DISCUSSION OF A HYPERBOLIC EQUATION RELATING TO INERTIA AND GRAVITATIONAL FLUID OSCILLATIONS

BY EINAR HØILAND

FREMLAGT I VIDENSKAPS-AKADEMIETS MØTE DEN 9DE FEBRUAR 1962

Summary. The hyperbolic partial differential equation describing (approximately) simple oscillations (or onset of convection) in a rotating homogeneous fluid and in a stratified incompressible fluid in static equilibrium is studied. Special characteristic lines are defined and applied to the discussion of certain boundary value problems. Finally an analytical approach is attempted for the case of rectangular boundaries.

1. The differential equations. The characteristic lines. The equation governing inertia oscillations symmetric around the axis of rotation in a homogeneous and incompressible fluid may be written [1]

$$(1.1)\qquad (\sigma^2 - 4\Omega_x{}^2)\,\frac{\partial^2\psi}{\partial x^2} - 8\Omega_x\Omega_y\,\frac{\partial^2\psi}{\partial x\partial y} + (\sigma^2 - 4\Omega_y{}^2)\,\frac{\partial^2\psi}{\partial y^2} = 0.$$

Here ψ is the velocity streamfunction, σ the frequency, and Ω_x and Ω_y the components along the x- and y-axis of the (constant) angular velocity $\mathbf{\Omega}$. The linear dimensions of the cell within which the oscillations occur are considered small in comparison with its distance from the axis of rotation.

The equation corresponding to equation (1.1) for gravitational two-dimensional oscillations in an incompressible fluid with a density decreasing exponentially upwards (along the vertical) is

$$(1.2)\quad (\sigma^2 - \Gamma g_y{}^2)\,\frac{\partial^2\psi}{\partial x^2} - 2\Gamma g_x g_y\,\frac{\partial^2\psi}{\partial x\partial y} + (\sigma^2 - \Gamma g_x{}^2)\,\frac{\partial^2\psi}{\partial y^2} + \Gamma\sigma^2\left(g_x\,\frac{\partial\psi}{\partial x} - g_y\,\frac{\partial\psi}{\partial y}\right) = 0.$$

Here Γ is the "coefficient of barotropy" introduced by V. BJERKNES, and g_x and $-g_y$ the components of the acceleration of gravity $\boldsymbol{g}$ along the X- and Y-axis respectively.

Our last equation will also for statically unstable stratification govern the onset of convection. By a somewhat changed interpretation of the quantities Ω_x and Ω_y, equation

(1.1) will likewise describe (approximately) onset of convection symmetric about the axis in a rotating fluid when Ω is decreasing sufficiently rapidly with increasing distance from the axis.

If the linear dimensions of the cell are small compared to the length l given by

$$l = \frac{1}{\Gamma(g_x^2 + g_y^2)^{\frac{1}{2}}} = \frac{1}{\Gamma g}, \tag{1.3}$$

it may be shown that equation (1.2) may approximately be written

$$(\sigma^2 - \Gamma g_y^2)\frac{\partial^2\psi}{\partial x^2} - 2\Gamma g_x g_y \frac{\partial^2\psi}{\partial x\partial y} + (\sigma^2 - \Gamma g_x^2)\frac{\partial^2\psi}{\partial y^2} = 0. \tag{1.4}$$

Considering now the equation

$$(\sigma^2 - \alpha_1^2)\frac{\partial^2\psi}{\partial x^2} - 2\alpha_1\alpha_2 \frac{\partial^2\psi}{\partial x\partial y} + (\sigma^2 - \alpha_2^2)\frac{\partial^2\psi}{\partial y^2} = 0, \tag{1.5}$$

we see that this equation gives the equation (1.1) for inertia oscillations when

$$\alpha_1 = 2\Omega_x \text{ and } \alpha_2 = 2\Omega_y, \tag{1.6}$$

and the equation (1.4) for gravitational oscillations when

$$\alpha_1 = g_y\sqrt{\Gamma} \text{ and } \alpha_2 = g_x\sqrt{\Gamma}. \tag{1.7}$$

In the following discussion we will often have to refer to a coordinate system ξ, η where $\alpha_2 = \alpha_\eta = 0, \alpha_1 = \alpha_\xi = \alpha$; i.e. where the equation (1.5) takes the form

$$(\sigma^2 - \alpha^2)\frac{\partial^2\psi}{\partial \xi^2} + \sigma^2 \frac{\partial^2\psi}{\partial \eta^2} = 0. \tag{1.8}$$

In this equation

$$\alpha^2 = 4\Omega^2 \text{ or } \alpha^2 = \Gamma g^2 \tag{1.9}$$

correspond to inertia and gravitational oscillations respectively. For the case of inertia oscillations the η-axis is perpendicular to the axis of rotation whereas for the case of gravitational oscillations it is directed vertically upwards.

The homogeneous equation (1.5) has no solutions within a cell (bounded by a closed curve) when it is of elliptic type. Only when it is of hyperbolic type may we expect solutions within cells, a fact which might also have been deduced by application of one of the circulation theorems of V. BJERKNES, see [1]. The equation is of hyperbolic type when

$$0 < \sigma^2 < \alpha^2 = \alpha_1^2 + \alpha_2^2. \tag{1.10}$$

From this condition it follows that for every admissible value σ of the frequency, we may always choose a coordinate system x', y' in such a way that $\alpha'_2 = \sigma$. In these systems of coordinates our equation may be written in the form

$$\frac{\partial}{\partial x'}\left\{(2\sigma^2-\alpha^2)\frac{\partial\psi}{\partial x'}\mp 2\sigma\sqrt{\alpha^2-\sigma^2}\frac{\partial\psi}{\partial y'}\right\}=0, \tag{1.11}$$

where the minus sign before the last term corresponds to $\alpha_1' = \sqrt{\alpha^2-\sigma^2}$ whereas the plus sign corresponds to $\alpha_1' = -\sqrt{\alpha^2-\sigma^2}$.

