A NUMERICAL METHOD OF SOLVING CERTAIN PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

BY RAGNAR FJØRTOFT

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Summary. In the meteorological problem of integrating the hydro-dynamic equations applicable to the large-scale atmospheric motions it has been customary to introduce the approximations which follow from the conditions that the atmosphere is in a state of approximate equilibrium between pressure, gravity, and Coriolis forces. By elimination a spatial second order partial differential equation is obtained for the tendency of geopotential. The numerical solution of this is difficult because of the intricate boundary conditions which the geopotential tendency must satisfy, and also because in extreme cases the equation may be hyperbolic in certain regions. In this article is proposed not to undertake the elimination, but rather apply a relaxation procedure to the system of equations itself. It is shown how the above mentioned difficulties then are removed.

1. Introduction. In the large-scale and meteorologically significant motions, free acoustic, inertia, and gravity waves and oscillations exist only as relatively shortperiodic superimposed motions of small amplitude. It has been shown by J. CHARNEY [1] and A. ELIASSEN [2] that simplified hydrodynamic equations exist in which these components are filtered out apriori. The integration problem then changes radically.

In order to illuminate this we adopt the quasi-static system of hydrodynamic equations with pressure as vertical coordinate. Neglecting friction and heating, these equations are:

$$\frac{d\boldsymbol{v}}{dt} = -\nabla\varphi - f\boldsymbol{k}\times\boldsymbol{v} \tag{1}$$

($\boldsymbol{v}$ = horizontal velocity; ∇ = horizontal nabla operator in isobaric surfaces).

$$0 = -\varphi_p + G\vartheta \tag{2}$$

$$\left(G = \frac{R}{100}\left(\frac{p}{100}\right)^{\frac{R}{c_p}-1};\ \vartheta = \text{pot. temp.};\ \varphi_p = \frac{\partial\varphi}{\partial p}\right)$$

$$\frac{d\vartheta}{dt} = 0 \tag{3}$$

In addition to these we have the following restraint conditions:

$$\nabla \cdot \boldsymbol{v} + \omega_p = 0 \qquad (4a)$$

$$(\omega = \frac{dp}{dt})$$

$$\alpha_0 \omega_0 + \gamma_0 = 0 \text{ for } p = p_0 = \text{pressure at the ground} \qquad (4b)$$

$$(\alpha = \text{specific volume}; \gamma = \frac{\partial \varphi}{\partial t})$$

$$\omega = 0 \text{ for } p = 0 \qquad (4c)$$

$$\gamma = 0 \text{ at the vertical boundary.}^1 \qquad (4d)$$

In Eq. (4b) the term of minor importance, $\boldsymbol{v} \cdot \nabla \varphi$, has been neglected.

Taking the individual time derivative of Eqs. (1)—(2) we obtain:

$$\frac{d^2\boldsymbol{v}}{dt^2} = -\frac{d}{dt} \nabla \varphi - f\boldsymbol{k} \times \frac{d\boldsymbol{v}}{dt} - \frac{df}{dt} \boldsymbol{k} \times \boldsymbol{v} \qquad (5)$$

$$0 = -\frac{d}{dt} \varphi_p + F\varphi_p \omega \qquad (6)$$

$$(F = \frac{\partial}{\partial p} ln\, G)$$

By a substitution in Eq. (5) from Eq. (1) we obtain

$$\frac{d^2\boldsymbol{v}}{dt^2} = -\frac{d}{dt} \nabla\varphi - f^2\boldsymbol{v} + f^2\overline{\boldsymbol{v}} \qquad (7)$$

$$(\overline{\boldsymbol{v}} = -f^{-1}\nabla\varphi \times \boldsymbol{k})$$

having ignored a term of minor importance.

