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ON PERIODIC MOTIONS OF AN IDEAL FLUID WITH AN ELASTIC BOUNDARY

By Torbjørn Ellingsen

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Summary. A two-layer model of an ideal fluid over an elastic bottom layer is considered. The boundary conditions are developed for simple harmonic wave motions superposed on an arbitrary linear fluid flow. The frequency equations are discussed for surface waves on a uniform stream and for perturbation of a flow with constant shear. Two response functions defined for the boundary are found useful for a graphical discussion of the frequency equations.

1. The boundary conditions. The present paper deals with some simple two-dimensional motions in a two-layer system. This system consists of an inviscid, incompressible and homogeneous fluid over an elastic bottom layer. The motions to be investigated are small perturbations in this system. The basic motion is a horizontal, linear and steady fluid flow under which the bottom layer is at rest. The coordinate system is chosen so that the xy-plane coincides with the horizontal interface and the z-axis is vertical, directed upwards. ρ_1 and ρ_2 are the densities of the fluid and the elastic medium respectively. λ and μ are the Lamé's constants. The displacement vector is $\boldsymbol{\rho} = \boldsymbol{i}\xi + \boldsymbol{k}\zeta$ and the stress components are

$$\begin{aligned} \sigma_x &= \lambda \nabla \cdot \boldsymbol{\rho} + 2\mu \frac{\partial \xi}{\partial x} \\ \sigma_z &= \lambda \nabla \cdot \boldsymbol{\rho} + 2\mu \frac{\partial \zeta}{\partial z} \\ \tau_{xz} &= \mu \left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \right). \end{aligned} \tag{1.1}$$

In the following an undisturbed quantity will be denoted by a superscript ^{0}.

origin at the bottom surface in its unstressed state; the other with its origin at its surface under the action of gravity. However, the approximations in the linear theory allow us to neglect the distinction between them.

When the system is perturbed, there will be displacements $\xi(x, z, t)$ and $\zeta(x, z, t)$ in addition to $\mathbf{p}^0$ and additional stresses σ_x, σ_z and τ_{xz} derived from (1.1). The interface is now given by $z=\zeta(x, o, t)=\zeta_0(x, t)$, and the dynamic boundary conditions are

$$\begin{aligned} &\sigma_n^0+\sigma_n+p^0+p=0 \\ &\tau_{nt}^0+\tau_{nt} \qquad\qquad =0\,, \quad z=\zeta_0\,. \end{aligned} \tag{1.7}$$

For a surface element making an angle θ with the positive x-axis, a decomposition yields

$$\begin{aligned} &\sigma_n^0=\sigma_z^0\cos^2\theta+\sigma_x^0\sin^2\theta-2\tau_{xz}^0\cos\theta\sin\theta \\ &\tau_{nt}^0=\tau_{xz}^0(\cos^2\theta-\sin^2\theta)+(\sigma_z^0-\sigma_x^0)\cos\theta\sin\theta \end{aligned} \tag{1.8}$$

and the analogous expressions for σ_n and τ_{nt}. Putting $\tan\theta=\dfrac{\partial\zeta_0}{\partial x}$ and assuming θ and ζ_0 to be small quantities, (1.7) becomes, to the first approximation,

$$\begin{aligned} &\sigma_z+(\rho_2-\rho_1)g\zeta_0+p=0 \\ &\tau_{xz}-\frac{2p^0(0)}{\lambda+2\mu}\mu\frac{\partial\zeta_0}{\partial x}=0\,, \quad z=0\,. \end{aligned} \tag{1.9}$$

Here use is made of (1.3) to (1.6).

