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PERTURBATIONS IN A BAROCLINIC MODEL ATMOSPHERE

BY PETER THRANE

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Summary. The simplest baroclinic model of the atmosphere (advective model) is considered. By means of linearized hydrodynamical equations sinusoidal waves and composite motions are studied with a view to finding stable motions where infinite amplitudes do not occur.

1. Introduction. It is the purpose of the present paper to study certain aspects of the problem of stable perturbations in the simplest baroclinic model of the atmosphere, the so-called "advective model".

This model is defined as having zero static stability. The curvature of the earth is ignored. The atmosphere is incompressible, horizontally unbounded and vertically bounded by rigid horizontal planes. The undisturbed motion is a steady zonal current which increases linearly from the bottom to the top of the atmosphere and has no variation with latitude. The vertical density gradient $\partial \rho_0 / \partial z$ is by definition zero.

Waves of small amplitude in the advective model have been studied by FJØRTOFT (1950), extending an investigation by CHARNEY (1947), by means of the linearized hydrodynamical equations. Fjørtoft started from a more general model where the vertical density gradient was different from zero and where the velocity perturbation was dependent on latitude, longitude and height, and he arrived at the advective model with velocity independent of latitude, as a limiting case. A more detailed study was made by HOLMBOE (1959) taking the advective model as a starting point.

The assumption $\partial \rho_0 / \partial z = 0$ makes it possible to study stability of waves independent of the effect of the static stability of the atmosphere and at the same time it leads to a great simplification in the equations of motion, whereby it becomes possible to establish a comparatively simple frequency formula from which stability conditions can be derived, as shown by Fjørtoft.

The simplifications involved in the model itself and in the assumptions regarding the nature of the motion facilitate the mathematical treatment, but, on the other hand, they limit the field of application. It also becomes necessary to guard against inconsistencies arising in a system of differential equations when restrictions are imposed on the form of the solutions. Such inconsistencies can be avoided here by ignoring the latitudinal variation of the Coriolis parameter.

Another difficulty encountered in studying waves of the advective model is that mathematical solutions derived from the linearized fundamental equations may have forms conflicting with the basic assumption of small amplitudes. Thus the stable sinusoidal waves have infinite amplitudes at certain levels. It will be attempted here to find forms of stable motion where no infinite amplitudes occur.

2. The linearized hydrodynamical equations applied to the simple wave motion. The motion of a frictionless, incompressible fluid on a rotating globe is governed by the equations

$$\frac{dV}{dt}+2\boldsymbol{\Omega}\times V+\nabla\phi = -S\nabla P\,, \tag{1}$$

$$\nabla\cdot V=0\,,\quad \frac{dS}{dt}=0\,, \tag{2}$$

Where V denotes velocity, $\boldsymbol{\Omega}$ angular velocity of the earth's rotation, ϕ gravitation potential, S specific volume, P pressure, t time, $\frac{d}{dt}=\frac{\partial}{\partial t}+V\cdot\nabla$.

These equations will be applied here to the advective model. Ignoring the curvature of the earth we introduce a rectangular system of coordinates, x, y, z, with x increasing eastward, y northward and z vertically upward. Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote unit vectors directed along these axes, let the velocity of the undisturbed current be $\boldsymbol{i}U(z)$ and let p_0, α_0, ρ_0 denote its pressure, specific volume and density. Putting $2\boldsymbol{\Omega}.\ \boldsymbol{k}= =2\Omega_z=f$ and denoting by g the gravity acceleration we obtain from (1) for the undisturbed current

$$\rho_0(\boldsymbol{j}fU+\boldsymbol{k}g) = -\nabla p_0\,. \tag{3}$$

Performing in (3) the curl operation $\nabla\times$ and recalling that $\partial\rho_0/\partial z=0$ we find

$$g\frac{\partial}{\partial y}\ln\rho_0 = fU'\,, \tag{4}$$

("thermal wind equation") where $U'=dU/dz$.

We now introduce a small perturbation defined by $\boldsymbol{v}, p, \alpha$, so that

$$V=\boldsymbol{i}U+\boldsymbol{v}\,,\quad P=p_0+p\,,\quad S=\alpha_0+\alpha\,. \tag{5}$$

We put $\boldsymbol{v} = \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w$. Further we assume that U' is constant. The linearized equations of perturbation deduced from (1, 2, 5) may be written

$$\frac{d\boldsymbol{v}}{dt} + \boldsymbol{i}U'w + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \alpha \nabla p_0 = -\alpha_0 \nabla p, \qquad (6)$$

$$\nabla \cdot \boldsymbol{v} = 0, \quad \frac{d}{dt}(\alpha_0 + \alpha) = 0, \qquad (7)$$

where $d/dt = \partial/\partial t + U\partial/\partial x + \boldsymbol{v} \cdot \nabla$.