A filtered equation in which free gravity and inertia oscillations and waves are discluded is obtained when the term $\frac{d^2\boldsymbol{v}}{dt^2}$ is ignored in this equation. A disclusion of this term will be justified for components of motions in which the orbital time variations are sufficiently slow compared to the variations in inertia oscillations. Denoting the meteorological variables in the filtered solution with capital letters and substituting $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla + \Omega \frac{\partial}{\partial p}$ in Eqs. (5)—(6), we then get:

[1] When we, as we shall do, deal with limited regions of the atmosphere, no natural boundary conditions exist at the boundaries defining the horizontal extension of the region. Condition (4d) is therefore somewhat arbitrary and could f. inst. have been replaced by a condition that $\boldsymbol{v}$ is directed along the boundary.

$$0 = f^2\bar{U} - (f^2 + \Phi_{xx})U - \Phi_{xy}V - \Phi_{xp}\Omega - \Gamma_x \tag{8a}$$

$$0 = f^2\bar{V} - \Phi_{xy}U - (f^2 + \Phi_{yy})V - \Phi_{yp}\Omega - \Gamma_y \tag{8b}$$

$$0 = -\Phi_{xp}U - \Phi_{yp}V - (\Phi_{pp} - F\Phi_p)\Omega - \Gamma_p \tag{8c}$$

$$(\Gamma = \Phi_t)$$

Together with Eqs. (4) applied to the filtered solution, these equations determine U, V, Ω, and Φ_t as a function of the Φ-distribution.

Assuming that the coefficient matrix in Eqs. (8) is non-singular, we can solve for U, V, and Ω, and then substitute into Eq. (4a). This gives a partial differential equation for Γ of second order of the form:

$$(A\Gamma_x + B\Gamma_y + C\Gamma_p)_x + (A'\Gamma_x + B'\Gamma_y + C'\Gamma_p)_y + (A''\Gamma_x + B''\Gamma_y + C''\Gamma_p)_p = H \tag{9}$$

Further, the boundary conditions (4b)–(4d) become:

$$\begin{cases} A''\Gamma_x + B''\Gamma_y + C''\Gamma_p + \alpha\Gamma = 0 \text{ for } p = p_0 \\ A''\Gamma_x + B''\Gamma_y + C''\Gamma_p = 0 \text{ for } p = 0 \\ \Gamma = 0 \qquad \text{at the vertical boundary.} \end{cases} \tag{10}$$

To solve Eqs. (9)–(10) numerically becomes difficult for several reasons: First the removal of the residuals in a straight forward relaxation method is laborious because of the variable coefficients in this equation. Next, the boundary conditions are indeed very complicated for numerical treatment, since they are conditions upon derivatives of Γ taken along directions which vary along the boundary. Finally, although the values of the coefficients in the atmosphere are such that Eq. (9) is predominatingly elliptic, cases in which the equation becomes hyperbolic in certain regions cannot be discluded altogether.

Because of these difficulties the filtered problem has hitherto been integrated only after a number of additional simplifications have been introduced, the exact nature of which has varied quite a lot from one experiment to another. In this paper a method of solution will be described which deals directly with the system of equations itself, Eqs. (8) and (4), rather than with Eqs. (9)–(10). It will be shown that the method eliminates the two difficulties first mentioned, and that it applies with some modification to the hyperbolic case, as well.

2. Description of the method. With $\mathscr{P}$ defined as the tensor

$$\mathscr{P} = \begin{Vmatrix} f^2 + \Phi_{xx} & \Phi_{xy} \\ \Phi_{xy} & f^2 + \Phi_{yy} \end{Vmatrix}, \tag{11}$$

and by integrating Eq. (8c) from p_0 to p, Eqs. (8) can be written as:

$$0 = f^2\bar{\boldsymbol{V}} - \boldsymbol{V}\cdot\mathscr{P} - \Omega\nabla\Phi_p - \nabla\Gamma \tag{12}$$

$$\Gamma = \Gamma_0 - \int_{p_0}^{p}(\boldsymbol{V}\cdot\nabla\Phi_p + \Omega\,(\Phi_{pp} - F\Phi_p))\,dp \tag{13}$$

We introduce now an iteration index r and a relaxation factor $\varkappa$ in Eqs. (12)—(13) as follows:

$$\boldsymbol{V}^{(r+1)} = \boldsymbol{V}^{(r)} + \varkappa\,(f^2\bar{\boldsymbol{V}} - \boldsymbol{V}^{(r)}\cdot\mathscr{P} - \Omega^{(r)}\nabla\Phi_p - \nabla\Gamma^{(r)}) \tag{14}$$

$$\Gamma^{(r+1)} = \Gamma_0^{(r+1)} - \int_{p_0}^{p}(\boldsymbol{V}^{(r+1)}\cdot\nabla\Phi_p + \Omega^{(r+1)}\,(\Phi_{pp} - F\Phi_p))\,dp \tag{15}$$