The kinematic boundary condition is

$$\frac{D\zeta_0}{dt}=\frac{\partial\zeta_0}{\partial t}+U\frac{\partial\zeta_0}{\partial x}=w\,, \quad z=0 \tag{1.10}$$

where w is the vertical component of the fluid velocity. In addition to the conditions at the surface $z=\zeta_0$, the bottom layer is assumed to be bounded by a rigid horizontal plane at $z=-H$. The boundary condition here is that the displacements vanish, i.e.

$$\xi=\zeta=0\,, \quad z=-H\,. \tag{1.11}$$

The appearance of the term $\dfrac{2p^0(0)}{\lambda+2\mu}\mu\dfrac{\partial\zeta_0}{\partial x}$ in the boundary conditions is due to the fact that although τ_{xz}^0 vanishes throughout the medium, $\tau_{nt}^0=(\sigma_z^0-\sigma_x^0)\cos\theta\sin\theta$, in general, does not. The effect of the term depends on the order of magnitude of $\dfrac{2p^0(0)}{\lambda+2\mu}$.

If the motion in our two-layer system is taken to be ocean waves under the influence of seismic waves (tsunami waves), the mean pressure term can be omitted. λ and μ will both be of order 10^5 atm., in a depth as great as 5 km. $p^0(0)$ is about $5 \cdot 10^2$ atm. and $\dfrac{2p^0(0)}{\lambda+2\mu}$ will be of order 10^{-3}. On the other hand, for a muddy bottom layer the term would perhaps be significant. For a discussion of tsunami waves, reference is made to a paper of NAKAMURA [1], and the papers cited therein. In the present investigation we will not restrict ourselves to special geophysical models, or to special values of λ and μ. However, the mean pressure will be omitted throughout most of the paper to simplify the algebra.

2. The boundary condition in terms of the stress function and the stream function. Expressions for the stresses and the displacements can be found independent of the motion of the fluid. Since the equations of motion for the elastic medium are linear, the additional stresses and displacements are governed by equations which are independent of the undisturbed stresses and of gravity. Hence we have

$$\rho_2 \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \tag{2.1}$$

$$\rho_2 \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} .$$

On the basis of these equations RADOK defines his stress function [2]. Using this we put

$$\sigma_x = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{2c_2{}^2}\frac{\partial^2}{\partial t^2}\right)\Phi \tag{2.2}$$

$$\sigma_z = \left(\frac{\partial^2}{\partial x^2} - \frac{1}{2c_2{}^2}\frac{\partial^2}{\partial t^2}\right)\Phi ,$$

where $\Phi = \Phi(x, z, t)$ satisfies the differential equation

$$\left(\nabla^2 - \frac{1}{c_1{}^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{1}{c_2{}^2}\frac{\partial^2}{\partial t^2}\right)\Phi = 0 . \tag{2.3}$$

∇^2 is the two-dimensional Laplace operator, $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$ and c_1 and c_2 are the dilatational and distortional wave velocities, defined by

$$\rho_2 c_1{}^2 = \lambda + 2\mu , \quad \rho_2 c_2{}^2 = \mu . \tag{2.4}$$

Radok considers some solutions of (2.3) assuming $\Phi = \Phi(x - ct, z)$ with c real and less than both c_1 and c_2. In our problem we have to allow for $c > c_2$ and $c > c_1$, and also for the possibility that c might be complex. However, we will consider only one Fourier component of the motion. According to this we assume $\Phi = \phi(z)e^{ik(x-ct)}$, and from (2.3) we find for the amplitude function $\phi(z)$

$$\phi(z) = A_1 \mathrm{Cos} k\beta_1 z + A_2 \mathrm{Sin} k\beta_1 z + A_3 \mathrm{Cos} k\beta_2 z + A_4 \mathrm{Sin} k\beta_2 z \tag{2.5}$$

when $\beta_1 \neq 0$ and $\beta_2 \neq 0$. These factors are defined by

$$\beta_1{}^2 = 1 - \frac{c^2}{c_1{}^2}, \quad \beta_2{}^2 = 1 - \frac{c^2}{c_2{}^2}. \tag{2.6}$$

