1) - f_{13}(\zeta_0) f_{24}(\zeta_1) + f_{14}(\zeta_0) f_{23}(\zeta_1) \\ & + f_{23}(\zeta_0) f_{14}(\zeta_1) - f_{24}(\zeta_0) f_{13}(\zeta_1) + f_{34}(\zeta_0) f_{12}(\zeta_1) = 0 \, . \end{aligned} \tag{3.8}$$

**4. Homogeneous incompressible fluid.** Some of the terms in the frequency equation which will be developed in this section did not occur in the frequency equation discussed by Hopf. These terms will be most dominant for long wave-perturbations. We shall therefore first discuss the case when $\varkappa \ll 1$. Since $s = 0$ in the case of a homogeneous fluid, the equation for the streamfunction is approximately given by equation (2.17) with $s = 0$,

$$i\zeta\Phi'''' + \zeta^2\Phi'' = 0, \tag{4.1}$$

where

$$\zeta = (\varkappa R)^{1/3} (z_1 - c_1). \tag{4.2}$$

The well-known solutions of equation (4.1) are

$$\Phi_1 = \zeta,\ \Phi_2 = 1,\ \Phi_3 = \int_{\zeta_0}^{\zeta} (\zeta - x)\, F_1(x)\, dx,\ \Phi_4 = \int_{\zeta_0}^{\zeta} (\zeta - x)\, F_2(x)\, dx\,, \tag{4.3}$$

where

$$\begin{cases} F_1(x) = x^{\frac{1}{2}} H_{1/3}{}^{(1)}\left(\dfrac{2}{3} x^{3/2} e^{-i\frac{\pi}{4}}\right), \\ F_2(x) = x^{\frac{1}{2}} H_{1/3}{}^{(2)}\left(\dfrac{2}{3} x^{3/2} e^{-i\frac{\pi}{4}}\right). \end{cases} \tag{4.4}$$

$F_1(x)$ and $F_2(x)$ are solutions of the equation

$$i\,F''(x) + xF(x) = 0\,. \tag{4.5}$$

$H_{1/3}{}^{(1)}$ and $H_{1/3}{}^{(2)}$ are the Hankel-functions. As lower limits in the integrals in the solutions (4.3) we have chosen $\zeta_0$, the value of $\zeta$ at the lower plane. The frequency equation (3.3) is found to be

$$\int_{\zeta_0}^{\zeta_1} xF_1(x)\, dx \int_{\zeta_0}^{\zeta_1} F_2(x)\, dx - \int_{\zeta_0}^{\zeta_1} F_1(x)\, dx \int_{\zeta_0}^{\zeta_1} xF_2(x)\, dx = 0\,. \tag{4.6}$$

The values of $\zeta_0$ and $\zeta_1$ are given by equation (4.2)

$$\zeta_0 = -(\varkappa R)^{1/3} c_1,\ \zeta_1 = (\varkappa R)^{1/3} (1 - c_1). \tag{4.7}$$

In order to evalute the integrals in equation (4.6) for large values of $|\zeta_0|$ and $|\zeta_1|$, we shall find the asymptotic expansions of $F_1(x)$ and $F_2(x)$. The asymptotic expansions of the Hankel-functions are ([8], page 198)

$$\left.\begin{aligned} H_{1/3}{}^{(1)}(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i\left(z - \frac{5\pi}{12}\right)} \sum_{n=o} \frac{(-1)^n \left(\frac{1}{3}, n\right)}{(2iz)^n}, \\ H_{1/3}{}^{(2)}(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i\left(z - \frac{5\pi}{12}\right)} \sum_{n=o} \frac{\left(\frac{1}{3}, n\right)}{(2iz)^n}, \end{aligned}\right\} \text{valid when } -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}\,.$$

The asymptotic expansions of $F_1(x)$ and $F_2(x)$ are then found to be

$$\left.\begin{cases} F_1(x) = \mathcal{N}_1(x), \\ F_2(x) = \mathcal{N}_2(x), \end{cases}\right\} \text{valid when } -\frac{\pi}{6} \leq \arg x \leq \frac{\pi}{2}\,, \tag{4.8}$$

where

$$(4.9)\quad \begin{cases} N_1(x) = \sqrt{\dfrac{3}{\pi}}\, e^{-i\frac{7\pi}{24}}\, e^{\frac{2}{3}x^{\frac{3}{2}}e^{i\frac{\pi}{4}}} \sum\limits_{n=o} (-1)^n \left(\dfrac{3}{4}\right)^n \left(\dfrac{1}{3}, n\right) e^{-i\frac{\pi}{4}n}\, x^{-\frac{3}{2}n-\frac{1}{4}} , \\[2ex] N_2(x) = \sqrt{\dfrac{3}{\pi}}\, e^{i\frac{13\pi}{24}}\, e^{-\frac{2}{3}x^{\frac{3}{2}}e^{i\frac{\pi}{4}}} \sum\limits_{n=o} \left(\dfrac{3}{4}\right)^n \left(\dfrac{1}{3}, n\right) e^{-i\frac{\pi}{4}n}\, x^{-\frac{3}{2}n-\frac{1}{4}} . \end{cases}$$

To find the asymptotic expansions for other values of arg $x$, we utilize the relations ([8], page 75)

$$H_{1/3}^{(1)}(z) = \frac{\sin(1-m)\frac{\pi}{3}}{\sin\frac{\pi}{3}} H_{1/3}^{(1)}(z\,e^{-im\pi}) - e^{-i\frac{\pi}{3}} \frac{\sin\frac{m\pi}{3}}{\sin\frac{\pi}{3}} H_{1/3}^{(2)}(ze^{-im\pi}) ,$$

$$H_{1/3}^{(2)}(z) = \frac{\sin(1+m)\frac{\pi}{3}}{\sin\frac{\pi}{3}} H_{1/3}^{(2)}(ze^{-im\pi}) + e^{i\frac{\pi}{3}} \frac{\sin\frac{m\pi}{3}}{\sin\frac{\pi}{3}} H_{1/3}^{(1)}(ze^{-im\pi}) ,$$

where $m$ is any positive or negative integer. The corresponding relations for $F_1(x)$ and $F_2(x)$ are found to be

$$(4.10)\quad \begin{cases} F_1(x) = e^{i\frac{m\pi}{3}} \left[ \dfrac{\sin(1-m)\frac{\pi}{3}}{\sin\frac{\pi}{3}} F_1\left(xe^{-i\frac{2m\pi}{3}}\right) - e^{-i\frac{\pi}{3}} \dfrac{\sin\frac{m\pi}{3}}{\sin\frac{\pi}{3}} F_2\left(xe^{-i\frac{2m\pi}{3}}\right) \right] , \\[2ex] F_2(x) = e^{i\frac{m\pi}{3}} \left[ \dfrac{\sin(1+m)\frac{\pi}{3}}{\sin\frac{\pi}{3}} F_2\left(xe^{-i\frac{2m\pi}{3}}\right) + e^{i\frac{\pi}{3}} \dfrac{\sin\frac{m\pi}{3}}{\sin\frac{\pi}{3}} F_1\left(xe^{-i\frac{2m\pi}{3}}\right) \right] . \end{cases}$$

Putting $m = 1$ the above equations read

$$(4.11)\quad \begin{cases} F_1(x) = -F_2\left(xe^{-i\frac{2\pi}{3}}\right) , \\ F_2(x) = e^{i\frac{\pi}{3}} F_2\left(xe^{-i\frac{2\pi}{3}}\right) - e^{-i\frac{\pi}{3}} F_1\left(xe^{-i\frac{2\pi}{3}}\right) . \end{cases}$$

Utilizing the asymptotic expansions (4.8) on the right hand sides of relations (4.11), we get

$$\left.\begin{aligned} F_1(x) &= -N_2\left(xe^{-i\frac{2\pi}{3}}\right) , \\ F_2(x) &= e^{i\frac{\pi}{3}} N_2\left(xe^{-i\frac{2\pi}{3}}\right) - e^{-i\frac{\pi}{3}} N_1\left(xe^{-i\frac{2\pi}{3}}\right) , \end{aligned}\right\} \begin{array}{l} \text{valid when} \\ \frac{\pi}{2} \le \arg x \le \frac{7\pi}{6} . \end{array}$$

These equations can be transformed further by equations (4.9) to

$$(4.12) \qquad \left.\begin{cases} F_1(x) = N_1(x) , \\ F_2(x) = N_2(x) - e^{i\frac{\pi}{3}} N_1(x) , \end{cases}\right\} \text{valid when } \frac{\pi}{2} \leq \arg x \leq \frac{7\pi}{6} .$$

In the following discussion we shall assume

$$(4.13) \qquad \frac{\pi}{2} \leq \arg \zeta_0 \leq \frac{7\pi}{6} , \quad -\frac{\pi}{6} \leq \arg \zeta_1 \leq \frac{\pi}{2} .$$

If these conditions are fulfilled, $\arg \zeta_1 > 0$ and $\arg \zeta_1 < 0$ correspond to stable and unstable wave-perturbations respectively.

To demonstrate a mistake made by HOPF, we shall evaluate the asymptotic values of $\int_{\zeta_0}^{\zeta_1} F_2 dx$ for large values of $|\zeta_1|$ and $|\zeta_0|$. According to equations (4.8) and (4.12) we have approximately

$$\int_{\zeta_0}^{\zeta_1} F_2 dx = \int_{\zeta_0}^{Ae^{i\frac{\pi}{2}}} [N_2(x) - e^{i\frac{\pi}{3}} N_1(x)] \, dx + \int_{Ae^{i\frac{\pi}{2}}}^{\zeta_1} N_2(x) \, dx ,$$

where $A \gg 1$.
Hence

$$\int_{\zeta_0}^{\zeta_1} F_2 \, dx = \int_{\zeta_0}^{\zeta_1} N_2 \, dx - e^{i\frac{\pi}{3}} \int_{\zeta_0}^{\infty e^{i\frac{\pi}{2}}} N_1 \, dx .$$

If $N_2$ is approximated by the first term in the last of expansions (4.9), we have to consider

$$\int_{\zeta_0}^{\zeta_1} x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}} e^{i\frac{\pi}{4}}} dx .$$

Integration by parts gives

$$\int_{\zeta_0}^{\zeta_1} x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}} e^{i\frac{\pi}{4}}} dx = e^{-i\frac{\pi}{4}} \Big[ - \zeta_1^{-\frac{3}{4}} e^{-\frac{2}{3} \zeta_1^{\frac{3}{2}} e^{i\frac{\pi}{4}}} + \zeta_0^{-\frac{3}{4}} e^{-\frac{2}{3} \zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}}$$

$$- \frac{3}{4} e^{-i\frac{\pi}{4}} \int_{\zeta_0}^{\zeta_1} x^{-\frac{7}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}} e^{i\frac{\pi}{4}}} dx \Big] .$$

HOPF neglected the last term in this equation. This can obviously lead to erroneous results. The first two terms on the right hand side of the equation will for instance tend to zero when $\zeta_0 \to \infty e^{i(\pi-\varphi)}$ and $\zeta_1 \to \infty e^{i\varphi}$ with $-\frac{\pi}{6} < \varphi < \frac{\pi}{6}$, whereas

$$\int_{\zeta_0}^{\zeta_1} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}} e^{i\frac{\pi}{4}}} dx = \int_{\zeta_0}^{0} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}} e^{i\frac{\pi}{4}}} dx + \int_{0}^{\zeta_1} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}} e^{\frac{i\pi}{4}}} dx$$

is easily seen to tend towards a finite constant in this case.