These are recovery formulae for $\boldsymbol{V}^{(r+1)}$ and $\Gamma^{(r+1)}$ for $r = 0, 1, 2, \ldots\ldots\ldots, \boldsymbol{V}^{(0)}$ representing an initial guess. Letting $\boldsymbol{V}^{(r)}$, $\Omega^{(r)}$, and $\Gamma^{(r)}$ be subject to the same restraint conditions as the ones in Eqs. (4) for $\boldsymbol{V}$, Ω, and Γ, the recovery formulae for $\Omega^{(r+1)}$ and $\Gamma_0^{(r+1)}$ are seen to be:

$$\Omega^{(r+1)} = -\int_{0}^{p}\nabla\cdot\boldsymbol{V}^{(r+1)}dp \tag{16}$$

$$\Gamma_0^{(r+1)} = -\alpha_0\Omega_0^{(r+1)} = \alpha_0\int_{0}^{p_0}\nabla\cdot\boldsymbol{V}^{(r+1)}dp \tag{17}$$

Eqs. (14)—(17) with the addition of the boundary condition in Eq. (4d), constitute the proposed complete iteration scheme. The arithmetical operations involved are rather similar to the ones which have to be used for the *time* integration of the original unfiltered equations, except that the above problem is linear in the variables which depend upon r. Provided therefore that it is possible to find values of $\varkappa$, if necessary as a function of r, which make $\boldsymbol{V}^{(r+1)}$, $\Omega^{(r+1)}$, $\Gamma^{(r+1)} \to \boldsymbol{V}$, Ω, Γ for $r \to \infty$, we are then in the possession of a method for solving the earlier defined filtered problem which is free of the difficulties mentioned earlier. Naturally it remains, however, to see if other difficulties appear instead before the proposed method can be finally judged.

3. Remarks on convergence criteria of the iteration scheme in Eqs. (14)—(17). We define first $\boldsymbol{v}^{(r)}$ and $\gamma^{(r)}$ from

$$\boldsymbol{v}^{(r)} = \boldsymbol{V}^{(r)} - \boldsymbol{V};\ \gamma^{(r)} = \Gamma^{(r)} - \Gamma. \tag{18}$$

Multiplying Eq. (12) by $\varkappa$ and subtracting from Eq. (14), we then get:

$$\boldsymbol{v}^{(r+1)} = \boldsymbol{v}^{(r)} + \varkappa\,(-\boldsymbol{v}^{(r)}\cdot\mathscr{P} - \omega^{(r)}\nabla\Phi_p - \nabla\gamma^{(r)}) \tag{19}$$

where the inhomogenious term $f^2\bar{\boldsymbol{V}}$ now is eliminated. Further, by subtracting Eq. (13) from Eq. (15) we get:

$$\gamma^{(r+1)} = \gamma_0^{(r+1)} - \int_{p_0}^{p}(\boldsymbol{v}^{(r+1)}\cdot\nabla\Phi_p + \omega^{(r+1)}\,(\Phi_{pp} - F\Phi_p))\,dp \tag{20}$$

Having assumed that $\boldsymbol{V}^{(r+1)}$, $\Omega^{(r+1)}$, and $\Gamma_0{}^{(r+1)}$ are subject to the restraint conditions valid for $\boldsymbol{V}$, Ω, and Γ_0, we also obtain:

$$\omega^{(r+1)} = -\int_0^p \nabla \cdot \boldsymbol{v}^{(r+1)}\, dp \tag{21}$$

$$\gamma_0{}^{(r+1)} = \alpha_0 \int_0^{p_0} \nabla \cdot \boldsymbol{v}^{(r+1)}\, dp \tag{22}$$

$$\gamma^{(r+1)} = 0 \text{ at the vertical boundaries.} \tag{23}$$

We shall now assume that $\boldsymbol{v}_n$, ω_n, γ_n are eigen-solutions satisfying Eq. (13), Eqs. (4), and

$$-\nu^2{}_n \boldsymbol{v}_n = -\boldsymbol{v}_n \cdot \mathscr{P} - \omega_n \nabla \Phi_p - \nabla \gamma_n \tag{24}$$