It is convenient to introduce

$$\kappa_0 = -g\ln\alpha_0, \quad \kappa_0 + \kappa = -g\ln(\alpha_0 + \alpha), \quad \kappa = -\frac{g}{\alpha_0}\alpha. \qquad (8)$$

From the definition of the model it follows that κ_0 is independent of x and z. We further adopt the assumption introduced by Rossby that $\boldsymbol{v}$ is independent of the meridional coordinate, i.e. $\partial \boldsymbol{v}/\partial y = 0$.

In the equation (6) we shall denote the lefthand side by $\boldsymbol{A}$ writing $\boldsymbol{A} = -\alpha_0 \nabla p$. From $\nabla \times (\rho_0 \boldsymbol{A}) = 0$ combined with (8) and (4) we find

$$\nabla \times \boldsymbol{A} = \frac{1}{g}\boldsymbol{A} \times \nabla\kappa_0 = \frac{fU'}{g}(-\boldsymbol{i}A_z + \boldsymbol{k}A_x). \qquad (9)$$

The component equations are

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{fU'}{g}A_z, \qquad (a)$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0, \qquad (b) \qquad (10)$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{fU'}{g}A_x, \qquad (c)$$

where according to equations (3) (6) and (8):

$$A_x = \frac{du}{dt} + U'w + 2\Omega_y w - fv, \qquad (a)$$

$$A_y = \frac{dv}{dt} + fu + \frac{fU}{g}\kappa, \qquad (b) \qquad (11)$$

$$A_z = \frac{dw}{dt} - 2\Omega_y u + \kappa. \qquad (c)$$

The vorticity equations (10) are interdependent owing to the relation $\nabla \cdot \nabla \times A = 0$.

The equations (10) and (7) will be taken as the starting point for the subsequent deductions.

In (11) and (10) we may introduce a stream-function $\psi(x, z)$ with

$$u = \frac{\partial \psi}{\partial z} = \psi', \quad w = -\frac{\partial \psi}{\partial x}. \tag{12}$$

The first equation (7) is thereby satisfied. The second equation (7) may be written, having regard to (4) and (8)

$$\frac{d}{dt}(\kappa_0 + \kappa) = \frac{d\kappa}{dt} + fU'v = 0\,. \tag{13}$$

From the above equations it follows that

$$\beta \frac{d^2}{dt^2} \nabla^2 \psi = 0\,, \tag{14}$$

where $\beta = df/dy$. This can be seen in the following way. Introduce in (11) the expressions (12) for u and w. Find from (11) $\partial A_z/\partial y$, $\partial A_x/\partial y$, recalling that $\partial v/\partial y = 0$. Apply to equation (10b) the operator $\partial/\partial y$ and then d/dt, having regard to (13) and the values just found from (11). This gives the relation

$$\beta\left(\frac{dv'}{dt} - U'\frac{\partial v}{\partial x}\right) = 0\,. \tag{15}$$

Applying the operator $\beta d/dt$ to (10b) and having regard to (15) we arrive at the relation (14).

We shall assume that ψ and v have the form

$$F(z, k, \gamma)e^{\,i(kx - \gamma t)}\,, \tag{16}$$

where k and γ are constants (k real, γ complex or real). From (14) and (16) it follows that

$$\beta \nabla^2 \psi = \beta(\psi'' - k^2 \psi) = 0\,. \tag{17}$$

Following Holmboe (1959) we introduce the hydrostatic approximation

$$\nabla^2 \psi = \psi'' \tag{18}$$

which is permissible if we consider waves that are long compared to the height of the atmosphere. Introducing this in (17) we obtain $\beta\psi'' = 0$. The condition $\psi'' = 0$ is obviously incompatible with the boundary conditions $\psi = 0$ for $z = 0$ and $z = h$ (This observation applies likewise to the condition $\psi'' - k^2\psi = 0$). Thus the fundamental equations and the properties ascribed to the model cannot be reconciled with the form (16) and the assumptions $\partial\psi/\partial y = 0$, $\partial v/\partial y = 0$. In order to avoid inconsistencies we put henceforward $\beta = 0$. It follows then from (13) that $\partial\kappa/\partial y = 0$.