To find approximate values of the integrals in equation (4.6) for large values of $|\zeta_0|$ and $|\zeta_1|$, we shall therefore use another method. We first notice that

$$\int_{\zeta_0}^{\zeta_1} xF_j(x)\,dx = -i\left[F_j'(\zeta_1) - F_j'(\zeta_0)\right], \qquad j = 1,2$$

since $F_1$ and $F_2$ are solutions of equation (4.5). The frequency equation (4.6) can be written as

$$(4.14) \qquad \left[F_1'(\zeta_1) - F_1'(\zeta_0)\right]\left(\int_0^{\zeta_1} F_2 dx - \int_0^{\zeta_0} F_2 dx\right) - \left[F_2'(\zeta_1) - F_2'(\zeta_0)\right]$$

$$\cdot\left(\int_0^{\zeta_1} F_1 dx - \int_0^{\zeta_0} F_1 dx\right) = 0 .$$

Utilizing equations (4.11), we make the following transformations

$$\int_0^{\zeta_0} F_1 dx = e^{-i\frac{\pi}{3}} \int_0^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_2 dx .$$

$$\int_0^{\zeta_0} F_2 dx = -\int_0^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_2 dx - e^{i\frac{\pi}{3}} \int_0^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_1 dx .$$

The first of these equations shows that

$$(4.15) \qquad \int_0^{\infty e^{i\frac{\pi}{2}}} F_1 dx = e^{-i\frac{\pi}{3}} \int_0^{\infty e^{-i\frac{\pi}{6}}} F_2 dx .$$

Utilizing this relation, $\int_0^{\zeta_0} F_1 dx$ and $\int_0^{\zeta_0} F_2 dx$ can further be transformed to

$$\int_0^{\zeta_0} F_1 dx = \int_0^{\infty e^{i\frac{\pi}{2}}} F_1 dx + e^{-i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_2 dx ,$$

$$\int_0^{\zeta_0} F_2 dx = -2e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} F_1 dx - e^{i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_1 dx - \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_2 dx .$$

Furthermore $F_1' (\zeta_0)$ and $F_2' (\zeta_0)$ can, by equations (4.11), be transformed to

$$F_1' (\zeta_0) = e^{i\frac{\pi}{3}} F_2' (\zeta_0 e^{-i\frac{2\pi}{3}}) ,$$

$$F_2' (\zeta_0) = e^{-i\frac{\pi}{3}} F_2' (\zeta_0 e^{-i\frac{2\pi}{3}}) + F_1' (\zeta_0 e^{-i\frac{2\pi}{3}}) .$$

By these transformations, the frequency equation (4.14) can be written as

$$\text{(4.16)} \quad \left[F_1' (\zeta_1) - e^{i\frac{\pi}{3}} F_2' (\zeta_0 e^{-i\frac{2\pi}{3}})\right] \left[C + \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_1} F_2 dx + e^{i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_1 dx \right.$$

$$\left. + \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_2 dx \right] - \left[F_2' (\zeta_1) - e^{-i\frac{\pi}{3}} F_2' (\zeta_0 e^{-i\frac{2\pi}{3}}) - F_1' (\zeta_0 e^{-i\frac{2\pi}{3}})\right]$$

$$\left[\int_{\infty e^{i\frac{\pi}{2}}}^{\zeta_1} F_1 dx - e^{-i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} F_2 dx\right] = 0,$$

where

$$\text{(4.17)} \qquad C = 3e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} F_1 dx = 2\sqrt{3} e^{i\frac{5\pi}{12}} .$$

The value of the constant $C$ is exact and evaluated in Appendix A.

Because of assumption (4.13), $F_1$ and $F_2$ in equation (4.16) can by equation (4.8) be approximated by $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively for large values of $|\zeta_0|$ and $|\zeta_1|$. From expansions (4.9) we find

(4.18)
$$\left\{\begin{aligned}
\mathcal{N}_1(\zeta) &= \sqrt{\frac{3}{\pi}}\, e^{-i\frac{7\pi}{24}}\, e^{\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \left[\zeta^{-\frac{1}{4}} + \frac{5}{48} e^{-i\frac{\pi}{4}} \zeta^{-\frac{7}{4}} + 0\left(\zeta^{-\frac{13}{4}}\right)\right],\\
\mathcal{N}_2(\zeta) &= \sqrt{\frac{3}{\pi}}\, e^{i\frac{13\pi}{24}}\, e^{-\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \left[\zeta^{-\frac{1}{4}} - \frac{5}{48} e^{-i\frac{\pi}{4}} \zeta^{-\frac{7}{4}} + 0\left(\zeta^{-\frac{13}{4}}\right)\right],\\
\mathcal{N}_1'(\zeta) &= \sqrt{\frac{3}{\pi}}\, e^{-i\frac{\pi}{24}}\, e^{\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \left[\zeta^{\frac{1}{4}} - \frac{7}{48} e^{-i\frac{\pi}{4}} \zeta^{-\frac{5}{4}} + 0\left(\zeta^{-\frac{11}{4}}\right)\right],\\
\mathcal{N}_2'(\zeta) &= \sqrt{\frac{3}{\pi}}\, e^{-i\frac{5\pi}{24}}\, e^{-\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \left[\zeta^{\frac{1}{4}} + \frac{7}{48} e^{-i\frac{\pi}{4}} \zeta^{-\frac{5}{4}} + 0\left(\zeta^{-\frac{11}{4}}\right)\right],\\
\int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} \mathcal{N}_1 dx &= \sqrt{\frac{3}{\pi}}\, e^{-i\frac{13\pi}{24}}\, e^{\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \left[\zeta^{-\frac{3}{4}} + \frac{41}{48} e^{-i\frac{\pi}{4}} \zeta^{-\frac{9}{4}} + 0\left(\zeta^{-\frac{15}{4}}\right)\right],\\
\int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta} \mathcal{N}_2 dx &= \sqrt{\frac{3}{\pi}}\, e^{-i\frac{17\pi}{24}}\, e^{-\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \left[\zeta^{-\frac{3}{4}} - \frac{41}{48} e^{-i\frac{\pi}{4}} \zeta^{-\frac{9}{4}} + 0\left(\zeta^{-\frac{15}{4}}\right)\right].
\end{aligned}\right.$$

The dominant terms in the frequency equation are found to be

(4.19)
$$\begin{aligned}
&\sqrt{\pi}\, e^{i\frac{3\pi}{8}}\left[\zeta_1^{\frac{1}{4}}\, e^{\frac{2}{3}\zeta_1^{\frac{3}{2}} e^{i\frac{\pi}{4}}} - \zeta_0^{\frac{1}{4}}\, e^{-\frac{2}{3}\zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\right] + \zeta_0^{-2} + \zeta_1^{-2}\\
&+ e^{i\frac{\pi}{4}}(\zeta_0\zeta_1)^{-\frac{3}{4}}(\zeta_1 - \zeta_0)\left\{-\frac{i}{2} e^{\frac{2}{3}\left(\zeta_0^{\frac{3}{2}} + \zeta_1^{\frac{3}{2}}\right) e^{i\frac{\pi}{4}}}\right.\\
&\left.+ \operatorname{Sin}\left[\frac{2}{3}\left(\zeta_1^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}}\right) e^{i\frac{\pi}{4}}\right]\right\} - (\zeta_0\zeta_1)^{-\frac{3}{4}}\left[\frac{7}{48}\left(\zeta_0^{-\frac{1}{2}} + \zeta_1^{-\frac{1}{2}}\right)\right.\\
&\left.+ \frac{41}{48}\zeta_0\zeta_1\left(\zeta_1^{-\frac{5}{2}} + \zeta_0^{-\frac{5}{2}}\right)\right] \operatorname{Cos}\left[\frac{2}{3}\left(\zeta_1^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}}\right) e^{i\frac{\pi}{4}}\right] = 0\,.
\end{aligned}$$

In solving equation (4.19) we must keep in mind that $\zeta_0$ and $\zeta_1$ must fulfil the relation (3.4) i.e.

(4.20)
$$\zeta_1 - \zeta_0 = (\varkappa R)^{\frac{1}{3}}\,.$$

We shall first find solutions of equations (4.19) — (4.20) where

$$\zeta_0 = \varrho e^{i\frac{\pi}{2}} e^{i\varphi}, \quad \zeta_1 = \varrho e^{i\frac{\pi}{2}} e^{-i\varphi}. \tag{4.21}$$

To fulfil relation (4.13) we must have

$$0 < \varphi \leqslant \frac{2\pi}{3}.$$

A solution of this kind represents a wave perturbation where the velocity of propagation of the wave is equal to the velocity of the mean flow in the middle of the layer, i.e. $R(c_1) = \frac{1}{2}$. This is easily seen from equations (4.7) and (4.21). For solutions where $0 < \varphi < \frac{\pi}{2}$, the wave-perturbations are stable, whereas they would be unstable in the case of $\frac{\pi}{2} < \varphi \leqslant \frac{2\pi}{3}$. With the assumptions (4.21), equations (4.19) — (4.20) become

$$\sqrt{\pi}\,\varrho^{\frac{3}{4}} e^{-\frac{2}{3}\varrho^{\frac{3}{2}}\cos\frac{3\varphi}{2}} \sin\left(\frac{2}{3}\varrho^{\frac{3}{2}}\sin\frac{3\varphi}{2} - \frac{\varphi}{4}\right) + \sin\varphi\left[\frac{1}{2}e^{-\frac{4}{3}\varrho^{\frac{3}{2}}\cos\frac{3\varphi}{2}} - \sin\left(\frac{4}{3}\varrho^{\frac{3}{2}}\sin\frac{3\varphi}{2}\right)\right]$$
$$+\varrho^{-\frac{3}{2}}\cos\frac{3\varphi}{2} - \varrho^{-\frac{3}{2}}\left(\frac{7}{48}\cos\frac{\varphi}{2} + \frac{41}{48}\cos\frac{5\varphi}{2}\right)\cos\left(\frac{4}{3}\varrho^{\frac{3}{2}}\sin\frac{3\varphi}{2}\right) = 0. \tag{4.22}$$

$$2\varrho\sin\varphi = (\varkappa R)^{\frac{1}{3}}. \tag{4.23}$$

We first notice that these equations have an infinite set of solutions with $\varphi \ll 1$ for any finite value of $\varkappa R$. Assuming $\varphi \ll 1$, the terms containing $e^{-\frac{3}{2}\varrho^{\frac{2}{3}}\cos\frac{3\varphi}{2}}$ and $e^{-\frac{4}{3}\varrho^{\frac{3}{2}}\cos\frac{3\varphi}{2}}$ as factors in equation (4.22) can be neglected. Equations (4.22)—(4.23) can be approximated by

$$\varphi\sin\left(2\varrho^{\frac{3}{2}}\varphi\right) - \varrho^{-\frac{3}{2}} + \varrho^{-\frac{3}{2}}\cos\left(2\varrho^{\frac{3}{2}}\varphi\right) = 0. \tag{4.22'}$$

$$2\varrho\varphi = (\varkappa R)^{\frac{1}{3}}. \tag{4.23'}$$

Equations (4.22')—(4.23') are the same as those discussed by HOPF for small values of $\varphi$. The solutions given by HOPF can be written as

$$\varrho = \frac{4}{(\varkappa R)^{\frac{2}{3}}}\left[\frac{n}{2}\pi - \Theta_n\right]^2, \quad \varphi = \frac{\varkappa R}{8\left(\frac{n}{2}\pi - \Theta_n\right)^2} \text{ for } n \geqslant 2. \tag{4.24}$$

In these solutions $\Theta_{2m} = 0$ and $0 < \Theta_{2m+1} < \frac{\pi}{2}$. $\Theta_{2m+1}$ decreases with increasing values of $m$.

When $\varkappa R \ll 1$, solutions (4.24) are valid for any integer $n$ greater than 1, since the assumption $\varphi \ll 1$ is fulfilled and $|\zeta_0|$ and $|\zeta_1|$ are large enough to justify the asymptotic approximations made above. As $\varkappa R$ increases, the lower modes starting with $n = 2$ given by the soluion (4.24) can not be utilized. The terms containing $e^{-\frac{2}{3}\varrho^{3/2}\cos\frac{3\varphi}{2}}$ and $e^{-\frac{4}{3}\varrho^{3/2}\cos\frac{3\varphi}{2}}$ in equations (4.22) have to be taken into account. An infinite set of solutions is, however, given by solutions (4.24) for any given value of $\varkappa R$, by making $n$ large enough.