The two sets of lines

$$y' = \text{const.}, \tag{1.12}$$

are in the ξ,η-system given by

$$\eta = \mp\frac{\sigma}{\sqrt{\alpha^2-\sigma^2}}\xi + C, \tag{1.13}$$

the minus sign again corresponding to $\alpha_1' > 0$, the plus sign to $\alpha'_1 < 0$. These two sets of lines are *the characteristic lines* of the hyperbolic equation (1.5). For convenience let us denote the first set of characteristic lines set I, the second set set II, and the corresponding coordinates x_1', y_1' and x_2', y_2'. For the angles β_1 and β_2 which the characteristic lines make with the ξ-axis, we have

$$\begin{aligned}\tan\beta_1 &= -\frac{\sigma}{\sqrt{\alpha^2-\sigma^2}} \text{ for set I,}\\ \tan\beta_2 &= \frac{\sigma}{\sqrt{\alpha^2-\sigma^2}} \text{ for set II,}\end{aligned} \tag{1.14}$$

or

$$\begin{aligned}\sin\beta_1 &= -\frac{\sigma}{\alpha} \text{ for set I,}\\ \sin\beta_2 &= \frac{\sigma}{\alpha} \text{ for set II.}\end{aligned} \tag{1.14'}$$

The two sets of characteristic lines are thus symmetric with regard to the ξ- and η-axis.

The equations (1.11) may be integrated to give

$$\begin{aligned}(2\sigma^2-\alpha^2)\frac{\partial\psi}{\partial x'_1} - 2\sigma\sqrt{\alpha^2-\sigma^2}\frac{\partial\psi}{\partial y'_1} &= f_1(y_1'),\\ (2\sigma^2-\alpha^2)\frac{\partial\psi}{\partial x'_2} + 2\sigma\sqrt{\alpha^2-\sigma^2}\frac{\partial\psi}{\partial y'_2} &= f_2(y'_2).\end{aligned} \tag{1.15}$$

We then obtain

$$\begin{aligned}(2\sigma^2-\alpha^2)\frac{\partial\psi}{\partial x_1'} - 2\sigma\sqrt{\alpha^2-\sigma^2}\frac{\partial\psi}{\partial y'_1} &= \text{const.},\\ (2\sigma^2-\alpha^2)\frac{\partial\psi}{\partial x'_2} + 2\sigma\sqrt{\alpha^2-\sigma^2}\frac{\partial\psi}{\partial y'_2} &= \text{const.}\end{aligned} \tag{1.16}$$

along characteristic lines of set I and set II respectively.

Introducing the velocity components

$$(1.17)\qquad \begin{aligned} u'_1 &= -\frac{\partial \psi}{\partial y'_1}, \quad v'_1 = \frac{\partial \psi}{\partial x'_1} \\ u'_2 &= -\frac{\partial \psi}{\partial y'_2}, \quad v'_2 = \frac{\partial \psi}{\partial x'_2} \end{aligned}$$

in the systems of coordinates corresponding to set I and II respectively, we get

$$(1.18)\qquad \begin{aligned} &(2\sigma^2 - a^2)\, v'_1 + 2\sigma \sqrt{a^2 - \sigma^2}\, u'_1 = \text{const.}, \\ &(2\sigma^2 - a^2)\, v'_2 - 2\sigma \sqrt{a^2 - \sigma^2}\, u'_2 = \text{const.} \end{aligned}$$

valid along the characteristic lines of set I and set II respectively.

2. The general solution of equation (1.5). Equation (1.5) may be written in the equivalent form:

$$(2.1)\qquad \left(\frac{\partial}{\partial x} - \frac{a_1 a_2 - \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{a_1 a_2 + \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2}\frac{\partial}{\partial y}\right)\psi = 0.$$

It is easily seen that this equation has the solution

$$(2.2)\qquad \psi = F_1\left(y + \frac{a_1 a_2 - \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2}\, x\right) + F_2\left(y + \frac{a_1 a_2 + \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2}\, x\right).$$

Since this solution involves two independent arbitrary functions, it is also the general solution of the equation.

The equations giving constant arguments in the two functions, i.e.

$$(2.3)\qquad \begin{aligned} y &= -\frac{a_1 a_2 - \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2}\, x + \text{const.}, \\ y &= -\frac{a_1 a_2 + \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2}\, x + \text{const.}, \end{aligned}$$

are the equations for the two sets of characteristic lines in the xy-system of coordinates. For later application we will in this connection especially draw attention to the fact that if we change the sign of $a_1\, a_2$ (say by changing the sign of a_2), i.e. change the sign of the coefficient for $\frac{\partial^2 \psi}{\partial x \partial y}$ in equation (1.5), the slope of the systems of characteristic lines will change sign in the xy-system.

We introduce the abbreviations

$$(2.4)\qquad \begin{aligned} c_1 &= \frac{a_1 a_2 - \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2} \\ c_2 &= \frac{a_1 a_2 + \sigma\sqrt{a^2 - \sigma^2}}{\sigma^2 - a_1^2}, \end{aligned}$$

into equation (2.2), and obtain

$$\psi = F_1 (y + c_1 x) + F_2 (y + c_2 x). \tag{2.5}$$

In the interval (1.10) of σ, c_1 and c_2 cannot be equal.