Having made this assumption we may now adopt the form (16) for κ as has already been done for ψ and v. For brevity we introduce

$$\frac{k}{f}=\mu\,,\quad \frac{\gamma}{k}=c\,,\quad \frac{U'}{g}=r\,,\quad \nu=\mu(U-c)=\frac{1}{f}(kU-\gamma)\,. \tag{19}$$

When differentiating ψ, v, κ we then have $d/dt=if\nu$.

We now turn to the equations (10, 11, 12). Introducing the form (16) for ψ, v, κ, having regard to (18, 19), neglecting in (10a) the term $(-fU'/g)\cdot(dw/dt-2\Omega_y u)$ and in (10c) the term $(fU'/g)2\Omega_y w$ and introducing from (13)

$$\kappa=iU'\frac{v}{\nu} \tag{20}$$

we obtain

$$(v\nu)'-(1-rU)\left(\frac{v}{\nu}\right)'=0\,, \tag{21}$$

$$i\psi''-\left(\frac{v}{\nu}\right)'=0\,, \tag{22}$$

$$-v\nu-rc\frac{v}{\nu}+i(1-rU+rc)\psi'+irU'\psi=0\,. \tag{23}$$

We note that equation (21) can be derived by differentiation of (23) with respect to z having regard to (22).

Integrating the first-order differential equation (21) we find

$$v=\frac{K\nu}{1-rU-\nu^2}\left(\frac{2\mu\nu+r-\sqrt{m}}{2\mu\nu+r+\sqrt{m}}\right)^{r/\sqrt{m}}e^{i(kx-\gamma t)}\,, \tag{24}$$

where

$$m=4\mu^2(1-rc)+r^2\,. \tag{25}$$

The expression found for v may be simplified when the following conditions are fulfilled

$$|rc|\ll 1\,,\quad |rU|\ll 1\,,\quad \left|\frac{r}{2\mu}\right|\ll 1\,. \tag{26}$$

We may then put $\sqrt{m}=2\mu$ and

$$v=\frac{K\nu}{1-rU-\nu^2}\left(\frac{\nu-1}{\nu+1}\right)^{r/2\mu}e^{i(kx-\gamma t)}\,. \tag{27}$$

Let us denote by the subscripts 0 and h the values of the variables at the levels $z=0$ and $z=h$, respectively.

According to equation (22) we may write that value of ψ which satisfies the boundary conditions $(\psi_0, \psi_h = 0)$, as follows

$$i\psi = \int_0^z \frac{v}{\nu} dz - \frac{z}{h}\int_0^h \frac{v}{\nu} dz \,. \tag{28}$$

If γ is complex, ν is complex, and it is seen from (27) that v/ν has no singularities. (The case of real γ will be treated in a later paragraph).

Introducing the expression (28) into the equation (23) we obtain for $z=0$, since $\psi_0 = 0$:

$$(1 - rU_0 - \nu_0^2)\frac{v_0}{\nu_0} - (1 - rU_0 + rc)\frac{1}{h}\int_0^h \frac{v}{\nu} dz = 0 \,. \tag{29}$$

A corresponding equation for $z=h$, $\psi_h = 0$ is obtained by replacing in (29) the subscript 0 by the subscript h. Combining these equations we obtain

$$\frac{1 - rU_0 - \nu_0^2}{1 - rU_0 + rc} \cdot \frac{v_0}{\nu_0} = \frac{1 - rU_h - \nu_h^2}{1 - rU_h + rc} \cdot \frac{v_h}{\nu_h} \,. \tag{30}$$

Using (27) and recalling that $U - c = \nu/\mu$ we finally obtain the frequency equation

$$\frac{1}{1 - \frac{r}{\mu}\nu_0}\left(\frac{1 - \nu_0}{1 + \nu_0}\right)^{r/2\mu} = \frac{1}{1 - \frac{r}{\mu}\nu_h}\left(\frac{1 - \nu_h}{1 + \nu_h}\right)^{r/2\mu} \,. \tag{31}$$

Taking the logarithm on both sides and putting $\ln(1 - r\nu/\mu) = -r\nu/\mu$ we find after a simple transformation

$$\tanh^{-1}\nu_h - \tanh^{-1}\nu_0 = \nu_h - \nu_0 \,. \tag{32}$$

Taking here coth on both sides and writing for brevity $\nu_h - \nu_0 = a$ we obtain $1 - \nu_0\,\nu_h = a\coth a$, whence it follows that

$$\left[c - \frac{1}{2}(U_0 + U_h)\right]^2 = \frac{1}{\mu^2}\left(1 + \frac{1}{4}a^2 - a\coth a\right). \tag{33}$$