We shall find the solutions corresponding to the lower modes $n = 2$, 3, 4 and 5 given by the solutions (4.24) for larger values of $\varkappa R$. For a given value of $\varkappa R$, we choose different values of $\varphi$, $\varrho$ is then given by equation (4.23). The values of $\varphi$ and $\varrho$ which satisfy equations (4.22)—(4.23) can then be found graphically. This has been done for different values of $\varkappa R$ and the results are shown in Table 1. For $\varkappa R$ less than about 305,

*Table 1.*

| | 1 mode ($n = 2$) | | 2 mode ($n = 3$) | | 3 mode ($n = 4$) | | 4 mode ($n = 5$) | |
|---|---|---|---|---|---|---|---|---|
| $\varkappa R$ | $\varrho$ | $\varphi$ | $\varrho$ | $\varphi$ | $\varrho$ | $\varphi$ | $\varrho$ | $\varphi$ |
| 1 | 39,5 | 43,′5 | 81,5 | 21′ | 158 | 10,′9 | 239 | 7,′2 |
| 200 | 3,14 | 68°,6 | 4,14 | 44°,9 | 5,34 | 33°,2 | 7,55 | 22°,8 |
| 300 | 3,47 | 74°,8 | 4,43 | 49° | 4,58 | 46°,9 | 6,25 | 32°,4 |
| 400 | 3,78 | 77° | — | | — | | 5,59 | 41°,2 |
| 500 | 4,08 | 76°,8 | — | | — | | 4,58 | 60° |
| 600 | 4,42 | 72°,7 | — | | — | | 4,48 | 70°,2 |

all modes exist. For $\varkappa R$ about 305, the second and third mode coincide and cease to exist for still larger values of $\varkappa R$. This means that they cease to exist as wave-solutions with the velocity of propagation equal to the velocity of the mean flow in the middle of the layer. As will be shown below, we get instead two solutions of another kind where the velocity of propagation of the waves will tend to the velocity of the mean flow at the lower or upper plane as $\varkappa R$ increases. For $\varkappa R$ about 605, the first and fourth mode will, in the same manner, coincide and cease to exist as solutions of the first kind for larger values of $\varkappa R$. Again, two solutions of the second kind will occur. As $\varkappa R$ increases further, more and more solutions of the first kind will cease to exist. Each time a pair of solutions of the first kind vanishes, two solutions of the second kind will appear.

In discussing the solutions of the second kind, we observe that because of the symmetry it is sufficient to discuss wave-solutions where the velocity of propagation is equal to the flow somewhere in, for example, the lower half of the layer. This is easily seen to correspond to $|\zeta_1| > |\zeta_0|$. We will seek solutions where $|\zeta_1| \gg |\zeta_0|$. We must, however, have so large values of $|\zeta_0|$ that the asymptotic expansions made above are valid. Since $|\zeta_1| \gg |\zeta_0|$ and $\zeta_1 - \zeta_0 = (\varkappa R)^{1/3}$, the dominant terms in equation (4.19) must be those having $e^{\frac{2}{3}\zeta_1^{3/2} e^{i\frac{\pi}{4}}}$ as a factor. The first approximation of equation

(4.19) is found to be

$$\sqrt{\pi}\, e^{i\frac{3\pi}{8}} + \frac{1}{2} e^{-i\frac{\pi}{4}}\, \zeta_0^{-\frac{3}{4}}\, e^{\frac{3}{2}\zeta_0^{\frac{2}{3}} e^{i\frac{\pi}{4}}} + \frac{1}{2} e^{i\frac{\pi}{4}}\, \zeta_0^{-\frac{3}{4}}\, e^{-\frac{2}{3}\zeta_0^{\frac{2}{3}} e^{i\frac{\pi}{4}}} = 0\,. \tag{4.25}$$

In Hopf's discussion of this case, the first term in this equation did not occur. This term must, however, dominate compared to one of the other terms in the equation. It is easily seen that the second term or the last term in equation (4.25) can be neglected when $\frac{\pi}{2} \leqslant \arg \zeta_0 < \frac{5\pi}{6}$ or $\frac{5\pi}{6} < \arg \zeta_0 \leqslant \frac{7\pi}{6}$, respectively. Accordlingy, we have to solve the two equations

$$2\sqrt{\pi}\, e^{i\frac{\pi}{8}}\, \zeta_0^{\frac{3}{4}} = -\, e^{-\frac{2}{3}\zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \quad \text{when } \frac{\pi}{2} \leqslant \arg \zeta_0 < \frac{5\pi}{6}\,,$$

$$2\sqrt{\pi}\, e^{i\frac{5\pi}{8}}\, \zeta_0^{\frac{3}{4}} = -\, e^{\frac{2}{3}\zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}} \quad \text{when } \frac{5\pi}{6} < \arg \zeta_0 \leqslant \frac{7\pi}{6}\,.$$

These equations, by putting $\zeta_0 = re^{i\left(\frac{5\pi}{6} - \theta\right)}$ in the first and $\zeta_0 = re^{i\left(\frac{5\pi}{6} + \theta\right)}$ in the last are transformed to

$$2\sqrt{\pi}\, r^{\frac{3}{4}} = e^{\frac{2}{3} r^{\frac{3}{2}} \sin\frac{3\theta}{2}}\, e^{i\left(\frac{2}{3} r^{\frac{3}{2}} \cos\frac{3\theta}{2} + \frac{\pi}{4} + \frac{3\theta}{4}\right)}\,,$$

$$2\sqrt{\pi}\, r^{\frac{3}{4}} = e^{\frac{2}{3} r^{\frac{3}{2}} \sin\frac{3\theta}{2}}\, e^{-i\left(\frac{2}{3} r^{\frac{3}{2}} \cos\frac{3\theta}{2} + \frac{\pi}{4} + \frac{3\theta}{4}\right)}\,.$$

These equations must be satisfied by the same values of $r$ and $\theta$. Only solutions where $0 < \theta \leqslant \frac{\pi}{3}$ can be used. Separating the last equations in their real and imaginary parts, we have to solve

$$\begin{cases} 2\sqrt{\pi}\, r^{\frac{3}{4}} = e^{\frac{2}{3} r^{\frac{3}{2}} \sin\frac{3\theta}{2}}\,, \\ \frac{2}{3} r^{\frac{3}{2}} \cos\frac{3\theta}{2} + \frac{\pi}{4} + \frac{3\theta}{4} = 2n\pi\,. \end{cases} \tag{4.26}$$

For small values of $\theta$, these equations approximate to

$$2\sqrt{\pi}\, r^{\frac{3}{4}} = e^{r^{\frac{3}{2}}\theta}\,,$$

$$\frac{2}{3} r^{\frac{3}{2}} = \left(2n - \frac{1}{4}\right)\pi\,,$$

which have the solutions

$$(4.27) \qquad r = \left[\left(3n - \frac{3}{8}\right)\pi\right]^{\frac{2}{3}}, \quad \theta = \frac{\ln\left[2\pi\sqrt{3n - \frac{3}{8}}\right]}{\left(3n - \frac{3}{8}\right)\pi}.$$

The solutions (4.27) can be shown to be fairly accurate solutions of equations (4.26).

With $n = 1$ in the solutions (4.27), the corresponding solutions of equation (4.25) are found to be

$$\zeta_0 = 4{,}08e^{i\,166^\circ\!.1}, \qquad \zeta_1 = (\varkappa R)^{\frac{1}{3}} + 4.08e^{i\,166^\circ\!.1},$$

and

$$\zeta_0 = 4.08e^{i\,133^\circ\!.9}, \qquad \zeta_1 = (\varkappa R)^{\frac{1}{3}} + 4{,}08e^{i\,133^\circ\!.9}.$$

For increasing values of the integer $n$ in solutions (4.27), $|\zeta_0|$ increases and arg $\zeta_0$ tends to 150°. The solutions are shown schematically in Fig. 2. In HOPF's discussion arg $\zeta_0 = \frac{5\pi}{6}$ for each solution in this case.

Only a finite set of solutions of this kind can exist for a finite value of $\varkappa R$. For any given value of $\varkappa R$, a finite number of the solutions (4.27) make $|\zeta_0| < |\zeta_1|$. For values of $n$ which make $|\zeta_0|$ only slightly less than $|\zeta_1|$, equation (4.19) has to be solved instead of equation (4.25). The number of solutions of this kind with $|\zeta_0| < |\zeta_1|$ is equal to the number of pair of solutions with $|\zeta_0| = |\zeta_1|$ which has ceased to exist.

The wave-solutions of both kinds found above are damped solutions since $\frac{\pi}{2} <$ arg $\zeta_0 < \pi$. The frequency equation we have discussed is only valid for values of $\zeta_0$ and $\zeta_1$ which fulfil the assumption (4.13). The frequency equations for other values of the argument of $\zeta_0$ and $\zeta_1$ can easily be obtained. They can, however. be shown to have no solutions for large values of $|\zeta_0|$ and $|\zeta_1|$. If $\zeta_0|$ or $|\zeta_1|$ or both are assumed not to be large enough to justify the approximations made above, expansions of $F_1(x)$ and $F_2(x)$ in power series in $x$ can be utilized to find frequency equations which are convenient for discussion in those cases. These equations will not be given here since no solution in addition to those given above has been found. They can, however, be used to verify the solutions having the least values of $|\zeta_0|$.

A discussion of the frequency equation for finite values of the wave-number $\varkappa$ will show that we have the same two kinds of solutions as for $\varkappa \ll 1$. This discussion will not be given here. We shall restrict ourselves to show that the same method of evaluating the frequency equation for large values of $|\zeta_0|$ and $|\zeta_1|$ can be utilized. The equation for the streamfunction (2.16) with $s = 0$ is

$$(4.28) \qquad i\Phi'''' + (\zeta - i\delta^2)\,\Phi'' - \delta^2\zeta\Phi = 0\,,$$

where $\zeta$ is defined by equation (2.12)

$$(4.29) \qquad \zeta = (\varkappa R)^{\frac{1}{3}}(z_1 - c_1) - i\delta^2\,.$$

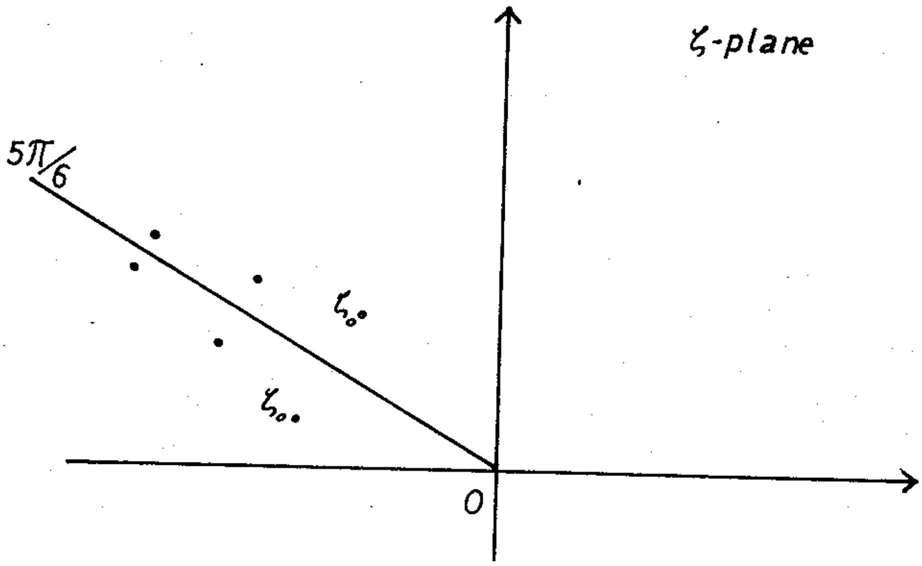


*Fig. 2.* Diagram showing the position of $\zeta_0$ when $|\zeta_1| >> |\zeta_0|$.