Let us now assume that the functions F_1 and F_2 may in a region including the cell and its boundary be developed in a convergent McLaurin series, i.e. that we have in this region

$$\psi = \sum_0^\infty A_n (y + c_1 x)^n + \sum_0^\infty B_n (y + c_2 x)^n. \tag{2.6}$$

Consider as an example that part of the boundary consists of the x-axis from $x = 0$ to $x = x_0$ and part of the y-axis from $y = 0$ to $y = y_0$. At these parts of the boundary the streamfunction $\psi = 0$, so that we have

$$\sum_0^\infty (A_n + B_n)\, y^n = 0 \qquad \text{for } 0 \leq y \leq y_0$$

$$\sum_0^\infty [A_n c_1{}^n x^n + B_n c_2{}^n x^n] = 0 \qquad \text{for } 0 \leq x \leq x_0.$$

These two equations can not be satisfied simultaneously when $|c_1| \neq |c_2|$. Since c_1 cannot be equal to c_2, the only possibility left is

$$c_1 = - c_2. \tag{2.7}$$

Then the equations will be satisfied simultaneously for

$$A_{2n} = - B_{2n},\ A_{2n+1} = B_{2n+1} = 0,$$

and the solution (2.6) takes the form

$$\psi = \sum_1^\infty A_{2n} \left[(\eta + c_2 \xi)^{2n} - (\eta - c_2 \xi)^{2n}\right]. \tag{2.8}$$

Here we have written η and ξ instead of y and x because as is readily seen, the condition $c_1 = - c_2$ is satisfied only in the $\xi\eta$-system.

Thus we have shown that a solution of the form (2.6) does not exist when the boundary has a corner where two pieces of straight lines belonging to the boundary meet at a right angle unless the pieces of straight lines are parallel to the ξ- and the η-axis.

3. The special characteristic lines. In section 1 we defined the two sets of characteristic lines for which the relations (1.18) were fulfilled. We now for each set define what we will call *the special characteristic lines* by the requirement that the relations

$$\begin{aligned} &(2\sigma^2 - \alpha^2) v_1' + 2\sigma \sqrt{\alpha^2 - \sigma^2}\, u_1' = 0, \\ &(2\sigma^2 - \alpha^2) v_2' - 2\sigma \sqrt{\alpha^2 - \sigma^2}\, u_2' = 0 \end{aligned} \tag{3.1}$$

shall be fulfilled along a special characteristic line of set I and set II respectively. Since the streamlines are closed and with the exception of contingent points where the velocity on a streamline vanishes, of finite curvature, there must always on a streamline be points where the first of equations (3.1) is satisfied and also points where the second of equations (3.1) is satisfied. The corresponding characteristic line through these points is a special characteristic line given by the appropriate relation of the pair (1.13).

From the relations (3.1) we see that *at all points of a special characteristic line the velocities are parallel* to each other. Assuming that they are not parallel to the characteristic line itself, they can not in a divergence-free motion have the same direction along the whole line through the interior of the cell. Therefore *the velocity must be zero at some point on a special characteristic line inside the cell.* At a point where the velocity is zero, both of the relations (3.1) will be fulfilled. Thus *there will pass a special characteristic line of each set through a point where the velocity is zero inside the cell.*

The first of the relations (3.1) is valid along a special characteristic line of set I with negative slope, the second along a special characteristic line of set II with positive slope in the $\xi\eta$-system. It is easily deduced that *the velocity in a point on a special characteristic line of set I is parallel to the characteristic lines of set II while the velocity in a point on a special characteristic line of set II is parallel to the characteristic lines of set I.* We now also have verified our assumption above that the velocity in a point of a special characteristic line is not parallel to the characteristic line itself. Above we saw that in a point where the velocity is zero inside the cell, two special characteristic lines must intersect. Now we also see that *in a point where two special characteristic lines intersect, the velocity must always be zero.*

4.. Some special boundaries.

a. Circular boundary. If there is only one zero point for the velocity (no zero point on the boundary) inside the circular boundary, we will have two special characteristic lines intersecting at the zero point. Each of the special characteristic lines will, according to what we deduced in the previous section, cut the circle in two points with parallel tangents. Thus the special characteristic lines will be diameters, and the velocity on each of them must be perpendicular to the line itself. Therefore, since the velocity at a point of a special characteristic line must be parallel to the other set of characteristic lines, the characteristic lines of the one set must be perpendicular to the characteristic lines of the other set. Thus the one set of characteristic lines make an angle $\beta_1 = -\frac{\pi}{4}$, the other set an angle $\beta_2 = \frac{\pi}{4}$ with the ξ-axis. From equations (1.4) we then obtain

$$\sigma = \frac{\sqrt{2}}{2}\, a, \tag{4.1}$$

corresponding to frequencies

(4.2)
$$\sigma = \sqrt{2}\Omega \quad \text{for inertia oscillations,}$$
$$\sigma = \frac{\sqrt{2}}{2} g\sqrt{\Gamma} \quad \text{for gravitational oscillations.}$$

It is easily verified that the solution considered here is given by

$$\psi = A(\xi^2 + \eta^2 - r^2), \tag{4.3}$$

where A is a constant and r the radius of the boundary circle.

Consider now the case that there are two zero points (for the velocity) inside the circular boundary. Then there must be two zero points on the boundary. From simple geometrical considerations it is easily seen that these two points will be the terminal points of a diameter either parallel or perpendicular to the ξ-axis. Consider the first case, it is readily seen that in order to satisfy the boundary conditions and the conditions of symmetry of the characteristic lines in the $\xi\eta$-system together with the rules for the directions of the velocities on the special characteristic lines, we must have $\beta_1 = -\frac{\pi}{6}$, $\beta_2 = \frac{\pi}{6}$. This corresponds to

$$\sigma = \frac{a}{2}, \tag{4.4}$$

or

(4.5)
$$\sigma = \Omega \quad \text{for inertia oscillations,}$$
$$\sigma = \frac{g}{2}\sqrt{\Gamma} \quad \text{for gravitational oscillations.}$$

It may be verified that the corresponding streamfunction is given by

$$\psi = A\eta(\xi^2 + \eta^2 - r^2), \tag{4.6}$$

the origin being at the centre of the circle.