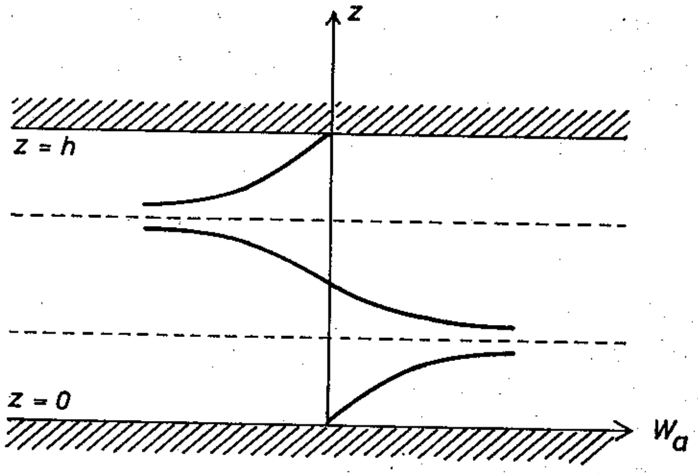
This equation was given by FJØRTOFT (1950) on the basis of a simplified version of the equations (21—23).

The frequency equation in the forms (31—33) is subject to the conditions (26). Therefore these forms cannot be used when $\mu \to 0$ $(k \to 0)$. In this case one has to go back to equation (30), where v_0, v_h are determined by (24).

The expression (27) may be simplified still further. If $|r/2\mu|$ is sufficiently small we may write $[(\nu - 1)/(\nu + 1)]^{r/2\mu} = 1$. Putting $1 - rU = 1$ we obtain

$$v = \frac{K\nu}{1 - \nu^2} e^{i(kx - \gamma t)} \tag{34}$$

Fig. 1. Amplitude of w as a function of z when k and γ have real values.



Introducing this into (28), having regard to (32) and noting that $z/h = (\nu - \nu_0)/(\nu_h - \nu_0)$ we find

$$i\psi = \frac{K}{\mu U'}\{\tanh^{-1}\nu - \tanh^{-1}\nu_0 - \nu + \nu_0\}e^{i(kx-\gamma t)} \tag{35}$$

We further have $w = -ik\psi$ and

$$iu = i\psi' = \frac{K\nu^2}{1-\nu^2}e^{i(kx-\gamma t)}. \tag{36}$$

It is seen from (34—36) that the amplitudes become infinite when $\nu = \pm 1$. This can only happen when γ is real. For any real values of k and γ there exists at least one level between $z=0$ and $z=h$, where $\nu^2=1$. Otherwise the real part of ψ' in (36) would have the same sign throughout the interval $(0, h)$, and the boundary conditions could not be satisfied. Let $U=U_1$ and $U=U_2$ denote "critical" levels where $\nu=\mu(U-c)=\pm 1$, i.e. $U_1=c+f/k$, $U_2=c-f/k$. If $U_h>U_0$ we have, since at least one of the critical levels must be within the interval $(0, h)$, two alternative conditions

$$U_0 < c \pm f/k < U_h. \tag{37}$$

One of these must be satisfied. Both may be satisfied if $U_0<c<U_h$ and $U_h-U_0>2f/k$. We then have two critical levels. This case is illustrated in Fig. 1.

It is evident that, if c is real and one of the conditions (37) is fulfilled, the boundary conditions can always be satisfied, and no frequency equation is needed. The significance of equation (32) for real values of ν_0 and ν_h can therefore be understood only when a continuous transition from complex to real c-values is considered. We shall return to this in a later paragraph.

The fact that v, ψ in (34—36) are infinite at certain levels, when ν is real, involves a contradiction, since the linearized equations (6—7) were deduced on the basic assumption that $\mathbf{v}, \kappa$ were small, so that product-terms could be neglected. The "stable waves" defined by (34—36) are therefore formal mathematical expressions which do not give an adequate description of possible fluid motion in the model considered. It would be of interest to seek forms of motion other than the form (16) with a view to finding stable motions where no infinite amplitudes occur. This will be attempted in the following paragraphs.