We shall next assume that these solutions represent a complete set, whereby any $\boldsymbol{v}$ can be developed after eigen-solutions $\boldsymbol{v}_n$ as

$$\boldsymbol{v} = \sum_{n=1}^{\infty} \boldsymbol{v}_n \tag{25}$$

Using Eq. (24), the n-th component of Eq. (19) can then be written:

$$\boldsymbol{v}_n^{(r+1)} = (1 - \varkappa \nu_n{}^2)\, \boldsymbol{v}_n^{(r)}, \tag{26}$$

or, after repeated eliminations:

$$\boldsymbol{v}_n^{(r+1)} = (1 - \varkappa \nu_n{}^2)^{r+1}\, \boldsymbol{v}_n^{(0)}. \tag{27}$$

Using Eq. (25), we may now write Eq. (19) correspondingly as:

$$\boldsymbol{v}^{(r+1)} = \boldsymbol{v}^{(r)} + \varkappa\,(-\boldsymbol{v}^{(r)} \cdot \mathscr{P} - \omega^{(r)} \nabla \Phi_p - \nabla \gamma^{(r)}) = \sum_{n=1}^{\infty} (1 - \varkappa \nu_n{}^2)^{r+1}\, \boldsymbol{v}_n^{(0)}. \tag{28}$$

As we shall soon see, the eigen-values $-\nu_n{}^2$ must all be real. Excluding the singular eigen-value $\nu_n{}^2 = 0$, the cases to be considered therefore are:

I. The *elliptic* case: $\nu_n{}^2$ are all of same sign.

II. The *hyperbolic* case: $\nu_n{}^2$ are *not* all of same sign.

In the *elliptic* case it is seen that all $|\, 1 - \varkappa \nu_n{}^2 \,| < 1$ provided $\varkappa$ is chosen such that

$$0 \leqslant \varkappa \leqslant \frac{2}{\nu_n{}^2} \text{ for all } \nu_n{}^2. \tag{29}$$

Under this condition $\boldsymbol{v}^{(r+1)} \to 0$, e.i. $\boldsymbol{V}^{(r+1)} \to \boldsymbol{V}$, when $r \to \infty$.

In the *hyperbolic* case, however, it is not possible to find a fixed value of $\varkappa$ which makes all $|\, 1 - \varkappa \nu_n{}^2 \,| < 1$. In this case some components must increase in amplitude indefinitely as $r \to \infty$. A slight modification of the method, however, makes it work

in this case, as well. This modification is to let $\varkappa$ depend upon r as $\varkappa^{(r)} = (-1)^r \varkappa$. The iteration scheme will then be one where Eq. (14) is replaced by

$$\boldsymbol{V}^{(r+1)} = \boldsymbol{V}^{(r)} + (-1)^r \varkappa \, (f^2 \overline{\boldsymbol{V}} - \boldsymbol{V}^{(r)} \cdot \mathscr{P} - \Omega^{(r)} \nabla \Phi_p - \nabla \Gamma^{(r)}), \tag{30}$$

but where otherwise Eqs. (13)—(15) of the earlier scheme are used. Eq. (26) now becomes

$$\boldsymbol{v}_n^{(r+1)} = (1 - (-1)^r \varkappa \nu_n^2) \, \boldsymbol{v}_n^{(r)}. \tag{31}$$

Increasing here r by one, we get

$$\boldsymbol{v}_n^{(r+2)} = (1 - (-1)^{r+1} \varkappa \nu_n^2) \, \boldsymbol{v}_n^{(r+1)} = (1 - \varkappa^2 \nu_n^4) \, \boldsymbol{v}_n^{(r)}. \tag{32}$$

Therefore, Eq. (30) now can be written

$$\begin{aligned} \boldsymbol{v}^{(r+1)} &= \boldsymbol{v}^{(r)} + (-1)^r \varkappa \, (- \boldsymbol{v}^{(r)} \cdot \mathscr{P} - \omega^{(r)} \nabla \Phi_p - \nabla \gamma^{(r)}) \\ &= \sum_{n=1}^{\infty} (1 - \varkappa^2 \nu_n^4) \, \boldsymbol{v}_n^{(0)} \end{aligned} \tag{33a}$$

when r = odd, and

$$\boldsymbol{v}^{(r+1)} = \sum_{n=1}^{\infty} (1 - \varkappa^2 \nu_n^4) \, \boldsymbol{v}_n^{(1} \tag{33b}$$

when r = equal.