Assuming that a solution of equation (1.5) also exists for an arbitrary number of zero points inside the circle, we may proceed to find the frequency, say for n zero points on a diameter parallel to the η-axis. The angles β_1 and β_2 will be given by

$$\beta_1 = -\frac{\pi}{2(n+1)}, \quad \beta_2 = \frac{\pi}{2(n+1)}, \tag{4.7}$$

leading to a frequency

$$\sigma = a \sin \frac{\pi}{2(n+1)}. \tag{4.8}$$

Where the zero points are situated, may also be quite easily determined for a given value of n. We find for instance that for $n = 2$, the zero points are at $\eta = \pm \frac{\sqrt{3}}{3} r$ and for $n = 3$, the zero points are at $\eta = 0$ and $\eta = \pm \sqrt{2 - \sqrt{2}}\, r$.

If the zero points are on a diameter belonging to the ξ-axis, we obtain for the frequency

$$(4.9) \qquad \sigma = \alpha \cos \frac{\pi}{2(n+1)} .$$

b. Elliptic boundary. It follows from the definition of conjugate diameters for an ellipse that two conjugate diameters will satisfy all conditions for being special characteristic lines through the centre of the ellipse when the ξ-axis is parallel to one of the lines bisecting the angle between the conjugate diameters. With the conjugate diameters as special characteristic lines we will have only one zero point inside the elliptic boundary.

The streamfunction is given by

$$(4.10) \qquad \psi = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right),$$

where a and b are the halfaxes of the ellipse. The corresponding frequency is

$$(4.11) \qquad \sigma = \left(\frac{b^2\alpha_1{}^2 + a^2\alpha_2{}^2}{a^2 + b^2}\right)^{\frac{1}{2}} .$$

It is easily verified that equation (1.8) has a solution given by

$$(4.12) \qquad \psi = \eta\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - 1\right),$$

being zero at the elliptic boundary

$$(4.13) \qquad \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1.$$

For the corresponding frequency we obtain

$$(4.14) \qquad \sigma = \alpha\left(\frac{b^2}{b^2 + 3a^2}\right)^{\frac{1}{2}} .$$

We note that by the transformation

$$(4.15) \qquad \xi = a\xi', \ \eta = b\eta',$$

the ellipse (4.13) transforms into a circle. By this transformation the differential equation (1.8) transforms to an equation which may be written

$$(4.16) \qquad (\sigma^2 - \alpha'^2)\frac{\partial^2\psi}{\partial\xi'^2} + \sigma^2\frac{\partial^2\psi}{\partial\eta'^2} = 0.$$

Here α'^2 is given by

$$(4.17) \qquad \alpha'^2 = \frac{1}{a^2}\left[b^2\alpha^2 + (a^2 - b^2)\sigma^2\right].$$

According to what we developed for a circular boundary above, the equation (4.16) gives for a circular boundary in the $\xi'\eta'$-plane with n zero points on the η'-axis (the centre of the boundary being at the origin) the frequency equation

$$\sigma^2 = a'^2 \sin^2 \frac{\pi}{2(n+1)} \,. \tag{4.18}$$

Inserting here our expression (4.17) for a'^2, and solving with respect to σ, we obtain

$$\sigma = ba \sin \frac{\pi}{2(n+1)} \left(\frac{1}{a^2 \cos^2 \frac{\pi}{2(n+1)} + b^2 \sin^2 \frac{\pi}{2(n+1)}} \right)^{\frac{1}{2}} . \tag{4.19}$$

Thus, this is the frequency for oscillations within an elliptic boundary with half-axes a and b having n zero points on the axis which is part of the η-axis.

For n zero points on the other axis of the ellipse we obtain the frequency simply by changing the sine to cosine and the cosine to sine in the above formula for σ.

As for the preceding case a solution of equation (4.16) for $n > 3$ must be assumed.

c. A general boundary of finite curvature and with no inflexion points. Just a few remarks will be given here on the case that the cell is bounded by a general boundary of finite curvature and with no inflexion points. Let us assume that there exists only one zero point for the velocity inside the boundary. Now, as our boundary is defined, there will to a given point on it correspond one and only one other point on the curve with tangent parallel to the tangent at the given point, i.e. with the same direction of the velocity at the two points. If we draw a line between the two points, say line I, this line may then be a special characteristic line. If there is a corresponding special characteristic line of the other set, this line, say line II, must connect the two points on the boundary where the tangents are parallel to line I. For these two lines to be corresponding special characteristic lines, it is, however, also required that line II is parallel to the velocities at the terminal points of line I, and it is evident that this requirement will, in general, not be fulfilled. Thus it seems probable that the general rule will be that no solutions of equation (1.5) exist for a general boundary of the type studied here with one zero point within the boundary. For specially constructed boundaries, we may, however, have solutions. In the case of the ellipse we saw that for every point of the ellipse we had corresponding lines which fulfilled all the requirements for being corresponding special characteristic lines.

d. A parallelogram as a boundary. At the corners of the parallelogram the velocity is zero, and therefore the diagonals of the parallelogram satisfy all the requirements for being corresponding special characteristic lines. The lines bisecting the diagonals must then be chosen as the ξ- and η-axis. The solution may be given in the explicit form

$$\psi = A \sin \frac{\pi}{L} (x - y \operatorname{cotan} \gamma) \quad \sin \frac{\pi}{H} y, \tag{4.20}$$

where L is the length of the "horizontal" sides (parallel to the x-axis) of the parallelogram, γ the angle the two other sides makes with the x-axis and H the height of the parallelogram measured along the y-axis.

