A CIRCULATION THEOREM FOR STATIONARY WIND CURRENTS

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In the following we are going to deduce a circulation theorem for stationary currents induced by wind. The theorem is somwhat special since we make some assumptions which are not always satisfied. We consider a stationary wind current in a sea of constant depth h, and assume that the wind system is such that closed stream lines are possible.

We shall use the following notation: u and v horizontal velocity components in the x and y directions, z height above the bottom, p pressure, ϱ density, λ Coriolis parameter and η eddy viscosity.

The equations of motion are supposed to be

$$-\varrho\lambda v + \frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right),$$

$$\varrho\lambda u + \frac{\partial p}{\partial y} = \frac{\partial}{\partial z}\left(\eta\frac{\partial v}{\partial z}\right),$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,.$$

We form an integration kernel defined by

$$K(z, s) = \int_0^s \frac{ds}{\eta}\ ;\ s \lesseqgtr z$$

$$= \int_0^z \frac{ds}{\eta}\ ;\ s > z$$

Then

$$\int_0^h K(z, s)\frac{\partial}{\partial s}\left(\eta\frac{\partial u}{\partial s}\right) ds = K\eta\frac{\partial u}{\partial s}\Big|_0^h - \int_0^h \frac{\partial u}{\partial s}\,\eta\,\frac{\partial K}{\partial s}\, ds\,.$$

At the bottom, $z = 0$, we have $K = 0$, and at the surface $s = h$, we have

$$\eta \frac{\partial u}{\partial s} = \tau_x \text{ and } K(z, h) = \int_0^z \frac{ds}{\eta}.$$

Furthermore

$$\eta \frac{\partial K}{\partial s} = 0; s > z \text{ and } \eta \frac{\partial K}{\partial s} = 1; s < z.$$

The result of the integration is then

$$\tau_x \int_0^z \frac{ds}{\eta} - u.$$

In the equations we replace z by s and integrate between 0 and h, giving the equations

$$\int_0^h K(z, s) \left(- \varrho\lambda v + \frac{\partial p}{\partial x}\right) ds = \tau_x \int_0^z \frac{\partial s}{\eta} - u,$$

$$\int_0^h K(z, s) \left(\varrho\lambda u + \frac{\partial p}{\partial y}\right) ds = \tau_y \int_0^z \frac{ds}{\eta} - v.$$

We multiply these equations by dx and dy and integrate along a stream line:

$$\int_0^h K(z, s) \left[\int \varrho\lambda \, (udy - vdx) + \int dp\right] ds = \int_0^z \frac{ds}{\eta} \left(\int \tau_x \, dx + \tau_y \, dy\right) - \int udx + vdy.$$

Along the stream line

$$udy - vdx = 0,$$

and if the integration contour is also closed, we have

$$\oint dp = 0.$$

$C = \oint (udx + vdy)$ is the circulation along the closed stream line, and we get

$$C = \int_0^z \frac{ds}{\eta} \oint (\tau_x \, dx + \tau_y \, dy).$$

$\oint (\tau_x \, dx + \tau_y \, dy)$ is the work done by the tangential stress of the wind along the same stream line. We then get the result that under these assumptions the circulation is proportional to the work done by the wind.

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