The solutions of equation (4.28) can be written as

$$\Phi_1 = e^{\delta\zeta}\,,\ \Phi_2 = e^{-\delta\zeta}\,,\ \Phi_3 = \int_{\zeta_0}^{\zeta} \mathrm{Sin}\,\delta\,(\zeta - x)\,F_1dx,\ \Phi_4 = \int_{\zeta_0}^{\zeta} \mathrm{Sin}\,\delta\,(\zeta - x)\,F_2dx\,.$$

Substituting these solutions in the frequency equation (3.3) we obtain

$$\int_{\zeta_0}^{\zeta_1} e^{\delta x}F_1dx \int_{\zeta_0}^{\zeta_1} e^{-\delta x}F_2dx - \int_{\zeta_0}^{\zeta_1} e^{-\delta x}F_1dx \int_{\zeta_0}^{\zeta_1} e^{\delta x}F_2dx = 0\,. \tag{4.30}$$

By the same method used for $\varkappa \ll 1$, we can make the following transformations

$$\begin{cases} \displaystyle\int_{\zeta_0}^{\zeta_1} e^{\delta x}F_1dx = -e^{-i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} e^{\delta x e^{i\frac{2\pi}{3}}} F_2dx + \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta_1} e^{\delta x}F_1dx\,, \\ \displaystyle\int_{\zeta_0}^{\zeta_1} e^{\delta x}F_2dx = C(\delta) + \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} e^{\delta x e^{i\frac{2\pi}{3}}} F_2dx + e^{i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta_0 e^{-i\frac{2\pi}{3}}} e^{\delta x e^{i\frac{2\pi}{3}}} F_1dx \\ \displaystyle + \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta_1} e^{\delta x}F_2dx\,, \end{cases} \tag{4.31}$$

where

$$(4.32)\qquad C(\delta) = e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} \left(e^{\delta x e^{i\frac{2\pi}{3}}} + e^{\delta x} + e^{\delta x e^{-i\frac{2\pi}{3}}}\right) F_1 dx = 2\sqrt{3}\, e^{i\frac{5\pi}{12}} e^{-i\frac{\delta^3}{3}} .$$

This value of $C(\delta)$ is evaluated in Appendix A. The other integrals in the frequency equation (4.30) are given by equations (4.31) and (4.32) by putting $-\delta$ instead of $\delta$ in these equations. In the integrals on the right hand sides of equations (4.31), $F_1$ and $F_2$ can be approximated by $N_1$ and $N_2$ respectively for large values of $|\zeta_0|$ and $|\zeta_1|$ if $\frac{\pi}{2} \leqslant \arg \zeta_0 \leqslant \frac{7\pi}{6}$ and $-\frac{\pi}{6} \leqslant \arg \zeta_1 \leqslant \frac{\pi}{2}$. The frequency equation we arrive at in this manner will again be different from the corresponding equation discussed by HOPF, due to the constants $C(\delta)$ and $C(-\delta)$ which did not occur in his calculations.

**5. Waves of infinite wave-length; slight static stability ($0 < s \leqslant 1$).** When $s$ is different from zero we shall, to simplify the mathematical treatment, first discuss waveperturbations where the wave-lengths are large in comparison with the height of the layer i.e. $\varkappa \ll 1$. The Reynolds number is, however, assumed to be so large that $\varkappa R$ may be large compared to unity. As shown in section 2, the equation for the stream-function is in this case approximated by

$$(5.1)\qquad i\zeta\Phi'''' + \zeta^2\Phi'' + s\Phi = 0 ,$$

where

$$(5.2)\qquad \zeta = (\varkappa R)^{\frac{1}{3}} (z_1 - c_1) .$$

Equation (5.1) has a logarithmic singularity at the point $\zeta = 0$. The logarithmic term can be shown to tend to zero as $\zeta^3 ln\zeta$ when $\zeta$ tends to zero. Accordingly, if $\Phi(\zeta)$ is a solution of equation (5.1) $\Phi(o)$, $\Phi'(o)$ and $\Phi''(o)$ exist and are singlevalued. Hence the velocity and the vorticity exist for $\zeta = 0$.

In order to find solutions of equation (5.1) which are convenient for discussion for large values of $|\zeta|$, we expand $\Phi(\zeta)$ in a power series in $s$

$$(5.3)\qquad \Phi(\zeta) = \sum_{n=0}^{\infty} s^n \Phi^{(n)}(\zeta).$$

Then equation (5.1) becomes

$$(5.4)\qquad \begin{cases} i\Phi^{(0)''''} + \zeta\, \Phi^{(0)''} = 0 , \\ i\Phi^{(n)''''} + \zeta\, \Phi^{(n)''} = -\dfrac{\Phi^{(n-1)}}{\zeta} , \quad n \geqslant 1. \end{cases}$$

The solutions of these equations can be written as

$$(5.5)\quad \Phi_1^{(0)} = \zeta,\ \Phi_2^{(0)} = 1,\ \Phi_3^{(0)} = \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} (\zeta - x)\, F_1 dx,\ \Phi_4^{(0)} = \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta} (\zeta - x)\, F_2 dx\ ,$$

$$\Phi_j^{(n)} = \frac{\pi}{6} \int^{\zeta} (\zeta - x)\, dx \Big[ F_1(x) \int^{x} \frac{F_2(y)\, \Phi_j^{(n-1)}(y)}{y}\, dy - F_2(x) \int^{x} \frac{F_1(y)\, \Phi_j^{(n-1)}(y)}{y}\, dy \Big], \qquad j = 1, 2, 3 \text{ or } 4,$$

where $F_1$ and $F_2$ are defined by equations (4.4). The lower limits in the integrals in the solutions (5.5) have to be suitably chosen. The asymptotic expansions (4.8)—(4.9) show that for large values of $|x|$ and $-\frac{\pi}{6} \leqslant \arg x \leqslant \frac{\pi}{2}$

$$F_1(x) \int_{\infty e^{-i\frac{\pi}{6}}}^{x} F_2(y)\, h(y)\, dy - F_2(x) \int_{\infty e^{i\frac{\pi}{2}}}^{x} F_1(y)\, h(y)\, dy = -\frac{6}{\pi} \frac{\mathrm{h}(\mathrm{x})}{x} [1 + 0(x^{-3})]\ ,$$

when the integrals exist. Utilizing this asymptotic behaviour, the solutions of equations (5.4) can be written as

$$(5.6)\quad \Phi_1^{(0)} = \zeta,\ \Phi_1^{(n)} = \frac{\pi}{6} \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} (\zeta - x)\, dx \Big[ F_1 \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2 \Phi_1^{(n-1)}}{y}\, dy - F_2 \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1 \Phi_1^{(n-1)}}{y}\, dy - \frac{6}{\pi} f''_n(x) \Big] + f_n(\zeta)\ ,$$

$$\Phi_2^{(0)} = 1,\ \Phi_2^{(n)} = \frac{\pi}{6} \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} (\zeta - x)\, dx \Big[ F_1 \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2 \Phi_2^{(n-1)}}{y}\, dy - F_2 \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1 \Phi_2^{(n-1)}}{y}\, dy - \frac{6}{\pi} g''_n(x) \Big] + g_n(\zeta)\ ,$$

$$\Phi_3^{(0)} = \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} (\zeta - x)\, F_1 dx,\ \Phi_3^{(n)} = \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} (\zeta - x)\, dx \Big[ F_1 \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_2 \Phi_3^{(n-1)}}{y}\, dy - F_2 \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1 \Phi_3^{(n-1)}}{y}\, dy \Big],$$

$$\Phi_4^{(0)} = \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta} (\zeta - x)\, F_2 dx,\ \Phi_4^{(n)} = \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta} (\zeta - x)\, dx \Big[ F_1 \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2 \Phi_4^{(n-1)}}{y}\, dy - F_2 \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_1 \Phi_4^{(n-1)}}{y}\, dy \Big],$$

where

$$f_n(x) = x\sum_{m=1}^{n} \alpha_m ln^m x\,, \qquad g_n(x) = \sum_{m=1}^{n} \beta_m ln^m x\,. \tag{5.7}$$

Furthermore we must have

$$f''_n(x) = -\frac{f_{n-1}(x)}{x^2}\,, \qquad g''_n(x) = -\frac{g_{n-1}(x)}{x^2}\,,$$

to make the integrals in equations (5.6) exist. We find

$$\alpha_1 = -1 \text{ and } \beta_1 = 1\,,$$

whereas $\alpha_m$ and $\beta_m$ for $m \geqslant 2$ can be calculated by integrating twice $f''_n(x)$ and $g''_n(x)$ as given by the relations above. The constants of integration must be chosen in such a manner that $f_n(x)$ and $g_n(x)$ take the form given by equations (5.7).

Asymptotic values of $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ given by equations (5.6) are seen to be

$$\Phi_1(\zeta) = \zeta + \sum_{n=1}^{\infty} s^n f_n(\zeta)\,,$$

$$\Phi_2(\zeta) = 1 + \sum_{n=1}^{\infty} s^n g_n(\zeta)\,, \qquad \text{when } -\frac{\pi}{6} \leqslant \arg\zeta \leqslant \frac{\pi}{2}\,.$$

The same series, which are seen to be convergent when $|s| < \frac{1}{4}$, are found if $\zeta^{\frac{1}{2}+\sqrt{\frac{1}{4}-s}}$ and $\zeta^{\frac{1}{2}-\sqrt{\frac{1}{4}-s}}$ are expanded in power series in $s$. Hence, we have asymptotically

$$\begin{cases} \Phi_1(\zeta) = \zeta^{\frac{1}{2}+\sqrt{\frac{1}{4}-s}}\,, \\ \Phi_2(\zeta) = \zeta^{\frac{1}{2}-\sqrt{\frac{1}{4}-s}}\,, \end{cases} \qquad \text{when } -\frac{\pi}{6} \leqslant \arg\zeta \leqslant \frac{\pi}{2}\,, \tag{5.8}$$

at least when $|s| < \frac{1}{4}$.

The asymptotic solutions $\Phi_1$ and $\Phi_2$ given by equations (5.8) are seen to be the solutions for the streamfunction in an inviscid fluid.

Substituting the expansions (4.8)—(4.9) in equations (5.6), we find the asymptotic values of $\Phi_3$ and $\Phi_4$ to be

$$\Phi_3(\zeta) = \sqrt{\frac{3}{\pi}}\, e^{-i\frac{19\pi}{24}}\, \zeta^{-\frac{5}{4}}\, e^{\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,, \tag{5.9}$$

$$\Phi_4(\zeta) = \sqrt{\frac{3}{\pi}}\, e^{i\frac{\pi}{24}}\, \zeta^{-\frac{5}{4}}\, e^{-\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,, \qquad \text{when } -\frac{\pi}{6} \leqslant \arg\zeta \leqslant \frac{\pi}{2}\,.$$

To find asymptotic values of the streamfunction for other values of arg $\zeta$ we observe that $\Phi_j\left(\zeta e^{-i\frac{2\pi}{3}}\right)$ is a solution of equation (5.1) if $\Phi_j(\zeta)$ is a solution of the same equation. Accordingly, we can write

$$\Phi_j(\zeta) = \sum_{k=1}^{4} A_{jk}\,\Phi_k\left(\zeta e^{-i\frac{2\pi}{3}}\right), \quad j = 1, 2, 3 \text{ or } 4. \tag{5.10}$$

The constants $A_{jk}$ have been determined in Appendix B. Equations (5.8)—(5.10) can then be utilized to find asymptotic values of $\Phi_j(\zeta)$ when $\frac{\pi}{2} \leqslant \arg\zeta \leqslant \frac{7\pi}{6}$.

If we put $\zeta e^{i\frac{2\pi}{3}}$ instead of $\zeta$ in equations (5.10), then these equations become

$$\Phi_j\left(\zeta e^{i\frac{2\pi}{3}}\right) = \sum_{k=1}^{4} A_{jk}\Phi_k(\zeta)\,.$$

Solving these equations with respect to $\Phi_j(\zeta)$ we obtain

$$\Phi_j(\zeta) = \sum_{k=1}^{4} B_{jk}\Phi_k\left(\zeta e^{i\frac{2\pi}{3}}\right), \tag{5.11}$$

where the constants $B_{jk}$ are determined by the constants $A_{jk}$. Equations (5.8), (5.9) and (5.11) can then be utilized to find asymptotic values of $\Phi_j(\zeta)$ when $-\frac{5\pi}{6} \leqslant \arg\zeta \leqslant -\frac{\pi}{6}$.

Equation (5.11) can also be written as

$$\Phi_j(\zeta) = \sum_{l=1}^{4}\sum_{m=1}^{4} B_{jl}B_{lm}\Phi_m\left(\zeta e^{i\frac{4\pi}{3}}\right) = \sum_{m=1}^{4} C_{jm}\Phi_m\left(\zeta e^{i\frac{4\pi}{3}}\right). \tag{5.12}$$

Equations (5.8), (5.9) and (5.12) can then be utilized to find asymptotic values of $\Phi_j(\zeta)$ when $-\frac{3\pi}{2} \leqslant \arg\zeta \leqslant -\frac{5\pi}{6}$.