For the frequency is obtained

$$\sigma = \left(\frac{2a_2{}^2L^2 + a^2H^2 \pm H\sqrt{4a_1{}^2a_2{}^2L^2 + a^4H^2}}{2(L^2 + H^2)}\right)^{\frac{1}{2}} . \tag{4.21}$$

The corresponding value of γ is given by

$$\operatorname{cotan} \gamma = \frac{a_1 a_2}{a_2{}^2 - \sigma^2} . \tag{4.22}$$

It appears that for given values of L and H, we get two values of σ with two different values of γ. It may be shown that the one value of γ corresponds to a positive slope of the non-horizontal sides, the other to a negative slope of the non-horizontal sides of the parallelogram. Since, however, a somewhat more detailed discussion of this case with a parallelogram as a boundary is given elsewhere [1], we will refrain from further discussion here.

e. Rectangular boundary. A special case of a parallelogram is a rectangle. For a rectangle the angle γ must be equal to $\frac{\pi}{2}$ and therefore cotan γ equal to zero. According to equation (4.22) above, we must then have either a_1 or a_2 equal to zero. That means that a solution with only one zero-point for the velocity inside the cell will, for a rectangular boundary, exist only when the sides of the rectangle are parallel to the ξ- and η-axis.

The solution is given by

$$\psi = A \sin \frac{\pi}{L} \xi \quad \sin \frac{\pi}{H} \eta, \tag{4.23}$$

with

$$\sigma = a \frac{H}{\sqrt{L^2 + H^2}} = a \frac{H}{D} , \tag{4.24}$$

where D is the diagonal of the rectangle, see Fig. 1.

Since the diagonals must be special characteristic lines, the angle β is given by (see Fig. 1).

$$\sin \beta = \frac{H}{D} . \tag{4.25}$$

Inserting from (1.14') we obtain again the relation (4.24) above.

The only zero point within the rectangle is the point where the two diagonals intersect.

Considering the rectangle made up of 2, 3, 4, ... rectangles, each of the smaller rectangles being congruent, the frequency formula (4.24) will apply to each of the small rectangles. Corresponding to a partition of our rectangle in rectangles with sides $\frac{L}{m}$ and $\frac{H}{n}$ (m and n positive integers), we obtain the solution

$$(4.26) \qquad \psi = A \sin\frac{m\pi}{L}\,\xi \;\; \sin\frac{n\pi}{H}\,\eta,$$

with a frequency

$$(4.27) \qquad \sigma = \alpha \frac{mH}{\sqrt{n^2L^2 + m^2H^2}}.$$

According to the theory of Fourier series, our system of solutions (4.26) forms a complete set of functions within the rectangular boundary.

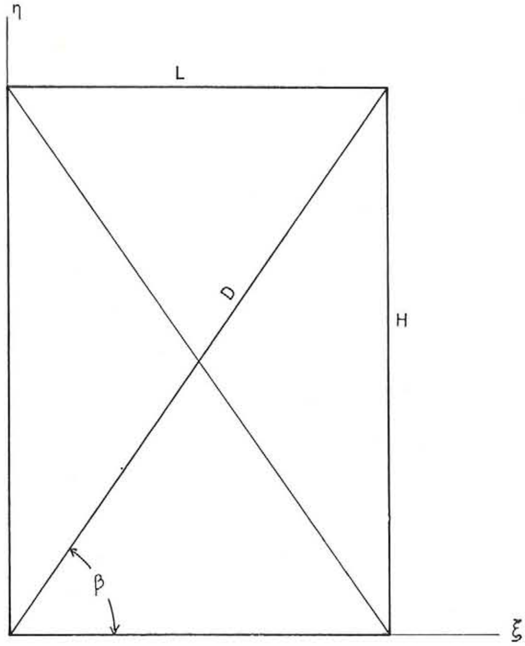

Fig. 1.

If $m = n$, i.e. if the small rectangles are similar to the rectangular boundary, we get the same frequency as for only one zero point within the cell. Thus we get an infinite number of solutions corresponding to this value of the frequency. And this is also true for all other values of σ corresponding to different values of m and n. Thus the Eigen-solution of equation (1.5) for a definite Eigen-value is far from uniquely determined in the case considered here.

The infinite numbers of solutions corresponding to a definite value of σ may be presented in a way very different from the above one. We consider only the case that we have one zero point for the velocity at each side of the rectangle between the corners. If we then select a point Q on one of the rectangle sides, Fig. 2, we can with this point as starting-point draw lines parallel to the diagonals as shown in the diagram. These lines together with the diagonals will be special characteristic lines for the definite value of σ when their intersection points are zero points for the velocity. We get four minor cells inside our boundary. The streamlines are sketched in the diagram. Letting the point Q pass from one terminal of the side to the other terminal, we will obtain a continuous spectrum of solutions for the definite value of σ all of them corresponding to a division of the rectangle in four minor cells. If the solution is uniquely determined by the number and positions of zero points for the velocity inside the rectangle, our spectrum will obviously contain all solutions for the definite value of σ corresponding to four minor cells inside the boundary. The suggestion that all the solutions for the definite value of σ might be obtained by some integration over this spectrum, might then perhaps be likely, but will not be discussed here. We notice

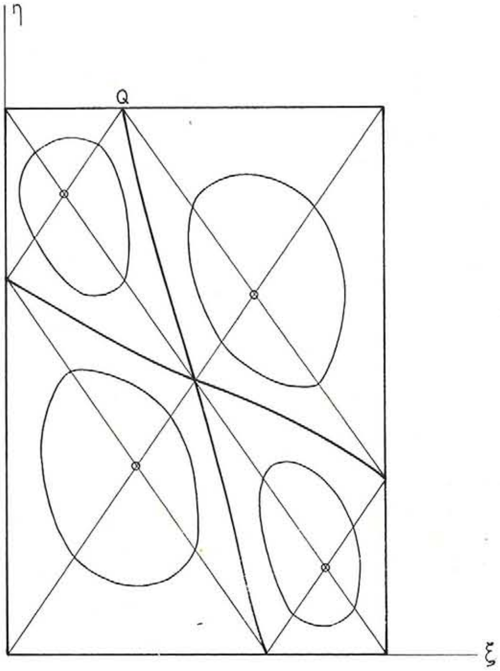

Fig. 2.