Now, all $\left| 1 - \varkappa^2 \nu_n^4 \right| < 1$, provided

$$0 < \varkappa < \frac{\sqrt{2}}{|\nu_n^2|} \text{ for all } \nu_n^2. \tag{34}$$

Under this condition therefore, also in the *hyperbolic* case

$$\boldsymbol{v}^{(r)} \to 0, \text{ e.i. } \boldsymbol{V}^{(r)} \to \boldsymbol{V} \text{ when } r \to \infty$$

provided the iteration scheme in Eqs. (30), (15)—(17) is used.[1]

4. Interpretation of the proposed iteration method. We shall now introduce a variable τ, of dimension time, in $\boldsymbol{V}$, Ω, and Γ, whereas Φ as before is assumed to depend on x, y, and p only. To distinguish these variables from the variables in the filtered solution, which at any instant are functions of x, y, and p only, we introduce the notations $\boldsymbol{V}^*$, Ω^*, and Γ^*. For these we now assume the following governing equations:

$$\boldsymbol{V}^*_{\tau\tau} = f^2 \overline{\boldsymbol{V}} - \boldsymbol{V}^* \cdot \mathscr{P} - \Omega^* \nabla \Phi_p - \nabla \Gamma^* \tag{35}$$

and in addition the equations analogous to Eqs. (20)—(23).

[1] The iteration equation (30) works of course also in the elliptic case. However, from considerations on convergence rates it follows that we should always use the iteration equation (14) whenever it is known that the problem is elliptic.

The filtered solution can now be considered as the equilibrium solution of Eq. (35), and conceivably be arrived at by carrying out a τ-integration in a specified way.

To see how this could be done we first use the filtered solution as reference motion by introducing

$$\boldsymbol{v}^* = \boldsymbol{V}^* - \boldsymbol{V};\ \omega^* = \Omega^* - \Omega;\ \gamma^* = \Gamma^* - \Gamma. \tag{36}$$

$\boldsymbol{v}^*$, ω^*, γ^* satisfy now the homogenious equation

$$\boldsymbol{v}^*_{\tau\tau} = -\boldsymbol{v}^* \cdot \mathscr{P} - \omega^* \nabla \Phi_p - \nabla \gamma^* \tag{37}$$

and Eqs. (20)—(23). We assume now that the general solution of Eqs. (37), (20)—(23) can be composed of eigen-solutions $\boldsymbol{v}_n^*$, ω_n^*, γ_n^* with a dependency upon τ given by a trigonometric factor $e^{i\nu_n\tau}$. Eq. (37) taken for the n-th eigen-solution then becomes:

$$-\nu_n^2 \boldsymbol{v}_n^* = -\boldsymbol{v}_n^* \cdot \mathscr{P} - \omega_n^* \nabla \Phi_p - \nabla \gamma_n^* \tag{38}$$

The squares of the frequencies ν_n of the above solutions are therefore, apart from a sign, identical with the eigen-values defined in the preceding section. The latter may therefore be interpreted correspondingly. In the particular *elliptic* case where all $\nu_n^2 > 0$, the solution of Eqs. (35), (20)—(23) are therefore composed of eigen-oscillations around corresponding component equilibriums $\boldsymbol{V}_n$, Ω_n, Γ_n.[1] It is therefore clear that by carrying out a stepwise integration in this case of Eqs. (35), (20)—(23) over a total period τ starting from arbitrary initial values of $\boldsymbol{V}^*$ and $\boldsymbol{V}_\tau^*$, then $\frac{1}{\tau}\int_0^\tau \boldsymbol{V}^* d\tau$ and $\frac{1}{\tau}\int_0^\tau \Gamma^* d\tau \to \boldsymbol{V}$ and Γ for increasing τ.