As shown in section 3, the frequency equation can be written as

$$\begin{aligned} &f_{12}(\zeta_0)f_{34}(\zeta_1) - f_{13}(\zeta_0)f_{24}(\zeta_1) + f_{14}(\zeta_0)f_{23}(\zeta_1) \\ &+ f_{23}(\zeta_0)f_{14}(\zeta_1) - f_{24}(\zeta_0)f_{13}(\zeta_1) + f_{34}(\zeta_0)f_{12}(\zeta_1) = 0, \end{aligned} \tag{5.13}$$

where

$$f_{jk}(\zeta) = \begin{vmatrix} \Phi_j(\zeta) & \Phi_k(\zeta) \\ \Phi_j'(\zeta) & \Phi_k'(\zeta) \end{vmatrix}. \tag{5.14}$$

If we substitute equations (5.8)—(5.9) in equations (5.14), then the asymptotic values of $f_{jk}(\zeta)$ are found to be

$$
(5.15)\quad \begin{cases}
f_{12}(\zeta) = -2\mu\,, \\
f_{13}(\zeta) = \sqrt{\frac{3}{\pi}}\, e^{-i\frac{13\pi}{24}}\, \zeta^{-\frac{1}{4}} + \mu\, e^{\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,, \\
f_{14}(\zeta) = \sqrt{\frac{3}{\pi}}\, e^{-i\frac{17\pi}{24}}\, \zeta^{-\frac{1}{4}} + \mu\, e^{-\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,, \\
f_{23}(\zeta) = \sqrt{\frac{3}{\pi}}\, e^{-i\frac{13\pi}{24}}\, \zeta^{-\frac{1}{4}} - \mu\, e^{\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,, \\
f_{24}(\zeta) = \sqrt{\frac{3}{\pi}}\, e^{-i\frac{17\pi}{24}}\, \zeta^{-\frac{1}{4}} - \mu\, e^{-\frac{2}{3}\zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,, \\
f_{34}(\zeta) = \frac{6}{\pi}\, i\zeta^{-2}\,,
\end{cases}
\qquad \text{when } -\frac{\pi}{6} \leqslant \arg\zeta \leqslant \frac{\pi}{2}\,.
$$

where

$$\mu = \sqrt{\frac{1}{4} - s.}$$

Utilizing equations (5.10)—(5.12), we find that $f_{jk}(\zeta)$ given by equations (5.14) also can be written as

$$(5.16)\qquad f_{jk}(\zeta) = e^{-i\frac{2\pi}{3}} \sum_{l=1}^{4} \sum_{m=1}^{4} A_{jl} A_{km} f_{lm}\left(\zeta e^{-i\frac{2\pi}{3}}\right).$$

$$(5.17)\qquad f_{jk}(\zeta) = e^{i\frac{2\pi}{3}} \sum_{l=1}^{4} \sum_{m=1}^{4} B_{jl} B_{km} f_{lm}\left(\zeta e^{i\frac{2\pi}{3}}\right).$$

$$(5.18)\qquad f_{jk}(\zeta) = e^{i\frac{4\pi}{3}} \sum_{l=1}^{4} \sum_{m=1}^{4} C_{jl} C_{km} f_{lm}\left(\zeta e^{i\frac{4\pi}{3}}\right).$$

Substituting equations (5.15) on the right hand sides of equations (5.16)—(5.18) we find the asymptotic values of $f_{jk}(\zeta)$ when $\frac{\pi}{2} \leqslant \arg\zeta \leqslant \frac{7\pi}{6}$, $-\frac{5\pi}{6} \leqslant \arg\zeta \leqslant -\frac{\pi}{6}$ and $-\frac{3\pi}{2} \leqslant \arg\zeta \leqslant -\frac{5\pi}{6}$, respectively.

We shall then seek such solutions of the frequency equation (5.13) that

$$(5.19)\qquad \frac{5\pi}{6} < \arg\zeta_0 \leqslant \frac{7\pi}{6} \text{ and } -\frac{\pi}{6} \leqslant \arg\zeta_1 < \frac{\pi}{6}\,.$$

Asymptotic values of $f_{jk}(\zeta_1)$ are therefore given by equations (5.15), whereas asymptotic values of $f_{jk}(\zeta_0)$ can be found from equations (5.15)—(5.16). Substituting these values of $f_{jk}(\zeta_0)$ and $f_{jk}(\zeta_1)$ in the frequency equation (5.13), we see that the dominant terms in this equation for sufficient large values of $|\zeta_0|$ and $|\zeta_1|$ are those having $e^{\frac{2}{3}\left(\zeta_0^{3/2} + \zeta_1^{3/2}\right) e^{i\frac{\pi}{4}}}$

as a factor. Retaining only these terms, we find that the frequency equation can be approximated by

$$(5.20)\qquad (A_{11}A_{44}-A_{14}A_{41})\, e^{-i\frac{2\pi}{3}\mu}\left(\frac{\zeta_0}{\zeta_1}\right)^{\mu}-(A_{22}A_{44}-A_{24}A_{42})\, e^{i\frac{2\pi}{3}\mu}\left(\frac{\zeta_1}{\zeta_0}\right)^{\mu}$$
$$+\,A_{24}A_{41}\, e^{-i\frac{2\pi}{3}\mu}\,(\zeta_0\zeta_1)^{\mu}-A_{14}A_{42}\, e^{i\frac{2\pi}{3}\mu}\,(\zeta_0\zeta_1)^{-\mu}=0\,.$$

We shall restrict ourselves to discussing the case when $0 < s \ll 1$. In this case approximate values of the constants $A_{jk}$ easily can be found. Approximate values of the constants occurring in equation (5.20) are (cf. Appendix B)

$$A_{11}A_{44}-A_{14}A_{41}=e^{i\frac{\pi}{3}}\,.$$

$$A_{22}A_{44}-A_{24}A_{42}=e^{i\frac{\pi}{3}}\,.$$

$$A_{24}A_{41}=-\frac{4\pi^2}{3^{7/6}\,\Gamma\left(\frac{2}{3}\right)}\,e^{i\frac{\pi}{3}}\,s\,.$$

$$A_{14}A_{42}=0\,(s^2)\,.$$

$$\mu=\frac{1}{2}-s\,.$$

Equation (5.20) can therefore be approximated by

$$(5.21)\qquad \left(\frac{\zeta_0}{\zeta_1}\right)^{\frac{1}{2}}-\left(\frac{\zeta_1}{\zeta_0}\right)^{\frac{1}{2}}-sK\,(\zeta_0\zeta_1)^{\frac{1}{2}}=0\,,$$

where

$$(5.22)\qquad K=\frac{4\pi^2}{3^{7/6}\,\Gamma\left(\frac{2}{3}\right)}=8{,}09\,.$$

Since

$$(5.23)\qquad \zeta_1-\zeta_0=(\varkappa R)^{\frac{1}{3}}\,,$$

equation (5.21) has no solution when $s \equiv 0$. As seen in section 4, other terms in the frequency equation than those having $e^{\frac{2}{3}\left(\zeta_0^{3/2}+\zeta_1^{3/2}\right)e^{i\frac{\pi}{4}}}$ as a factor hád to be taken into account in order to give solutions in this case. When $s \neq 0$ equations (5.21)—(5.23) have the two solutions

$$(5.24)\qquad \zeta_0=-\frac{(\varkappa R)^{\frac{1}{3}}}{2}\pm\frac{(\varkappa R)^{\frac{1}{3}}}{2}\sqrt{1-\frac{4}{Ks\,(\varkappa R)^{1/3}}}\,,\quad \zeta_1=\zeta_0+(\varkappa R)^{\frac{1}{3}}\,.$$

When

$$(\varkappa R)^{\frac{1}{3}} = \frac{4}{Ks}, \tag{5.25}$$

we have a solution with $\zeta_0 = \frac{(\varkappa R)^{\frac{1}{3}}}{2} e^{i\pi}$ and $\zeta_1 = \frac{(\varkappa R)^{\frac{1}{3}}}{2}$. This solution represents a neutral wave-perturbation with a velocity of propagation equal to the velocity of the mean flow in the middle of the layer.

When

$$(\varkappa R)^{\frac{1}{3}} < \frac{4}{Ks},$$

we have $|\zeta_0| = |\zeta_1|$ for the two solutions (5.24). When the relations (5.19) are fulfilled we have one solution with $\frac{5\pi}{6} < \arg \zeta_0 < \pi$ which represents a stable wave-perturbation. For the second solution we have $\pi < \arg \zeta_0 < \frac{7\pi}{6}$. This solution is only a mathematical solution of equation (5.24) and does not represent an unstable wave-perturbation, (see conditions (3.5)). In order to find unstable wave-solutions we have to find the asymptotic expression for the frequency equation for negative values of $\arg \zeta_0$ and $\arg \zeta_1$. This can be done by use of equations (5.15)—(5.18). As no unstable wave-solutions have been found for large values of $|\zeta_0|$ and $|\zeta_1|$ when $|s| \leqslant 1$, the frequency equations obtained in this manner will not be given here.

When

$$(\varkappa R)^{\frac{1}{3}} > \frac{4}{Ks},$$

further approximations have to be made to find whether the two solutions (5.24) represent stable wave-perturbations ($\arg \zeta_0 < \pi$) or if they are non-existent ($\arg \zeta_0 > \pi$).

We observe the great difference between the solutions in a non-stratified fluid and a stratified fluid. In the first case we have, as shown in section 4, always damped oscillations. In a stratified fluid, we can always find neutral oscillations however small the value of $s$, by making the Reynolds number large enough to satisfy equation (5.25). As mentioned above, the asymptotic expressions (5.8) for $\Phi_1$ and $\Phi_2$ are the solutions for the streamfunction for an inviscid fluid. In the case of a non-stratified fluid $\Phi_1$ and $\Phi_2$ represent the streamfunction for an inviscid fluid in the entire $\zeta$-plane. For a stratified fluid, however, this is not the case as shown by equations (5.10). The asymptotic value of $\Phi_1$ for instance is found to be

$$\Phi_1 = \zeta^{\frac{1}{2} + \mu}, \qquad \text{when } -\frac{\pi}{6} \leqslant \arg \zeta \leqslant \frac{\pi}{2},$$

$$\Phi_1 = \zeta^{\frac{1}{2} + \mu} + A_{14} \sqrt{\frac{3}{\pi}} e^{i\frac{7\pi}{8}} \zeta^{-\frac{5}{4}} e^{\frac{2}{3} \zeta^{\frac{3}{2}} e^{i\frac{\pi}{4}}}, \quad \text{when } \frac{\pi}{2} \leqslant \arg \zeta \leqslant \frac{7\pi}{6}.$$

The last term will dominate compared to the inviscid solution $\zeta^{\frac{1}{2}+\mu}$ for sufficiently large values of $|\zeta|$ when $\frac{5\pi}{6} < \arg \zeta \leqslant \frac{7\pi}{6}\cdot$ This is true however small we make $s$.