Now (2.2) gives the normal stresses, from which the displacements are found by an integration, and finally the tangential stress is found. For the elimination of the constants A_1, A_2, A_3 and A_4 we have at our disposal the five equations (1.9), (1.10) and (1.11). Remembering that $\dfrac{\partial \zeta_0}{\partial x} = ik\zeta_0$ and $\dfrac{\partial \zeta_0}{\partial t} = -ikc\zeta_0$, we may write the result of the elimination in the following form

$$p(0)\Delta_1 - \frac{w(0)}{ik(c-U(0))}\left[g(\rho_2-\rho_1)\Delta_1 - \Delta_2 + \mu ik\frac{2p^0(0)}{\lambda+2\mu}\Delta_3\right] = 0 \tag{2.7}$$

where Δ_1, Δ_2 and Δ_3 are the determinants for the three sets of equations

$$\begin{array}{lll} \tau_{xz}(0) = 0 & \sigma_z(0) = 0 & \sigma_z(0) = 0 \\ \zeta(0) = 0 & \tau_{xz}(0) = 0 & \zeta(0) = 0 \\ \xi(-H) = 0 & \xi(-H) = 0 & \xi(-H) = 0 \\ \zeta(-H) = 0 & \zeta(-H) = 0 & \zeta(-H) = 0 \end{array} \tag{2.8}$$

respectively. The arguments 0 and $-$H refer to the z-coordinate. The determinants can now be found, and may be written

$$\Delta_1 = QD_1, \quad \Delta_2 = k\mu(1-\beta_2{}^2)QD_2, \quad \Delta_3 = -i(1-\beta_2{}^2)QD_3, \tag{2.9}$$

where Q is an immaterial constant. After some calculations, the functions D_1, D_2, D_3 are found to be

$$\begin{aligned} D_1 &= \beta_1(1-\beta_2{}^2)^2[\mathrm{Cos}kH\beta_1 \mathrm{Sin}kH\beta_2 - \beta_1\beta_2 \mathrm{Sin}kH\beta_1 \mathrm{Cos}kH\beta_2] \\ D_2 &= [(1+\beta_2{}^2)^2 + 4\beta_1{}^2\beta_2{}^2]\mathrm{Sin}kH\beta_1 \mathrm{Sin}kH\beta_2 \\ &\quad -\beta_1\beta_2[(1+\beta_2{}^2)^2+4]\mathrm{Cos}kH\beta_1 \mathrm{Cos}kH\beta_2 + 4\beta_1\beta_2(1+\beta_2{}^2) \\ D_3 &= (1+\beta_2{}^2+2\beta_1{}^2\beta_2{}^2)\mathrm{Sin}kH\beta_1 \mathrm{Sin}kH\beta_2 \\ &\quad -\beta_1\beta_2(3+\beta_2{}^2)\mathrm{Cos}kH\beta_1 \mathrm{Cos}kH\beta_2 + \beta_1\beta_2(3+\beta_2{}^2). \end{aligned} \tag{2.10}$$

It should be noted that the integrations determining ξ and ζ give two arbitrary functions. In the calculations leading to (2.10) these functions are put equal to zero.

This is necessary in order for the equations of motion to be satisfied when $\beta_1 \neq 0$ and $\beta_2 \neq 0$. For $\beta_1 = 0$, (2.5) becomes

$$\phi(z) = A_1 kz + A_2 + A_3 \mathrm{Cos} k\beta_2 z + A_4 \mathrm{Sin} k\beta_2 z \,, \tag{2.5'}$$

and for $\beta_2 = 0$

$$\phi(z) = A_1 \mathrm{Cos} k\beta_1 z + A_2 \mathrm{Sin} k\beta_1 z + A_3 kz + A_4 \,. \tag{2.5''}$$

On the basis of these functions, calculations similar to those above can be carried out to determine the correct values of D_1, D_2 and D_3 for $c = c_1$ and for $c = c_2$. Apart from a factor β_1 (resp. β_2), (2.10) are found to give the correct limits. Hence, by using (2.10) in the boundary condition (2.7), we do not lose any solutions with velocities c_1 and c_2. But we will find (2.7) to be satisfied by $c = c_1$ and $c = c_2$ for all wave lengths. These solutions are false solutions.