3. Forms of motion with bounded amplitudes.

3.1. *Solutions of the partial differential equations.* Let us consider the system (S) consisting of the four linear partial differential equations (10) and (13). Since the coefficients are independent of x and t we may seek solutions of (S) of the form

$$F_n(x,t,z,k,\gamma) = \int_{\gamma_0}^{\gamma} f_n(z,k,\eta) e^{i(kx-\eta t)} d\eta \,, \tag{38}$$

where F_n may denote ψ, v, κ for $n=1,2,3$ respectively and the functions f_n have to be determined. k may be independent of γ, or a relation $k=k(\gamma)$ may exist. We shall consider the latter case. γ_0 is a constant independent of γ.

It is evident that, if the functions F_n satisfy the system (S) then the integrands in (38) must also satisfy (S). We may therefore choose for the functions $f_n \exp[i(kx-\gamma t)]$ those particular values of ψ, v given by (34, 35) and the corresponding value of κ from (20), provided that the conditions (26) are fulfilled. K may be any function of γ.

Since the functions in (34—36), constituting the integrands in (38), satisfy (S) for all values of k and γ, this must be the case also for F_n. This leads to the introduction of new solutions of (S)

$$G_n(x,t,z,k,\gamma) = \int_{\gamma_0}^{\gamma} F_n(x,t,z,k,\eta) d\eta \tag{39}$$

and similarly

$$H_n(x,t,z,k,\gamma) = \int_{\gamma_0}^{\gamma} G_n(x,t,z,k,\eta) d\eta \,. \tag{40}$$

This procedure can be iterated any number of times.

It is convenient to introduce ν as variable instead of η in the integrals (38—40). Let us consider for instance the case where f_n is identical with the amplitude of v in (34). Introducing $k=k(\gamma)$, $dk/d\gamma = k'$, $f\nu = kU-\gamma$, $(k'U-1)d\gamma = fd\nu$, we obtain from (38) (replacing γ by η under the integral sign)

$$\int v\, d\eta = \int \frac{L(x,t,U,\nu)}{1-\nu^2} d\nu$$

where L is a regular function. Abbreviating $L(x,t,U,\nu) = L(\nu)$ we may write:

$$\int v\, d\eta = \frac{1}{2} L(1) \int \frac{d\nu}{1-\nu} - \frac{1}{2} L(-1) \int \frac{d\nu}{1+\nu} + \int L_1(x,t,U,\nu) d\nu \,, \tag{41}$$

where L_1 is another regular function. L and L_1 are known, when a value of $K=K(\gamma)$ in (34) has been chosen.

The integrals in (41) are taken along a path of integration in the complex ν-plane. This path must not pass through or terminate in the points $\nu = \pm 1$ on the real axis.

If the integral (38) has the form (41), then F_n contains terms with the factor $\ln(\nu \pm 1)$. The next integration, corresponding to (39) will then render terms containing $(\nu \pm 1)\ln(\nu \pm 1)$, and (40) will render terms containing $(\nu \pm 1)^2 \ln(\nu \pm 1)$.

Summing up, it is seen that, if we replace in (38) the functions f_n by the expressions (34—36) and perform three successive integrations like (38—40) we arrive at a solution of the system (S) of the form H_n. Let this solution be denoted by $\bar{\psi}, \bar{v}, \bar{\kappa}$. It is seen that these variables contain $\ln(\nu \pm 1)$ in terms of the form $(\nu \pm 1)^m \ln(\nu \pm 1)$, where $m=3$ for $\bar{\psi}$, $m=2$ for $\bar{v}$ and $\bar{\kappa}$, $m=1$ for $\bar{\psi}''$ and $\bar{v}'$, so that all terms in (S) are continuous for $\nu = \pm 1$. In the motion determined by $\bar{\psi}, \bar{v}, \bar{\kappa}$ no infinite amplitudes occur.

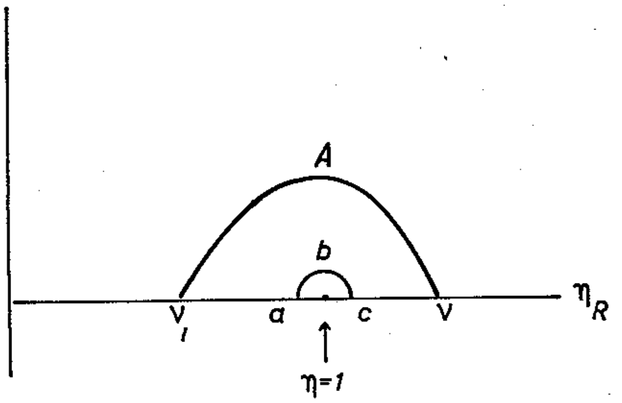


Fig. 2. Paths of integration in the complex η-plane.