**6. Waves of finite wave-length; slight static stability ($0 < s \ll 1$).** In this section we shall find neutral wave-solutions for any value of the wave-number $\varkappa$. Accordingly, we have the equation for the streamfunction given by equation (2.16) which reads

$$i\,(\zeta + i\delta^2)\,\Phi'''' + (\zeta + i\delta^2)\,(\zeta - i\delta^2)\,\Phi'' + [s - \delta^2\,\zeta\,(\zeta + i\delta^2)]\,\Phi = 0\,, \tag{6.1}$$

where

$$\zeta = (\varkappa R)^{\frac{1}{3}}\,(z_1 - c_1) - i\delta^2\,. \tag{6.2}$$

Expanding $\Phi$ in power series in $s$, we put

$$\Phi\,(\zeta) = \sum_{n=0}^{\infty} s^n \Phi^{(n)}\,(\zeta) \tag{6.3}$$

in equation (6.1). The solutions of equation (6.1) can then be written as

$$\left\{\begin{aligned} &\Phi_1^{(0)} = \operatorname{Sin}\delta\zeta,\ \Phi_2^{(0)} = \operatorname{Cos}\delta\zeta,\ \Phi_3^{(0)} = \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} \operatorname{Sin}\,[\delta\,(\zeta - x)]\,F_1 dx\,,\\ &\Phi_4^{(0)} = \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta} \operatorname{Sin}\,[\delta\,(\zeta - x)]\,F_2 dx\,,\\ &\Phi_j^{(n)} = \frac{\pi}{6\delta}\int_{\infty e^{i\alpha_j}}^{\zeta} \operatorname{Sin}\,[\delta\,(\zeta - x)]\left[F_1\,(x)\int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2(y)\,\Phi_j^{(n-1)}\,(y)}{y + i\delta^2}\,dy\right.\\ &\qquad\left. - F_2\,(x)\int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1(y)\,\Phi_j^{(n-1)}\,(y)}{y + i\delta^2}\,dy\right] \quad j = 1, 2, 3 \text{ or } 4\,, \end{aligned}\right. \tag{6.4}$$

where

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{2}, \qquad \alpha_4 = -\frac{\pi}{6}\cdot \tag{6.5}$$

By this choice of $\alpha_j$, the integrals in equations (6.4) exist. In the following discussion the Reynolds number will be assumed to be so large that

$$\delta = \left(\frac{\varkappa^2}{R}\right)^{\frac{1}{3}} \ll 1\,. \tag{6.6}$$

Due to this assumption, we can, for large values of $|\zeta|$ approximate equations (6.4) by

$$
(6.7)\quad \begin{cases}
\Phi_1^{(0)} = \operatorname{Sin} \delta\zeta,\ \Phi_2^{(0)} = \operatorname{Cos} \delta\zeta,\ \Phi_3^{(0)} = \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} \operatorname{Sin} [\delta (\zeta - x)] F_1 dx\,, \\
\Phi_4^{(0)} = \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta} \operatorname{Sin} [\delta (\zeta - x)] F_2 dx\,, \\
\Phi_j^{(n)} = \frac{\pi}{6\delta} \int_{\infty e^{ia_j}}^{\zeta} \operatorname{Sin} [\delta (\zeta - x)]\, dx \left[ F_1 \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2 \Phi_j^{(n-1)}}{y} dy \right. \\
\left. \qquad - F_2 \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1 \Phi_j^{(n-1)}}{y} dy \right].
\end{cases}
$$

Furthermore, equation (6.2) can be approximated by

$$
(6.8)\qquad \zeta = (\varkappa R)^{\frac{1}{3}} (z_1 - c_1)\,.
$$

We shall seek solutions of the frequency equation given by equation (5.13) where

$$
(6.9)\qquad \zeta_0 = \frac{(\varkappa R)^{\frac{1}{3}}}{2} e^{i\pi}\,, \qquad \zeta_1 = \frac{(\varkappa R)^{\frac{1}{3}}}{2}\,.
$$

Solutions of this kind represent neutral wave-perturbations. We shall furthermore assume

$$
(6.10)\qquad s \ll 1\,.
$$

In evaluating $\Phi_j(\zeta_0)$ and $\Phi_j(\zeta_1)$, given by equation (6.7), we shall assume $(\varkappa R)^{1/3}$ so large that $F_1$ and $F_2$ can be approximated by their asymptotic values. As in section 5, the frequency equation will be approximated by putting the terms having $e^{\frac{2}{3}\left(\zeta_0^{3/2} + \zeta_1^{3/2}\right) e^{i\frac{\pi}{4}}}$ as a factor equal to zero.

If we put $\zeta = \zeta_1$ in equations (6.7), $F_1$ and $F_2$ can be approximated by $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. Approximating $\mathcal{N}_1$ and $\mathcal{N}_2$ by the first term in their expansions (4.9) we find as a first approximation

$$
(6.10)\quad \begin{cases}
\Phi_1(\zeta_1) = \operatorname{Sin} \delta\zeta_1\,, \\
\Phi_2(\zeta_1) = \operatorname{Cos} \delta\zeta_1\,, \\
\Phi_3(\zeta_1) = \sqrt{\frac{3}{\pi}}\, e^{-i\frac{19\pi}{24}} \delta\zeta_1^{-\frac{5}{4}} e^{\frac{2}{3}\zeta_1^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,, \\
\Phi_4(\zeta_1) = \sqrt{\frac{3}{\pi}}\, e^{i\frac{\pi}{24}} \delta\zeta_1^{-\frac{5}{4}} e^{-\frac{2}{3}\zeta_1^{\frac{3}{2}} e^{i\frac{\pi}{4}}}\,.
\end{cases}
$$

If we put $\zeta = \zeta_0$ in equation (6.7), $F_1$ and $F_2$ can not be approximated by $\mathcal{N}_1$ and $\mathcal{N}_2$ since arg $\zeta_0$ is assumed to be equal to $\pi$. We shall again find that $\Phi_1$ and $\Phi_2$ contain terms having $e^{\frac{2}{3}\zeta^{3/2} e^{i\frac{\pi}{4}}}$ as a factor when $\frac{\pi}{2} \leqslant \arg \zeta \leqslant \frac{7\pi}{6}$ · Taking for example $\Phi_1^{(1)}$, given by equations (6.7), we have

$$\Phi_1^{(1)}(\zeta) = \frac{\pi}{6\delta} \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} \text{Sin}\,[\delta(\zeta - x)]\, dx \Bigg[ F_1(x) \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2(y)\,\text{Sin}\,\delta y}{y}\, dy - F_2(x) \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta} \frac{F_1(y)\,\text{Sin}\,\delta y}{y}\, dy \Bigg].$$

This equation can be transformed to

$$\Phi_1^{(1)}(\zeta) = \frac{\pi}{6\delta} e^{i\frac{2\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta e^{-i\frac{2\pi}{3}}} \text{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] dx$$

$$\cdot \Bigg[ F_1\left(x e^{i\frac{2\pi}{3}}\right) \left\{ \int_{\infty e^{-i\frac{\pi}{6}}}^{0} \frac{F_2(y)\,\text{Sin}\,\delta y}{y}\, dy + \int_{0}^{x} \frac{F_2\left(y e^{i\frac{2\pi}{3}}\right) \text{Sin}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\, dy \right\}$$

$$- F_2\left(x e^{i\frac{2\pi}{3}}\right) \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1\left(y e^{i\frac{2\pi}{3}}\right) \text{Sin}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\, dy \Bigg].$$

Since, by equation (4.11)

$$F_1\left(x e^{i\frac{2\pi}{3}}\right) = -F_2(x)\ ,$$

$$F_2\left(x e^{i\frac{2\pi}{3}}\right) = e^{i\frac{\pi}{3}} F_2(x) - e^{-i\frac{\pi}{3}} F_1(x)\ ,$$

we find

$$\Phi_1^{(1)}(\zeta) = \frac{\pi}{6\delta} e^{-i\frac{\pi}{3}} C_1(\delta) \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta e^{-i\frac{2\pi}{3}}} \text{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] F_2 dx \tag{6.11}$$

$$- \frac{\pi}{6\delta} e^{i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta e^{-i\frac{2\pi}{3}}} \text{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] dx \Bigg[ F_1 \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2\,\text{Sin}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\, dy$$

$$- F_2 \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1\,\text{Sin}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\, dy \Bigg].$$

The constant $C_1(\delta)$ is given by

$$C_1(\delta) = \int_{\infty e^{-i\frac{\pi}{6}}}^{0} \frac{F_2(y)\,\mathrm{Sin}\,\delta y}{y}\,dy - e^{i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{0} \frac{F_2(y)\,\mathrm{Sin}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\,dy$$
$$+ e^{-i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{0} \frac{F_1(y)\,\mathrm{Sin}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\,dy = 2\sqrt{3}\,e^{-i\frac{7\pi}{12}}\,\delta + 0\,(\delta^7)\,. \tag{6.12}$$

This value of $C_1(\delta)$ is evaluated in Appendix A. Transforming $\Phi_2^{(1)}(\zeta)$ given by equations (6.7) in the same manner, we arrive at

$$\Phi_2^{(1)}(\zeta) = \frac{\pi}{6\delta} e^{-i\frac{\pi}{3}} C_2(\delta) \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta e^{-i\frac{2\pi}{3}}} \mathrm{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] F_2 dx$$
$$- \frac{\pi}{6\delta} e^{i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta e^{-i\frac{2\pi}{3}}} \mathrm{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] dx \left[F_1 \int_{\infty e^{-i\frac{\pi}{6}}}^{x} \frac{F_2\,\mathrm{Cos}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\,dy\right.$$
$$\left. - F_2 \int_{\infty e^{i\frac{\pi}{2}}}^{x} \frac{F_1\,\mathrm{Cos}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\,dy\right], \tag{6.13}$$

where

$$C_2(\delta) = \int_{\infty e^{-i\frac{\pi}{6}}}^{a} \frac{F_2\,\mathrm{Cos}\,\delta y}{y}\,dy - e^{i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{ae^{-i\frac{2\pi}{3}}} \frac{F_2\,\mathrm{Cos}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\,dy$$
$$+ e^{-i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{ae^{-i\frac{2\pi}{3}}} \frac{F_1\,\mathrm{Cos}\left(\delta y e^{i\frac{2\pi}{3}}\right)}{y}\,dy\,. \tag{6.14}$$

The value of $C_2(\delta)$ is independent of the choice of the constant a and is in Appendix A evaluated to be

$$C_2(\delta) = \frac{2}{3^{1/6}\,\Gamma\left(\frac{2}{3}\right)} e^{i\frac{11}{12}\pi} + 0\,(\delta^4)\,. \tag{6.15}$$

When $\frac{\pi}{2} \leqslant \arg \zeta \leqslant \frac{7\pi}{6}$, $F_1$ and $F_2$ in equations (6.11) and (6.13) can be approximated by $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. We then see that the first term on the right hand side of these equations has $e^{\frac{2}{3}\zeta^{3/2} e^{i\frac{\pi}{4}}}$ as a factor and will be the dominating term for large values of $|\zeta|$ when $\frac{5\pi}{6} < \arg \zeta \leqslant \frac{7\pi}{6}$. Accordingly, approximate values of $\Phi_1(\zeta_0)$ and $\Phi_2(\zeta_0)$ are found to be

$$(6.16)\quad \begin{cases} \Phi_1(\zeta_0) = \operatorname{Sin} \delta\zeta_0 + s\frac{\pi}{6}\sqrt{\frac{3}{\pi}}\, C_1(\delta)\, e^{i\frac{29\pi}{24}}\, \zeta_0^{-\frac{5}{4}}\, e^{\frac{2}{3}\zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}}, \\ \Phi_2(\zeta_0) = \operatorname{Cos} \delta\zeta_0 + s\frac{\pi}{6}\sqrt{\frac{3}{\pi}}\, C_2(\delta)\, e^{i\frac{29\pi}{24}}\, \zeta_0^{-\frac{5}{4}}\, e^{\frac{2}{3}\zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}}. \end{cases}$$

To find approximate values of $\Phi_3^{(0)}(\zeta)$ and $\Phi_4^{(0)}(\zeta)$ given by equations (6.7) we can easily transform them to

$$\Phi_3^{(0)}(\zeta) = e^{-i\frac{\pi}{3}} \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta e^{-i\frac{2\pi}{3}}} \operatorname{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] F_2 dx ,$$

$$\Phi_4^{(0)}(\zeta) = C_3(\delta) \operatorname{Sin} \delta\zeta - C_4(\delta) \operatorname{Cos} \delta\zeta - \int_{\infty e^{-i\frac{\pi}{6}}}^{\zeta e^{-i\frac{2\pi}{3}}}$$

$$\cdot \operatorname{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] F_2 dx - e^{i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{\zeta e^{-i\frac{2\pi}{3}}} \operatorname{Sin}\left[\delta e^{i\frac{2\pi}{3}}\left(\zeta e^{-i\frac{2\pi}{3}} - x\right)\right] F_1 dx ,$$

where

$$(6.17)\quad \begin{cases} C_3(\delta) = \int_{\infty e^{-i\frac{\pi}{6}}}^{0} \operatorname{Cos}(\delta x) F_2 dx + \int_{\infty e^{-i\frac{\pi}{6}}}^{0} \operatorname{Cos}\left(\delta x e^{i\frac{2\pi}{3}}\right) F_2 dx \\ \quad + e^{i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{0} \operatorname{Cos}\left(\delta x e^{i\frac{2\pi}{3}}\right) F_1 dx = 2\sqrt{3}\, e^{-i\frac{7\pi}{12}} + 0(\delta^6). \\ C_4(\delta) = \int_{\infty e^{-i\frac{\pi}{6}}}^{0} \operatorname{Sin}(\delta x) F_2 dx + \int_{\infty e^{-i\frac{\pi}{6}}}^{0} \operatorname{Sin}\left(\delta x e^{i\frac{2\pi}{3}}\right) F_2 dx \\ \quad + e^{i\frac{\pi}{3}} \int_{\infty e^{i\frac{\pi}{2}}}^{0} \operatorname{Sin}\left(\delta x e^{i\frac{2\pi}{3}}\right) F_1 dx = 0(\delta^3). \end{cases}$$