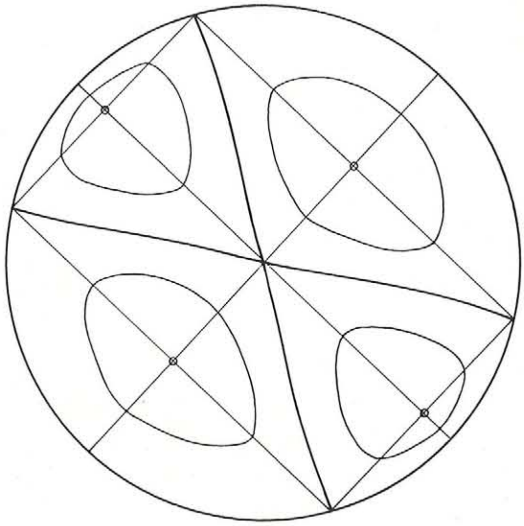

Fig. 3.

that when Q is at the middle point of the side, we obtain a solution of the set discussed previously.

If we select two points on the side as starting-points, we obtain nine minor cells, and so on.

The same procedure as used above for the rectangle, may also be applied to the circular and elliptic boundary. In Fig. 3 is sketched a solution for circular boundary with four minor cells and with frequency equal to the frequency obtained for only one zero point of the velocity inside the boundary.

It should be noted that above we have assumed the existence of the solution when appropriate special characteristic lines can be drawn.

Consider now a rectangular boundary with pair of sides which are not parallel to the ξ respectively the η-axis, which are, however, parallel to the x respectively the y-axis. Let the angle between the x-axis and the ξ-axis be denoted by φ, see Fig. 4.

The diagonals are now not symmetric with regard to the $\xi\eta$-system, and can therefore not form corresponding special characteristic lines. Therefore, with this boundary there are, as already stated above, no solutions of equation (1.5) with only one zero-point for the velocity inside the rectangle. There may, however, be solutions with two zero points. In Fig. 4 are drawn two pairs of parallel lines satisfying all the requirements for being corresponding special characteristic lines when the velocities at the boundary points E and F are zero. The zero points for the velocity inside the boundary are at G and J.

The lower limit for the slope of the special characteristic lines is φ (the point E coincides with the point A Fig. 4). To a given value of L and H (the sides AB and BC respectively), the angle φ must therefore satisfy the relation

$$|\tan 2\varphi| < \frac{H}{L}, \tag{4.28}$$

if a solution with two zero points for the velocity inside the rectangle and one zero point at each of the sides BC and AD may exist. For $\varphi = 45°$ the left hand side of our formula becomes infinite. In this case there is therefore no solutions of the considered type.

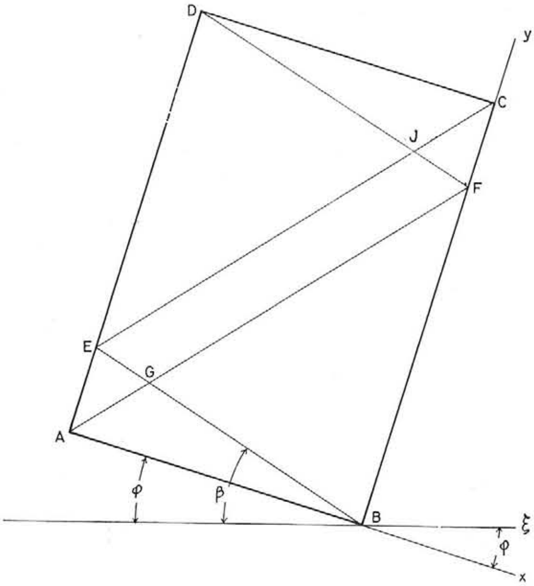

Fig. 4.

The condition that solutions with two zero points inside the rectangle and one zero point on each of the sides AB and DC may be possible is easily seen to be given by the relation

$$|\tan 2\varphi| < \frac{L}{H}. \tag{4.29}$$

From formulae (4.28) and (4.29) it appears that both types of solutions considered here may exist for the same rectangular boundary when $0 \leq \varphi < \frac{\pi}{8}$ or $\frac{\pi}{2} \geq \varphi \geq \frac{3\pi}{8}$. When, however, $\frac{\pi}{8} \leq \varphi \leq \frac{3\pi}{8}$, the two types of solutions cannot exist simultaneously.

For the angle β determining the frequency we must for the case illustrated in Fig. 4 have

$$\varphi + \arctan \frac{H}{L} > \beta > \varphi. \tag{4.30}$$

β is determined by the equation

$$\tan(\beta - \varphi) + \tan(\beta + \varphi) = \frac{H}{L}. \tag{4.31}$$

This equation enables us to determine β as a function of φ and $\frac{H}{L}$. Thus we also can find the frequency if the corresponding solution exists. β, of course, must also fulfill the condition (4.30) above and φ must fulfill the condition (4.28). Also for the zero point situated at each of the two other sides of the rectangle the frequency equation and a condition similar to (4.30) may be deduced. It will, however, not be given here.