3.2. *Stability in relation to boundary conditions.* Let $\phi(x, t, z, \eta)$ be an analytic function and suppose that $|\phi| \to \infty$ when $t \to \pm\infty$ if the parameter η is complex, whereas ϕ remains bounded if η has real values. The function may then be called "stable" for real values of η. Obviously an integral of the form

$$I = \int_{\nu_1}^{\nu} \phi(x, t, z, \eta) d\eta$$

is also "stable" if η passes through real values from ν_1 to ν.

If $\phi = \phi_1(x, t, z, \eta)(\eta - 1)^{-1}$ where ϕ_1 is regular, we may choose the path of integration in the complex η-plane as shown in Fig. 2, i.e. either the path $\nu_1 abc\nu$ or some path like $\nu_1 A \nu$.

The small semicircle contributes to the integral with the term $i\pi\phi_1(x, t, z, 1)$. A sufficient condition for the "stability" of I is that ν_1 and ν are real. This condition is also necessary.

Applying this to the integrals (38—40) we conclude that they can be made stable by a suitable choice of the path of integration in the complex η-plane. Choosing a suitable path of integration we can thus ensure that the solution $\bar{\psi}, \bar{v}, \bar{\kappa}$ mentioned in paragraph 3.1 above is stable and bounded. It remains to see if the boundary condition $\bar{\psi} = 0$ for $z=0$, $z=h$ can be satisfied.

It is necessary that these conditions are fulfilled for the function f_1 in (38). If we had, for instance, $\psi_0 = f_1(0, k, \eta) \neq 0$, then $F_1(x, t, 0, k, \gamma)$ could not be zero for all values of $x, t,$ but only for particular values. On the other hand, if f_1 fulfills the boundary conditions, this applies also to F_1, G_1, H_1 in (38—40). We conclude that it is necessary and sufficient that the expression (35) should vanish for $z=0$ and $z=h$, which leads to the equation (33) as shown in the previous section.

In the integrals (38—40) the existence of a relation $k = k(\gamma)$ was assumed. We shall now specify this relation, taking it to be identical with the frequency equation (33). For brevity we write

$$1 + \frac{1}{4}a^2 - a\coth a = M\,, \tag{42}$$

where $a=\nu_h-\nu_0=\mu(U_h-U_0)$. We further introduce

$$\frac{1}{2}(U_0+U_h)=U_m\,,\quad \zeta=\frac{1}{h}\left(z-\frac{1}{2}h\right)=\frac{U-U_m}{U_h-U_0}\,. \tag{43}$$

Equation (33) may be written, having regard to (19)

$$\gamma=kU_m\pm f\sqrt{M} \tag{44}$$

When this relation between k and γ is fulfilled, ν may be written in the form

$$\nu=\mu U-\frac{\gamma}{f}=\zeta a\pm\sqrt{M} \tag{45}$$

This gives ν as a function of k and z. It is convenient here to use a and ζ instead of k and z. We shall consider the case $U_h>U_0$ whereby a is positive and proportional to k and μ. ζ determines a level between $z=0(\zeta=-\frac{1}{2})$ and $z=h(\zeta=\frac{1}{2})$.

We further note the following properties of M:

$$\begin{aligned} &(a) && M=0 \quad \text{for } a=0 \text{ and for } a=a_0=2{,}4 \text{ (approx.)}\\ &(b) && M<0 \quad \text{for } 0<a<a_0\,,\ M>0 \text{ for } a>a_0\\ &(c) && \left(\sqrt{M}-\frac{1}{2}a\right)\to-\coth a\to-1\,,\ \text{when } a\to\infty\,. \end{aligned} \tag{46}$$

Combining this with (45) it is seen that $\nu=0$ for $a=a_0$, ν complex when $0<a<a_0$, ν real when $a>a_0$. If $+\sqrt{M}$ is chosen we have from (45) and (46) (c) when $a\to\infty$

$$\begin{aligned} &\nu\to\left(\frac{1}{2}+\zeta\right)a-\coth a\to\infty\,,\ \text{when } \zeta>-\frac{1}{2}\,,\\ &\nu\to-\coth a\to-1\,,\ \text{when } \zeta=-\frac{1}{2}\,. \end{aligned} \tag{47}$$

According to (45) ν depends on a and ζ. In Fig. 3 $\nu=\nu_R+i\nu_i$ is represented by a point in the complex ν-plane. Let a vary from 0 to a_0, while ζ is kept constant. The point $P(\zeta)$ representing ν then describes a curve, e.g. OAB. To each value of ζ corresponds one curve. OCD corresponds to $\zeta=\frac{1}{2}$, i.e. the upper boundary level and OC_1D_1 corresponds to $\zeta=-\frac{1}{2}$, i.e. the lower boundary. The point B where the curve meets the axis can have any position between D_1 and D depending on the choice of ζ. For $\zeta=0$, $\nu_R=\zeta a$ is zero and the curve reduces to the straight line OEO.