This would therefore be *one* method of successively getting at the filtered solution. It is thought, however, to be more efficient to carry out the τ-integration in the following way: The integration is carried out as before except that at the end of each time-interval of length τ' the "energy" $(\boldsymbol{V}_\tau^*)^2$ is removed by putting $\boldsymbol{V}_\tau^* = 0$ before the integration is continued. If further τ' is taken sufficiently small compared to the length of the period of the most rapidly oscillating eigen-solution, we do not risk to get τ' equal to a half-period of any of the eigen-solutions, and as a result we must obviously approach the equilibrium after each partial integration, most rapidly for the high-frequency components, and less rapidly for the low-frequency ones. It can be shown that this method of getting at the filtered solution is essentially the same as the iteration method described in the previous section for the *elliptic* case. We shall confine ourselves to an indication of this by pointing out the following facts:

[1] In the other *elliptic* case, where all $\nu_n^2 < 0$, the eigen-solutions will contain factors increasing exponentially with τ. However, by changing Eq. (35) into

$$\boldsymbol{V}^*_{\tau\tau} = -(f^2 \overline{\boldsymbol{V}} - \boldsymbol{V}^* \cdot \mathscr{P} - \Omega^* \nabla \Phi_p - \nabla \Gamma^*) \tag{35}$$

we have again the stable case.

Noting that $\boldsymbol{V}_\tau^*$ has been put equal to zero at the beginning of each partial integration we have approximately, if τ' is sufficiently small:

$$\boldsymbol{V}^*(r\tau' + \tau') = \boldsymbol{V}^*(r\tau') + \frac{1}{2}\tau'^2 \boldsymbol{V}_{\tau\tau}^*(r\tau') \qquad (r = 0, 1, 2, \ldots.) \tag{39a}$$

where, according to Eq. (35),

$$\boldsymbol{V}_{\tau\tau}^*(r\tau') = f^2\bar{\boldsymbol{V}} - \boldsymbol{V}^*(r\tau') \cdot \mathscr{P} - \Omega^*(r\tau')\nabla\Phi_p - \nabla\Gamma^*(r\tau') \tag{39b}$$

Identifying $\boldsymbol{V}^*(r\tau')$ and $\frac{1}{2}\tau'^2$ above with $\boldsymbol{V}^{(r)}$ and $\varkappa$ in Eq. (14), it is seen that Eqs. (39) above and the earlier Eq. (14) are identical recovery formulae for $\boldsymbol{V}^{(r)}$.[1]

5. Discussion of eigen-values and possible choices of relaxation factors. Above we have got some insight into why and when the proposed iteration scheme applies to our problem. It was seen that the permitted values of $\varkappa$ were bounded upwards by a limit $\frac{\sqrt{2}}{|\nu_n^2|_{\max}}$ if the hyperbolic as well as the elliptic cases were to be included. On the other hand, however, the convergence rate is slowed down very much if $|\varkappa|$ is taken much below this limit. If we want to study how we can maximize the total convergence rate of the proposed iterations it is necessary to find the range covered by the eigen-values ν_n^2 under general atmospheric conditions. However, it will also be valuable for a fruitful discussion of this problem to bring in a number of different considerations. These are connected with the possibilities of speeding up the convergence rate by letting the relaxation factor depend on the iteration number, and also in the *time*-integration of the filtered solution, with the practicability of using as initial guess of the variables of the filtered solution at any time-step, their values at the preceding time-step. As a third possibility we may even let the relaxation factor be a function of the iteration number in the *time*-integration. As it would lead us too far in this paper to undertake a full discussion of these problems we shall confine ourselves to derive a formula for ν_n^2 by means of which upper bounds for $|\nu_n^2|$ may be found.