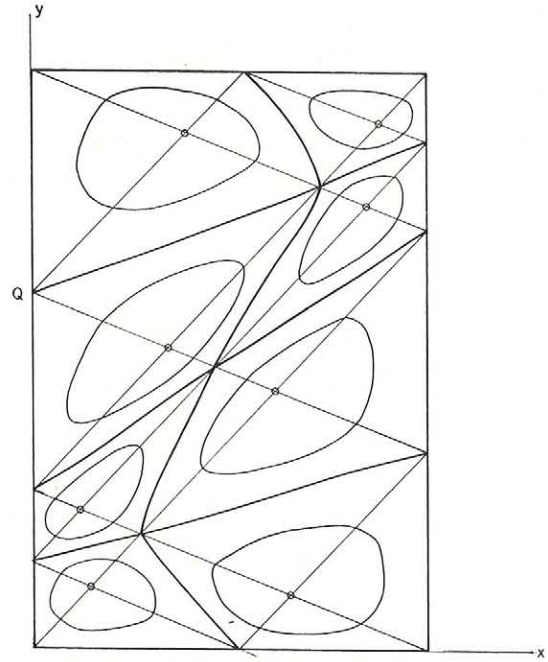

Fig. 5.

It is easily seen that no solutions with an equal number of zero points on two of the (parallel) sides of the rectangle (between the corners) and none at the two other sides can exist. For an unequal number (an equal number of cells within the rectangle), say $2n + 1$, we may have solutions. Corresponding to the formula (4.28) and (4.29) we obtain the corresponding relations

$$(4.32) \qquad \begin{aligned} |\tan 2\varphi| &< \frac{H}{(2n+1)L}, \\ |\tan 2\varphi| &< \frac{L}{(2n+1)H}. \end{aligned}$$

The formulae corresponding to the relations (4.30) and (4.31) are easily written down.

It will be noted that for the case considered here, just as in the case when the sides of the rectangle were parallel respectively perpendicular to the ξ-axis, we may select a point Q on a side, and with this point as starting-point draw lines parallel to the characteristic lines. Considering only the case that we have one zero point for the velocity at each of the one pair of rectangle sides and three zero points at each of the other pair of sides, these parallels will form two parallelograms, see Fig. 5. It appears that the parallels, together with the original special characteristic lines, may represent a new set of special characteristic lines. A sketch of the corresponding solution (assuming again that it exists) is given in Fig. 5. As in the preceding cases we may let the point Q travel along the side, and we may obtain a continuous spectrum of solutions corresponding to a definite value of σ, and with eight minor cells within the rectangle. Again we also may choose two or more points on the side as starting-points for a similar procedure, thus obtaining solutions with 21, 40, ... minor cells inside the boundary.

5. An analytical approach for a rectangular boundary. In the last part of the preceding section we discussed by the method of special characteristic lines the possibility of having solutions of equations (1.5) within rectangular boundaries. We found that when certain conditions were fulfilled, we might have such solutions. In no cases with φ different from zero, was the existence of such solutions demonstrated. And even if they really existed in all cases when an appropriate system of special characteristic lines might be drawn, we obviously did not get a complete system of fundamental solutions as in the case when $\varphi = 0$. For instance we did not in any case with φ different from zero have a possible solution with only one zero point for the velocity inside the boundary. And for $\varphi = 45°$ we do not have possibility for a single solution of equation (1.5) with a finite number of zeros inside the boundary.

The problem as to whether fundamental solutions exist when an appropriate system of special characteristic lines can be drawn, may be of interest both from a mathematical and a physical point of view.

For $\varphi = 0$ we had fundamental solutions of our equation which may directly be continued to the whole $\xi\eta$-plane representing oscillations or onset of convections within cells obtained by dividing the plane in rectangles congruent with our original rectangle. In neighbouring cells the streamlines will be identical, the velocity, however, will be in opposite rotational direction.

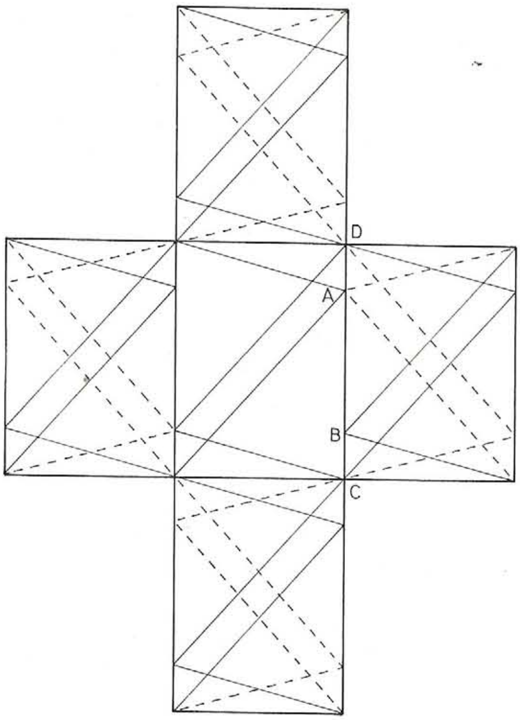

Fig. 6.

A similar direct continuation of a contingent fundamental solution is not possible when φ is different from zero. The diagram Fig. 6 shows the geometric pattern of the corresponding characteristic lines in neighbouring cells for the case with two zero points for the velocity inside the cells. Looking now at for instance the points A and B, we see that the velocity there is zero at one side of the line CD and different from zero at the other side. If the pattern of the special characteristic lines in the cell to the right of CD had been as shown by the dotted lines in the diagram, the streamlines in the neighbouring cells to the right and left of the side CD would be symmetric with regard to this side, of course again assuming the solution to exist. This change of pattern of special characteristic lines in the cell to the right of CD corresponds, as emphasized in section 2, to a change of the sign of a_1a_2 in equation (1.5). This change of sign will not change the value of σ. The direction of the ξ- and η-axis will however be changed. Thus applying equation (1.5) with say $-a_2$ substituted for a_2, in the cell to the right, we may continue periodically the contingent solution in the middle cell to the cell to the right. And it is easily seen that for all possible solutions in the originally considered cell, a periodic regular continuation of the solutions is possible if, by passing from a cell to the neighbouring cell we change the sign of the coefficient of $\frac{\partial^2\psi}{\partial x \partial y}$ in equation (1.5).