We note that $OD=\frac{1}{2}a_0=1{,}2$ (appr.) and $OD_1=-\frac{1}{2}a_0$. The points F and F_1 correspond to $\nu=1$ and $\nu=-1$, respectively.

The curves in Fig. 3 are for the greater part determined by equation (45). However, it should be recalled that this equation is valid under the conditions (26) which are

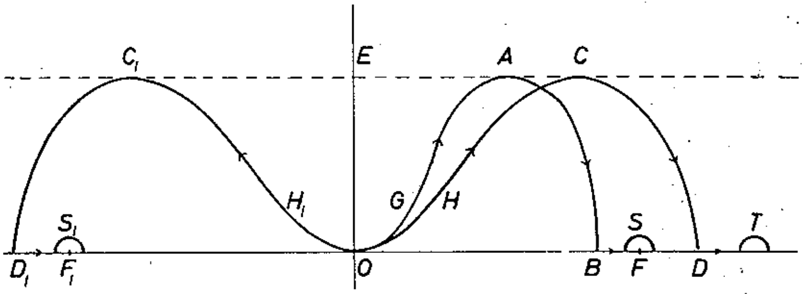


Fig. 3. The complex ν-plane, $\nu=\nu_R+i\nu_i$ (ν_R abscissa, ν_i ordinate).

obviously not fulfilled when $a\to 0$ $(\mu\to 0)$. Therefore, a certain part of the curves, indicated on Fig. 3 by OG, OH, OH_1, is not in conformity with (45). In order to study this part we go back to the expression (24). Introducing this into the equation (30) we obtain a form of the frequency equation which may be written

$$\left(\frac{1-rU_h+rc}{1-rU_0+rc}\right)^{\sqrt{m}/r}=\frac{2\mu\nu_h+r-\sqrt{m}}{2\mu\nu_0+r-\sqrt{m}}\cdot\frac{2\mu\nu_0+r+\sqrt{m}}{2\mu\nu_h+r+\sqrt{m}}. \tag{48}$$

This is a relation between γ and μ which replaces equation (44) for small k-values.

The exact form of the curves will not be studied here. It is sufficient for our purpose to know that the curves pass through origo, i.e. $\nu=0$ for $\mu=0$ $(a=0)$. This is seen as follows. In order to have $\nu=\mu(U-c)\to 0$ when $\mu\to 0$, we must have $\mu c\to 0$. We may then expand $\sqrt{m}$ in a series:

$$\frac{\sqrt{m}}{r}=1-\frac{2}{r}\mu^2c+\frac{2}{r^2}\mu^2-\frac{2}{r^2}\mu^4c^2+\cdots \tag{49}$$

Introducing this in (48) we easily find that the right-hand side approaches to the value $(rU_h-1)/(rU_0-1)$ when $\mu\to 0$. The left-hand side approaches to the same value provided that $c\to 0$ when $\mu\to 0$.

The curves in Fig. 3 illustrate the variation of ν when a varies from 0 to a_0 while ζ is kept constant. The arrows indicate the direction of the motion of the point $P(\zeta)$ representing ν. When a increases beyond a_0, $\sqrt{M}$ becomes real, and since we have chosen its positive value, the points P, leaving the curves, move along the real axis to the right as indicated by the arrows. The point $P(\zeta)$ coming from OAB will pass through the singular point F. The point $P(\frac{1}{2})$ does not pass through F or F_1. The point $P(-\frac{1}{2})$ moves from D_1 to the right but, according to (47), it does not reach F_1 for any value of a. If we put $\zeta=-\frac{1}{2}+\varepsilon$ and if $a=a_1>a_0$ is given, we can always find a value of ε such that the point $P(-\frac{1}{2}+\varepsilon)$ coincides with F_1 when $a=a_1$.