We arrive at a formulae for ν_n^2 by multiplying Eq. (24) by $-\boldsymbol{v}_n$ and then integrate over the entire region inside the boundaries. This gives:

$$\nu_n^2 \int \boldsymbol{v}_n^2 = \int (\boldsymbol{v}_n \cdot \mathscr{P} + \omega_n \nabla\Phi_p + \nabla\gamma_n) \cdot \boldsymbol{v}_n \qquad \left(\int = \iiint dxdydp\right) \tag{40}$$

[1] A closer examination shows that the exact form of Eq. (39a), when applied to an eigen-solution in the elliptic stable case is

$$\boldsymbol{V}_n^*(r\tau' + \tau') = \boldsymbol{V}_n^*(r\tau') + \frac{1 - \cos \nu_n \tau'}{\nu_n^2} \boldsymbol{V}_{n\,\tau\tau}^*(r\tau')$$

Differentiating next Eq. (20) with respect to p, we get by a subsequent multiplication by $-\omega_n$:

$$0 = \int (-\boldsymbol{v}_n \cdot \nabla \Phi_p + \omega_n (\Phi_{pp} - F\Phi_p))\,\omega_n$$

Adding this to Eq. (40) we get:

$$\nu_n^2 \int \boldsymbol{v}_n^2 = \int (\boldsymbol{v}_n - \omega_n \boldsymbol{k}) \cdot \mathscr{P}_3 \cdot (\boldsymbol{v}_n - \omega_n \boldsymbol{k}) + \int (\nabla \gamma_n - \gamma_{np} \boldsymbol{k}) \cdot (\boldsymbol{v}_n - \omega_n \boldsymbol{k}) \tag{41}$$

$\mathscr{P}_3$ being defined as the tensor:

$$\mathscr{P}_3 = \begin{Vmatrix} f^2 + \Phi_{xx} & \Phi_{xy} & \Phi_{xp} \\ \Phi_{xy} & f^2 + \Phi_{yy} & \Phi_{yp} \\ \Phi_{xp} & \Phi_{yp} & \Phi_{pp} - F\Phi_p \end{Vmatrix} \tag{42}$$

In view of the restraint conditions in Eqs. (21)—(23) we obtain

$$\int (\nabla \gamma_n - \gamma_{np} \boldsymbol{k}) \cdot (\boldsymbol{v}_n - \omega_n \boldsymbol{k}) = \int_{\sigma_0} a_0 \omega_{n0}^2 d\sigma_0 = \int p_0^{-1} a_0 \omega_{n0}^2, \text{ where } \int_{\sigma_0} d\sigma_0$$

symbolizes a surface integral at the ground. Substituting this into Eq. (41) we get after division by $\int \boldsymbol{v}_n^2$,

$$\nu_n^2 = \frac{\int [(\boldsymbol{v}_n - \omega_n \boldsymbol{k}) \cdot \mathscr{P}_3 \cdot (\boldsymbol{v}_n - \omega_n \boldsymbol{k}) + p_0^{-1} a_0 \omega_{n0}^2]}{\int \boldsymbol{v}_n^2} \tag{43}$$

We make next in each point (x, y, p) a transformation of $\mathscr{P}_3$ to its principal axes $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ whereby Eq. (43) can be written:

$$\nu_n^2 = \frac{\int [a_1 u_n'^2 + a_2 v_n'^2 + a_3 \omega_n'^2 + p_0^{-1} a_0 \omega_{n0}^2]}{\int \boldsymbol{v}_n^2} \tag{43'}$$

Here u_n', v_n' and ω_n' are the components of $\boldsymbol{v}_n$ along the principal axes of $\mathscr{P}_3$, while a_1, a_2, a_3 are the three roots of

$$\begin{vmatrix} f^2 + \Phi_{xx} - a & \Phi_{xy} & \Phi_{xp} \\ \Phi_{xy} & f^2 + \Phi_{yy} - a & \Phi_{yp} \\ \Phi_{xp} & \Phi_{yp} & \Phi_{pp} - F\Phi_p - a \end{vmatrix} = 0. \tag{44}$$