The values of $C_3(\delta)$ and $C_4(\delta)$ are evaluated in Appendix A. Approximating $\Phi_3(\zeta_0)$ and $\Phi_4(\zeta_0)$ by $\Phi_3^{(0)}(\zeta_0)$ and $\Phi_4^{(0)}(\zeta_0)$, respectively, approximate values of $\Phi_3(\zeta_0)$ and $\Phi_4(\zeta_0)$ are then found to be

$$(6.18)\quad \begin{cases} \Phi_3(\zeta_0) = \sqrt{\dfrac{3}{\pi}}\, e^{-i\frac{19\pi}{24}}\, \delta\zeta_0^{-\frac{5}{4}}\, e^{\frac{3}{2}\zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}}, \\ \Phi_4(\zeta_0) = C_3(\delta)\,\mathrm{Sin}\,\delta\zeta_0 - C_4(\delta)\,\mathrm{Cos}\,\delta\zeta_0 - \sqrt{\dfrac{3}{\pi}}\, e^{-i\frac{11\pi}{24}}\, \delta\zeta_0^{-\frac{5}{4}}\, e^{\frac{2}{3}\zeta_0^{\frac{3}{2}} e^{i\frac{\pi}{4}}}. \end{cases}$$

Substituting $\Phi_j(\zeta_1)$ and $\Phi_j(\zeta_0)$ given by equations (6.10), (6.16) and (6.18) in equations (5.13)—(5.14), we find an approximate form of the frequency equation. Approximating further by putting the terms having $e^{\frac{2}{3}\left(\zeta_1^{3/2} + \zeta_0^{3/2}\right) e^{i\frac{\pi}{4}}}$ as a factor equal to zero we finally arrive at

$$(6.19)\qquad \mathrm{Sin}\,[\delta(\zeta_1 - \zeta_0)] + \frac{sK}{\delta}\,\mathrm{Sin}\,\delta\zeta_0\,\mathrm{Sin}\,\delta\zeta_1 = 0\,,$$

where $K$ is defined by equation (5.22). Since $(\zeta_1 - \zeta_0) = (\varkappa R)^{\frac{1}{3}}$ and $\delta = \left(\frac{\varkappa^2}{R}\right)^{\frac{1}{3}}$ we find

$$\delta(\zeta_1 - \zeta_0) = \varkappa$$

For neutral wave-solutions given by relations (6.9) we have furthermore

$$\delta\zeta_0 = -\frac{\varkappa}{2}, \qquad \delta\zeta_1 = \frac{\varkappa}{2}.$$

Equation (6.19) then shows that we have neutral wave-solutions when

$$(6.20)\qquad Tg\,\frac{\varkappa}{2} = \frac{2\delta}{Ks} = \frac{2}{Ks}\left(\frac{\varkappa^2}{R}\right)^{\frac{1}{3}}.$$

For a given value of $s$, equation (6.20) represents a curve in a $(\varkappa, R)$-plane which is schematically given in Fig. 3.

The two asymptotic branches are given by

$$(\varkappa R)^{\frac{1}{3}} = \frac{4}{Ks} \qquad \text{when } \varkappa \ll 1\,,$$

$$\frac{\varkappa^2}{R} = \left(\frac{Ks}{2}\right)^3 \qquad \text{when } \varkappa \gg 1\,.$$

The asymptotic branch when $\varkappa \ll 1$ is seen to be the same as that found in section 5. Neutral wave-solutions are found when $R \geqslant R_{\min}$. This minimum value of $R$ is found from equation (6.20) to be

$$R_{\min} = \left(\frac{2}{Ks}\right)^3 \frac{\varkappa_m{}^2}{Tg^3\,\frac{\varkappa_m}{2}}\,,$$

where

$$\varkappa_m \approx 1{,}62$$

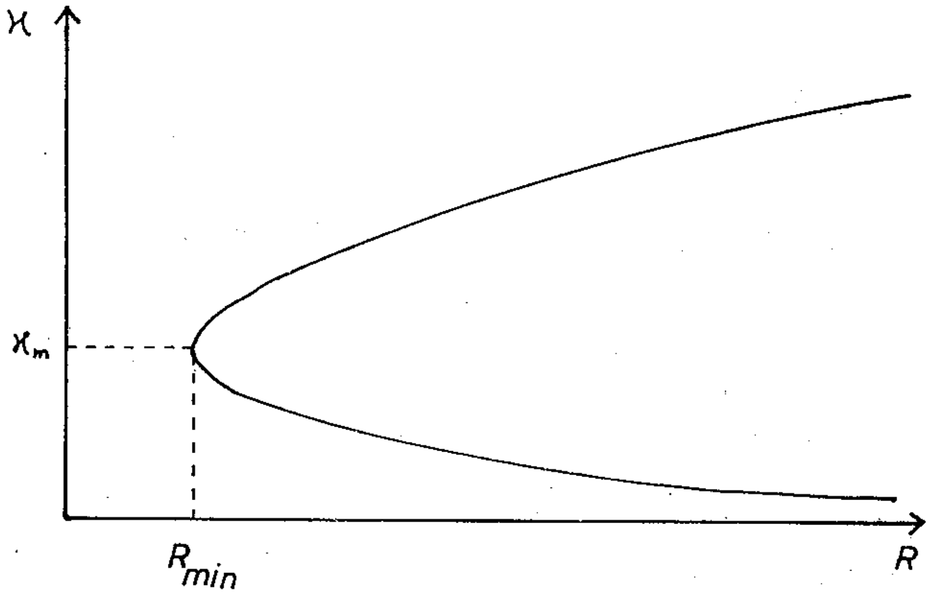


*Fig. 3.* Diagram showing the curve of neutral stability.

is a solution of

$$Tg^2 \frac{\varkappa}{2} + \frac{4}{3\varkappa} Tg \frac{\varkappa}{2} - 1 = 0 \,.$$

We must keep in mind the assumption $s \ll 1$. We then have along the entire neutral curve in Fig. 3 that $|\zeta_0| = |\zeta_1| = \frac{(\varkappa R)^{1/3}}{2} \gg 1$ and $\delta \ll 1$ in accordance with the approximations made above. For decreasing values of $s$, $R_{min}$ is seen to increase and tend to infinity when $s$ tends to zero. This is in accordance with the fact that no neutral wave-solutions are found in the homogeneous case. For finite values of $s$, the solutions found above can not be utilized. A brief discussion of neutral wave-solutions in this case will be given in section 7.

As shown in section 5 only stable wave-perturbations can exist when $\varkappa \ll 1$. A discussion of the proper frequency equations for finite values of $\varkappa$ will give the same result. The regions on both sides of the curve of neutral stability in Fig. 3 are regions for stable wave-solutions.

**7. Waves of infinite wave-length; finite values of s.** For finite values of $s$, we shall restrict ourselves to discuss long waves i.e. $\varkappa \ll 1$. The solutions of the equation for the streamfunction in this case

$$i\zeta\Phi'''' + \zeta^2\Phi'' + s\Phi = 0 \,, \tag{7.1}$$

can be written as

$$(7.2)\qquad \Phi_1 = \sum_{n=0}^{\infty} i^n a_m \zeta^{3n+1}, \qquad \Phi_2 = \sum_{n=0}^{\infty} i^n b_n \zeta^{3n+2},$$

$$\Phi_3 = \sum_{n=0}^{\infty} i^n c_n \zeta^{3n+3}, \qquad \Phi_4 = s\Phi_3 ln\zeta + i\sum_{n=0}^{\infty} i^n d_n \zeta^{3n},$$

The real constants $a_n$, $b_n$, $c_n$ and $d_n$ can easily be determined. For neutral wave-perturbations with

$$(7.3)\qquad \zeta_0 = \varrho e^{i\pi}, \quad \zeta_1 = \varrho, \quad \varrho = \frac{(\varkappa R)^{\frac{1}{3}}}{2},$$

the frequency equation is by equations (3.7)—(3.8) found to be

$$(7.4)\quad Im\left\{f_{12}^*(\varrho) f_{34}(\varrho) + f_{13}^*(\varrho) f_{24}(\varrho) + f_{23}^*(\varrho) f_{14}(\varrho)\right\} = s\pi Re\left\{f_{13}^*(\varrho) f_{23}(\varrho)\right\}.$$

In this equation $f_{jk}^*(\varrho)$ is the complex conjugate of $f_{jk}(\varrho)$. Furthermore *Im* and *Re* stand for the imaginary part and the real part respectively. The left hand side of equation (7.4) can be seen to be a power series in $\varrho$, starting with the term $\varrho^4$, whereas the right hand side can be seen to be a power series in $\varrho$, starting with the term $\varrho^7$. When $s \ll 1$ equation (7.4) is inconvenient for discussion since $\varrho = \frac{(\varkappa R)^{1/3}}{2} \gg 1$ as shown in section 5. For $s = \frac{1}{10}$, the solution of equation (7.4) is found to be $\varrho = 3{,}7$. This value of $\varrho$ is seen to correspond to an asymptotic branch of the neutral curve in $(\varkappa, R)$-plane given by

$$(\varkappa R)^{\frac{1}{3}} = 7{,}4 \qquad \text{when } \varkappa \ll 1 .$$

Equation (5.25) shows that this branch is given by

$$(\varkappa R)^{\frac{1}{3}} = 4{,}9$$

when the asymptotic method is utilized. For increasing values of $s$, the solution of equation (7.4) gives decreasing values of $(\varkappa R)^{\frac{1}{3}}$. For $s > \frac{1}{4}$, these solutions can not be compared to the results found in the preceeding sections.

**8. Final remarks.** In an inviscid fluid, a neutral wave-solution with a velocity of propagation equal to the velocity of the mean flow somewhere in the layer can not be utilized due to the fact that velocity and vorticity become infinite. Singular solutions of this kind can, however, by an integration process be utilized to find the development of an arbitrary infinitesimal disturbance, A. Eliassen, E. Høiland and E. Riis [7]. In the inviscid case the value of $s = \frac{1}{4}$ is emphasized. When $\varkappa \ll 1$, the solutions for the streamfunction can be written as

$$\Phi_1 = (u_0 - c)^{\frac{1}{2} + \sqrt{\frac{1}{4} - s}}, \quad \Phi_2 = (u_0 - c)^{\frac{1}{2} - \sqrt{\frac{1}{4} - s}}$$

which, as mentioned in section 4, are proportional to $\zeta^{\frac{1}{2} + \sqrt{\frac{1}{4} - s}}$ and $\zeta^{\frac{1}{2} - \sqrt{\frac{1}{4} - s}}$ respectively. When $s = \frac{1}{4}$, $\Phi_1$ and $\Phi_2$ coincide. When $0 < s < \frac{1}{4}$, no ordinary wave-solutions exists, whereas when $s > \frac{1}{4}$ an infinite set of neutral wave-solutions are found. The velocity of propagation of these waves do not coincide with the velocity of the mean flow anywhere in the layer.

The particular value $s = \frac{1}{4}$ does not seem to be emphasized in a viscous fluid. The four solutions in the power series in $\zeta$ (7.2) are four independent solutions for any value of $s$. Furthermore the series are convergent for any finite value of $\zeta$. As shown in section 7, we have for instance, neutral wave-solutions for $s < \frac{1}{4}$ and for $s > \frac{1}{4}$.

## APPENDIX A

The constant $C$ was given by equation (4.17) as

(A 1) $$C = 3e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} F_1 dx \,.$$

Due to the definition of $F_1(x)$ as given by the first of equations (4.4), we have

$$C = 3e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} x^{\frac{1}{2}} H_{1/3}^{(1)}\left(\frac{2}{3} x^{\frac{3}{2}} e^{-i\frac{\pi}{4}}\right) dx = 3e^{i\frac{13\pi}{12}} \int_0^{\infty} H_{1/3}^{(1)}\left(y e^{i\frac{\pi}{2}}\right) dy \,.$$

Since

$H_{1/3}^{(1)}\left(y e^{i\frac{\pi}{2}}\right) = \frac{2}{\pi} e^{-i\frac{2\pi}{3}} K_{1/3}(y)$ (see for instance G. N. Watson [8], page 78).