It follows, therefore, that, for any value $a_1>a_0$ there always exist certain levels ζ for which the point $P(\zeta)$ has to pass through F or F_1 when a varies from $a<a_1$ to $a>a_1$.

It should be noted that a cannot be too large. Owing to the approximation (18) k must not exceed a certain maximum value, whence it follows that a must be kept below a certain value a_m.

If $a=a_0$ and ζ varies from $-\frac{1}{2}$ to $\frac{1}{2}$ the point P, representing ν, moves along the real axis from D_1 to D. The stream function ψ defined by (35) becomes infinite at the points F_1 and F. Not let P deviate from the axis following the semi-circles S_1 and S. ψ then obtains a continuous variation from $\psi=\psi_0=0$ at D_1 to $\psi=\psi_h$ at D. It follows from equation (32), which is identical with (44), that $\psi_h=0$. The γ-value in (32) is real in this case, and it has been shown previously that, when γ is real ψ can be defined so that $\psi_0, \psi_h=0$ regardless of equation (32). This would, however, entail a discontinuity in ψ, not only at F and F_1 but also on S and S_1 or on any curve joining two points on the real axis separated by F or F_1.

The role of the frequency equation for real γ is to ensure continuous variation of ψ, κ, v when ν varies from a real value $\nu_1<\pm 1$ through complex values to a real value $\nu_2>\pm 1$.

It was shown above that the integrals (38—40) can be made "stable" by performing the integration along a path beginning and ending on the real axis of the complex η-plane. The same obviously applies to the complex ν-plane when ν has been introduced in stead of η, as shown in (41).

Further the path of integration must conform to the equations (44, 45). We have therefore to choose one of the following paths:

(a) The curves OAB, OAC etc. letting a vary from zero to a_0,
(b) these curves with addition of a part of the real axis, letting a vary from zero to $a_1>a_0$, (50)
(c) parts of the real axis only, obtained by letting a increase from $a_1>a_0$ to $a_2 \gtreqless a_m$.

In all cases ζ-values exist for which the path of integration has to pass through one of the points F or F_1, $(\nu=\pm 1)$, whereby the integral (41) becomes meaningless. We conclude that, in a strict mathematical sense, stable integrals of the form (38—40) are incompatible with the boundary conditions.

Let us consider the error made by deviating from the path of integration determined by the equation (45), using the semicircle S (or S_1) as a part of the path. When $a>a_0$, $\gamma=\gamma_R$ determined by (44) and $\nu=\nu_R$ determined by (45) are real. Let the corresponding point $P(\zeta)$ be situated on the axis within S when $\zeta=\zeta_1$. We now replace $P(\zeta_1)$ by a point $P'(\zeta_1)$ on S having the same abscissa. γ_R is then replaced by $\gamma_R+i\gamma_i$ and ν_R by $\nu_R+i\nu_i$, where γ_i and ν_i depend on a and on the radius of S but are independent of ζ. For any ζ-value the point $P(\zeta)$ is then replaced by $P'(\zeta)$ having the same abscissa and describing a semicircle when $P'(\zeta_1)$ moves on S. Thus, for $\zeta=\frac{1}{2}$ the point $P'(\frac{1}{2})$ describes the semicircle T.

The value of ψ_h determined by (35) is different from zero on T, and since T is part of the path of integration for $\zeta=\frac{1}{2}$, the integral $\int\psi_h d\nu$ is also different from zero. Thus the upper boundary condition is not satified. However, if the radius ρ of S is small T is small and $\int\psi_h d\nu \to 0$ when $\rho\to 0$.

Further we note that, if v and ψ are given by (34, 35) the integral $\int\psi d\nu$ taken along S contains the factor $\rho\ln\rho$, thus tending to zero when $\rho\to 0$. The integral

$\int v dv$ taken along S or S_1 renders a term of the form iR (R real) which does not tend to zero when $\rho \to 0$. The next integration (corresponding to (39)) taken along S renders a term containing the factor $iR\rho \to 0$, whereby the continuity at $\nu = \pm 1$ is ensured. It should be noted, however, that a discontinuity will always be present in certain higher derivatives like $\partial^n \psi / \partial \nu^n$.

Going back to the integrals (38—40) and the solution $\bar{\psi}$, $\bar{\kappa}$, $\bar{v}$ defined in paragraph 3.1 we may now affirm that, if one of the paths of integration (50) is chosen, this solution represents a stable motion with bounded amplitudes satisfying the boundary conditions, save for an error due to the semicircles S_1, S, T. This error is always present but can be made arbitrarily small.

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