The Fourier development for a function changing sign in this manner is easily found to be given by

$$-\frac{32\alpha_1\alpha_2}{\pi^2}\sum_0^\infty\sum_0^\infty \frac{1}{(2m+1)(2n+1)} \sin(2m+1)\pi\frac{x}{L}\sin(2n+1)\pi\frac{y}{H}.$$

Thus, the contingent fundamental solutions of the hyperbolic equation

$$(5.1)\qquad \begin{aligned} &(\sigma^2-\alpha_1^2)\frac{\partial^2\psi}{\partial x^2} \\ &-\frac{32\alpha_1\alpha_2}{\pi^2}\sum_0^\infty\sum_0^\infty \frac{1}{(2m+1)(2n+1)} \sin(2m+1)\pi\frac{x}{L}\sin(2n+1)\pi\frac{y}{H}\frac{\partial^2\psi}{\partial x\partial y} \\ &+(\sigma^2-\alpha_2^2)\frac{\partial^2\psi}{\partial y^2}=0 \end{aligned}$$

within a rectangular boundary may, as in the case of $\varphi = 0$, be continued directly to the whole xy-plane.

Introducing in the above equation a double Fourier development for ψ, i.e. putting

$$(5.2)\qquad \psi=\sum_0^\infty\sum_0^\infty B_{pq}\sin p\,\pi\frac{x}{L}\quad\sin q\,\pi\frac{y}{H},$$

we obtain an equation which may be written

$$(5.3)\qquad \begin{aligned} &\frac{\pi^2}{8}\sum_0^\infty\sum_0^\infty B_{pq}\left[(\sigma^2-\alpha_1^2)\frac{p^2H}{L}+(\sigma^2-\alpha_2^2)\frac{q^2L}{H}\right]\sin p\,\pi\frac{x}{L}\quad\sin q\,\pi\frac{y}{H} \\ &+\alpha_1\alpha_2\sum_0^\infty\sum_0^\infty\sum_0^\infty\sum_0^\infty B_{pq}\frac{pq}{(2m+1)(2n+1)}\left[\sin(2m+1+p)\pi\frac{x}{L}+\sin(2m+1-p)\pi\frac{x}{L}\right] \\ &\times\left[\sin(2n+1+q)\pi\frac{y}{H}+\sin(2n+1-q)\pi\frac{y}{H}\right]=0 \end{aligned}$$

The coefficient for every term $\sin r\,\pi\frac{x}{L}\sin s\,\pi\frac{y}{H}$ (r and s definite positive numbers) must be zero. Simple calculations then lead to the equation

$$(5.4)\qquad \begin{aligned} &\alpha_1\alpha_2\sum_{p>0}^\infty\sum_{q>0}^\infty B_{pq}\frac{pq}{(p^2-r^2)(q^2-s^2)} \\ &+\frac{\pi^2}{32}\left[(\sigma^2-\alpha_1^2)\frac{r^2H}{L}+(\sigma^2-\alpha_2^2)\frac{s^2L}{H}\right]\frac{B_{rs}}{rs}=0. \end{aligned}$$

Here it is summed for all even positive numbers (except zero) of p respectively q if r respectively s are odd numbers, and for all odd numbers of p respectively q if r respectively s are even numbers.

Since p and q never take the value zero we may introduce

$$D_{pq} = p^{n+1}\, q^{n+1}\, B_{pq}, \tag{5.5}$$

n being an arbitrary number.

The system of equation (5.4) then assumes the form

$$\begin{aligned} &\alpha_1\alpha_2 \sum_{p>0}^{\infty} \sum_{q>0}^{\infty} D_{pq} \frac{1}{p^n q^n (p^2 - r^2)(q^2 - s^2)} \\ &+ \frac{\pi^2}{32}\left[(\sigma^2 - \alpha_1{}^2)\frac{r^2 H}{L} + (\sigma^2 - \alpha_2{}^2)\frac{s^2 L}{H}\right]\frac{D_{rs}}{r^{n+2}\, s^{n+2}} = 0. \end{aligned} \tag{5.6}$$

To this doubly infinite system of homogeneous linear equations it corresponds a determinant equation where the determinant has a double infinity of terms in each row and in each column. So far as the author knows, no methods are available for solving an equation of this type. There will therefore be no attempt to discuss it here. The aim of this section was only to present a formulation of a consistent attack of the problem of finding solutions of the hyperbolic equation (1.5) within rectangular cells. From the discussion in the previous section it appears that the determinant equation will have no solutions for $\sigma^2 \geq \alpha_1{}^2 + \alpha_2{}^2$. Nor will it have solutions when $\alpha_1 = \alpha_2$. It is seen that for $\alpha_1\alpha_2 = 0$, we recover the frequency equation (4.27).

The necessity of changing the sign of α_2 from cell to cell in order to obtain a solution which may be continued periodically is from a physical point of view readily understood. By this change of sign we obtain a density distribution which is symmetric in neighbouring cells and the corresponding vector $\boldsymbol{\alpha}$ is also symmetric. It is then evident that when a solution giving oscillations (or onset of convections) exists in one cell, a solution with symmetric streamline pattern will exist in the neighbouring cells.

REFERENCE

1. Høiland, Einar: *On the interpretation and application of the circulation theorems of* V. Bjerknes, Arch. Math. og Nat. XLII, Nr. 5, Oslo 1939.

(Manuscript received 17 November 1961)