$\int_0^{\infty} K_{1/3}(y)\, dy = \frac{1}{2} \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sqrt{3}}$ (G. N. Watson [8], page 388),

we arrive at

(A 2) $$C = 2\sqrt{3}\, e^{i\frac{5\pi}{12}} \,.$$

The constant $C(\delta)$ was given by equation (4.32) as

$$\text{(A 3)} \qquad C(\delta) = e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} F_1\left(e^{\delta x e^{i\frac{2\pi}{3}}} + e^{\delta x} + e^{\delta x e^{-i\frac{2\pi}{3}}}\right) dx\,.$$

It follows from equation (4.5) that

$$C'(\delta) = -ie^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} F_1''\left(e^{i\frac{2\pi}{3}} e^{\delta x e^{i\frac{2\pi}{3}}} + e^{\delta x} + e^{-i\frac{2\pi}{3}} e^{\delta x e^{-i\frac{2\pi}{3}}}\right) dx\,.$$

Integrating twice by parts, we find

$$C'(\delta) = -i\delta^2 C(\delta)\,.$$

Accordingly, $C(\delta)$ is seen to be

$$\text{(A 4)} \qquad C(\delta) = C\,e^{-i\frac{\delta^3}{3}} = 2\sqrt{3}\,e^{i\frac{5\pi}{12}}\,e^{-i\frac{\delta^3}{3}},$$

since $C(0) = C$ is given by (A 2). Utilizing the first of equations (4.11) which reads

$$\text{(A 5)} \qquad F_1(x) = -F_2\left(xe^{-i\frac{2\pi}{3}}\right),$$

the constant $C_1(\delta)$ given by equation (6.12), can be written as

$$\text{(A 6)} \qquad C_1(\delta) = \int_0^{\infty e^{i\frac{\pi}{2}}} \frac{F_1}{x}\left(\text{Sin}\,\delta x e^{-i\frac{2\pi}{3}} - e^{i\frac{\pi}{3}}\,\text{Sin}\,\delta x - e^{-i\frac{\pi}{3}}\,\text{Sin}\,\delta x e^{i\frac{2\pi}{3}}\right) dx\,.$$

It follows that

$$C_1'(\delta) = -\frac{1}{2}C(\delta) - \frac{1}{2}C(-\delta) = -2\sqrt{3}\,e^{i\frac{5\pi}{12}}\cos\frac{\delta^3}{3}$$

Accordingly, we find

$$\text{(A 7)} \qquad C_1(\delta) = 2\sqrt{3}\,e^{-i\frac{7\pi}{12}} \int_0^{\delta} \cos\frac{x^3}{3}\,dx = 2\sqrt{3}\,\delta e^{-i\frac{7\pi}{12}} + 0(\delta^7)\,.$$

The constant $C_2(\delta)$, given by equation (6.14), can be written as

$$\text{(A 8)} \qquad C_2(\delta) = \int_a^{\infty e^{i\frac{\pi}{2}}} \frac{F_1}{x}\left(\text{Cos}\,\delta x e^{-i\frac{2\pi}{3}} - e^{i\frac{\pi}{3}}\,\text{Cos}\,\delta x - e^{-i\frac{\pi}{3}}\,\text{Cos}\,\delta x e^{i\frac{2\pi}{3}}\right) dx$$

$$- \int_a^{ae^{i\frac{2\pi}{3}}} \frac{F_1\,\text{Cos}\left(\delta x e^{-i\frac{2\pi}{3}}\right)}{x}\,dx + e^{-i\frac{\pi}{3}} \int_a^{ae^{-i\frac{2\pi}{3}}} \frac{F_1\,\text{Cos}\left(\delta x e^{i\frac{2\pi}{3}}\right)}{x}\,dx\,,$$

if we make use of equation (A 5).

If $a \to o$, equation (A 8) gives

$$C_2(\delta) = \int_0^{\infty e^{i\frac{\pi}{2}}} \frac{F_1}{x}\left(\mathrm{Cos}\,\delta x e^{-i\frac{2\pi}{3}} - e^{i\frac{\pi}{3}}\,\mathrm{Cos}\,\delta x - e^{-i\frac{\pi}{3}}\,\mathrm{Cos}\,\delta x e^{i\frac{2\pi}{3}}\right) dx - e^{i\frac{\pi}{3}} F_1(o)\,\frac{2\pi}{\sqrt{3}}\,.$$

It follows that

$$C_2'(\delta) = -\frac{1}{2} C(\delta) + \frac{1}{2} C(-\delta) = 2\sqrt{3}\, e^{i\frac{11\pi}{12}} \sin\frac{\delta^3}{3}\,.$$

Hence,

$$C_2(\delta) = 2\sqrt{3}\, e^{i\frac{11\pi}{12}} \int_0^{\delta} \sin\frac{x^3}{3}\, dx - e^{i\frac{\pi}{3}} F_1(o)\,\frac{2\pi}{\sqrt{3}}\,.$$

Since $F_1(x) = x^{\frac{1}{2}} H^{(1)}_{1/3}\left(\frac{2}{3} x^{\frac{3}{2}} e^{-i\frac{\pi}{4}}\right)$, it is easily found that

$$F_1(o) = \frac{2}{3^{\frac{1}{6}}\,\Gamma\left(\frac{2}{3}\right)}\, e^{-i\frac{5\pi}{12}}\,.$$

Accordingly

(A 9) $$C_2(\delta) = \frac{4\pi}{3^{\frac{2}{3}}\,\Gamma\left(\frac{2}{3}\right)}\, e^{i\frac{11\pi}{12}} + 0\,(\delta^4)\,.$$

Equations (6.17) and (A 5) give

$$C_3(\delta) = -e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} F_1\left(\mathrm{Cos}\,\delta x e^{i\frac{2\pi}{3}} + \mathrm{Cos}\,\delta x + \mathrm{Cos}\,\delta x e^{-i\frac{2\pi}{3}}\right) dx\,.$$

$$C_4(\delta) = -e^{i\frac{\pi}{3}} \int_0^{\infty e^{i\frac{\pi}{2}}} F_1\left(\mathrm{Sin}\,\delta x e^{i\frac{2\pi}{3}} + \mathrm{Sin}\,\delta x + \mathrm{Sin}\,\delta x e^{-i\frac{2\pi}{3}}\right) dx\,.$$

It follows that

(A 10) $$C_3(\delta) = -\frac{1}{2} C(\delta) - \frac{1}{2} C(-\delta) = 2\sqrt{3}\, e^{-i\frac{7\pi}{12}} \cos\frac{\delta^3}{3} = 2\sqrt{3}\, e^{-i\frac{7\pi}{12}} + 0\,(\delta^6)\,.$$

(A 11) $$C_4(\delta) = -\frac{1}{2} C(\delta) + \frac{1}{2} C(-\delta) = 2\sqrt{3}\, e^{i\frac{11\pi}{12}} \sin\frac{\delta^3}{3} = 0\,(\delta^3)\,.$$

## APPENDIX B

The constants $A_{jk}$ were given by equations (5.10) which read

$$\Phi_j(\zeta) = \sum_{k=1}^{4} A_{jk}\Phi_k\left(\zeta e^{-i\frac{2\pi}{3}}\right), \qquad j = 1, 2, 3 \text{ or } 4 . \tag{B 1}$$

If we put $\zeta = ae^{i\frac{\pi}{2}}$, these equations then become

$$\Phi_j\left(ae^{i\frac{\pi}{2}}\right) = \sum_{k=1}^{4} A_{jk}\Phi_k\left(ae^{-i\frac{\pi}{6}}\right).$$

When $a \to \infty$, the asymptotic expansions (5.8)—(5.9) can be utilized on both sides of these equations. We then find

$$\begin{cases} A_{12} = A_{13} = A_{21} = A_{23} = A_{31} = A_{32} = A_{33} = 0 , \\ A_{11} = e^{i\frac{2\pi}{3}\left(\frac{1}{2}+\mu\right)}, A_{22} = e^{i\frac{2\pi}{3}\left(\frac{1}{2}-\mu\right)}, A_{34} = e^{i\frac{\pi}{3}}, A_{43} = 1 . \end{cases} \tag{B 2}$$

From equations (B 1) it follows that

$$\begin{cases} \Phi_j(o) = \sum_{k=1}^{4} A_{jk}\Phi_k(o) , \\ \Phi_j'(o) = e^{-i\frac{2\pi}{3}} \sum_{k=1}^{4} A_{jk}\Phi_k'(o) , \\ \Phi_j''(o) = e^{-i\frac{4\pi}{3}} \sum_{k=1}^{4} A_{jk}\Phi_k''(o) . \end{cases} \tag{B 3}$$

Solving these equations we find

$$\begin{cases} A_{14} = (1 - A_{11}) \dfrac{\Phi_1(o)}{\Phi_4(o)} = \left(e^{i\frac{2\pi}{3}} - A_{11}\right) \dfrac{\Phi_1'(o)}{\Phi_4'(o)} = \left(e^{i\frac{4\pi}{3}} - A_{11}\right) \dfrac{\Phi_1''(o)}{\Phi_4''(o)}, \\ A_{24} = (1 - A_{22}) \dfrac{\Phi_2(o)}{\Phi_4(o)} = \left(e^{i\frac{2\pi}{3}} - A_{22}\right) \dfrac{\Phi_2'(o)}{\Phi_4'(o)} = \left(e^{i\frac{4\pi}{3}} - A_{22}\right) \dfrac{\Phi_2''(o)}{\Phi_4''(o)}, \\ A_{14}A_{41} = e^{i\frac{\pi}{3}} \dfrac{(1 - A_{11}{}^3)\left(e^{i\frac{2\pi}{3}} - A_{22}\right)}{A_{22} - A_{11}}, \\ A_{24}A_{42} = - e^{i\frac{\pi}{3}} \dfrac{(1 - A_{22}{}^3)\left(e^{i\frac{2\pi}{3}} - A_{11}\right)}{A_{22} - A_{11}}, \\ A_{44} = e^{-i\frac{\pi}{3}} + e^{i\frac{\pi}{3}}\left(e^{i\frac{2\pi}{3}} - A_{11}\right)\left(e^{i\frac{2\pi}{3}} - A_{22}\right) . \end{cases} \tag{B 4}$$

The constants $A_{jk}$ are seen to be determined by equations (B 2) and (B 4).

For the particular case $s \ll 1$, equations (5.3) and (5.6) give

$$\text{(B 5)} \qquad \begin{cases} \Phi_1'(o) = 1 + 0(s) , \\ \Phi_2(o) = 1 + 0(s) , \\ \Phi_4(o) = -\int_{\infty e^{-i\frac{\pi}{6}}}^{0} xF_2 dx + 0(s) = \dfrac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{\pi} e^{i\frac{\pi}{4}} + 0(s) , \\ \Phi_4'(o) = \int_{\infty e^{-i\frac{\pi}{6}}}^{0} F_2 dx + 0(s) = \dfrac{2}{\sqrt{3}} e^{-i\frac{7\pi}{12}} + 0(s) . \end{cases}$$

Furthermore we have

$$\text{(B 6)} \qquad \mu = \sqrt{\frac{1}{4} - s} = \frac{1}{2} - s + 0(s^2) .$$

Equations (B 2), (B 4), (B 5) and (B 6) finally give the approximate values

$$A_{11} = e^{i\frac{2\pi}{3}}\left(1 - i\frac{2\pi}{3}s\right), A_{22} = 1 + i\frac{2\pi}{3}s, A_{14} = \frac{\pi}{\sqrt{3}} e^{-i\frac{\pi}{4}} s ,$$

$$A_{24} = -\frac{2\pi^2}{3^{\frac{5}{3}}\Gamma\left(\frac{2}{3}\right)} e^{i\frac{\pi}{4}} s, A_{41} = 2\sqrt{3}\, e^{i\frac{\pi}{12}}, A_{42} = 2 \cdot 3^{\frac{1}{6}}\, \Gamma\left(\frac{2}{3}\right) e^{-i\frac{\pi}{12}} s, A_{44} = e^{-i\frac{\pi}{3}} .$$

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