These roots must all be real because of the symmetri of $\mathscr{P}_3$. The sufficient and necessary condition that a_1, a_2, a_3 are all of same sign is found to be the simultaneous ones:

$$(f^2 + \Phi_{xx})(f^2 + \Phi_{yy}) - \Phi_{xy}^2 + (2f^2 + \nabla^2 \Phi)(\Phi_{pp} - F\Phi_p) - (\nabla \Phi_p)^2 > 0$$

$$\begin{vmatrix} f^2 + \Phi_{xx} & \Phi_{xy} & \Phi_{xp} \\ \Phi_{xy} & f^2 + \Phi_{yy} & \Phi_{yp} \\ \Phi_{xp} & \Phi_{yp} & \Phi_{pp} - F\Phi_p \end{vmatrix} > 0 \tag{44}$$

This is also, as can be shown, the condition that Eq. (9) for Φ_t is elliptic. If we also have

$$2f^2 + \nabla^2\Phi + \Phi_{pp} - F\Phi_p > 0 \tag{45}$$

then a_1, a_2, a_3 are all positive. The conditions in Eqs. (44)—(45) are as a rule satisfied in the atmosphere. This is because of the dominating influence from the inertia and gravitational stability in the atmosphere. To a first approximation we have in fact

$$\mathscr{P}_3 \approx \begin{Vmatrix} f^2 & 0 & 0 \\ 0 & f^2 & 0 \\ 0 & 0 & \Phi_{pp} - F\Phi_p \end{Vmatrix}$$

Correspondingly

$$a_1 \approx a_2 \approx f^2;\ a_3 \approx \Phi_{pp} - F\Phi_p,$$

while the principal axes become:

$$\boldsymbol{i}' \approx \boldsymbol{i},\ \boldsymbol{j}' \approx \boldsymbol{j},\ \boldsymbol{k}' \approx \boldsymbol{k}$$

In accordance herewith the formula in Eq. (43)' approximately becomes

$$\nu_n^2 \approx \frac{\int f^2 \boldsymbol{v}_n^2}{\int \boldsymbol{v}_n^2} + \frac{\int (\Phi_{pp} - F\Phi_p)\omega_n^2}{\int \boldsymbol{v}_n^2} + \frac{\int p_0^{-1}\alpha_0\omega_{0n}^2}{\int \boldsymbol{v}_n^2} \tag{46}$$

This formula is equal to a formula in integral form which can be derived for the frequencies in certain systems of oscillations where intertia stability, and internal and external gravitionally stabilities are acting together. This is of course not so surprising when account is taken of the identity between the above eigen-values and the squares of the eigen-frequencies of the oscillation problem formulated in Eqs. (37), (20)—(23), equations which are obviously closely connected with the original unfiltered equations, Eqs. (4)—(6).

6. Application to certain systems of linear, algebraic equations. In this final section we wish to point out that the proposed iteration method can be applied to systems of linear, algebraic equations in which the coefficient matrix is symmetric and non singular. Let $\boldsymbol{R}$ be the N-dimensional vector, $\boldsymbol{R} = [x_1, x_2, \ldots . x_N]$ where $x_1, x_2, \ldots x_N$ are the unknowns, $\boldsymbol{H}$ an other N-dimensional vector, and $\mathscr{P}$ the coefficient matrix of N-th order,

$$\mathscr{P} = \begin{Vmatrix} A_{11}, A_{12}, \ldots, A_{1N} \\ \vdots \\ A_{N1}, A_{N2}, \ldots, A_{NN} \end{Vmatrix} \quad ;\ A_{rs} = A_{sr}$$

The equations can then be written

$$0 = \boldsymbol{R} \cdot \mathscr{P} + \boldsymbol{H}$$

to which the following iteration scheme can be applied:

$$\boldsymbol{R}^{(r+1)} = \boldsymbol{R}^{(r)} + (-1)^r \varkappa \, (\boldsymbol{R}^{(r)} \cdot \mathscr{P} + \boldsymbol{H}).$$

For $\varkappa$ we have to take a value

$$0 < \varkappa < \frac{\sqrt{2}}{|a_n|_{\max}},$$

where a_n are the eigen-values given as the roots of the algebraic equation

$$\begin{vmatrix} A_{11} - a, & A_{12}, & \ldots, & A_{1N} \\ A_{21}, & A_{22} - a, & \ldots, & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1}, & A_{N2}, & \ldots, & A_{NN} - a \end{vmatrix} = 0$$

LIST OF REFERENCES

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