For the perturbed motion in the fluid we introduce the stream function $\Psi = \Psi(x, z, t)$ by

$$u = -\frac{\partial \Psi}{\partial z}, \quad w = \frac{\partial \Psi}{\partial x}, \quad \Psi = \psi(z) e^{ik(x-ct)} \,. \tag{2.11}$$

Retaining only terms of the first order, the equations of motion give for the fluid pressure

$$p(z) = -\rho_1 [(c - U(z))\psi'(z) + U'(z)\psi(z)] e^{ik(x-ct)} \,. \tag{2.12}$$

According to this, (2.7) becomes

$$\psi'(0) + \left[\frac{U'(0)}{c - U(0)} + \frac{g}{(c - U(0))^2}(B-1)\right]\psi(0) = 0 \,, \tag{2.13}$$

where the function B is defined by

$$B = \frac{\rho_2}{\rho_1}\left[1 - \frac{kc^2}{gD_1}\left(D_2 - \frac{2p^0(0)}{\lambda + 2\mu} D_3\right)\right]. \tag{2.14}$$

3. Elastic response and the free waves. Since the variable part of the fluid pressure acting on the interface $z = \zeta_0(x, t)$ is given by $p(\zeta_0) = p(0) - \rho_1 g \zeta_0$, and the vertical velocity component is $w(0) = -ik(c - U(0))\zeta_0$, (2.7) can be rewritten in the form

$$p(\zeta_0)\Delta_1 + \zeta_0\left[\rho_2 g \Delta_1 - \Delta_2 + ik\mu \frac{2p^0(0)}{\lambda + 2\mu}\Delta_3\right] = 0 \,. \tag{3.1}$$

From this we find

$$\frac{p(\zeta_0)}{\rho_1 g \zeta_0} = -B \tag{3.2}$$

Thus B is a stiffness coefficient for the elastic boundary as introduced by BROOKE BENJAMIN [3]. B is nondimensional, and is to a certain extent arbitrary, depending on the normalizing factor used. It is also worth noting that B does not depend on the properties of the elastic medium alone, but also on the boundary conditions.

When the fluid pressure has its maxima at the wave throughs, B will be positive, while the maxima at the wave crests correspond to B negative; otherwise B is complex. From (2.10) and (2.14) it is seen that this can happen only for complex values of c.

Brooke Benjamin also considers a flexible boundary to be characterized by a surface tension T and an effective mass m per unit area. We will shortly discuss how these quantities must be chosen for our elastic bottom.

Let us consider a fluid motion with a pressure $p(\zeta_0)$ at the interface $z=\zeta_0$, $p(\zeta_0)$ and ζ_0 both varying as $e^{ik(x-ct)}$. (3.2) will then be the frequency equation. Further we assume that the elastic bottom can be replaced by a membrane with a mass m per unit area and with a tension T, m and T depending upon k and c in such a way that this membrane will cause the same effect on the fluid motion as does the elastic bottom. The equation of motion for the membrane is

$$m\frac{\partial^2\zeta_0}{\partial t^2}=T\frac{\partial^2\zeta_0}{\partial x^2}-p(\zeta_0)=\left(T+\frac{1}{k^2}\frac{p(\zeta_0)}{\zeta_0}\right)\frac{\partial^2\zeta_0}{\partial x^2}, \tag{3.3}$$

which gives

$$T=mc^2+\frac{\rho_1 g}{k^2}B\,. \tag{3.4}$$

Since this is the only equation determining T and m, we first conclude that one of them can be chosen arbitrarily. Further we note that the expressions for T and m can usually not be found without a knowledge of the motion in the bottom layer. However, if the boundary really was a membrane with known values for T and m, (3.4) would define the stiffness coefficient B for this membrane. We may introduce the free wave velocity c_0 for the membrane by putting $T=mc_0^2$. The expression for B can then be written

$$B=\frac{mk^2}{\rho_1 g}(c_0{}^2-c^2)=\frac{Tk^2}{\rho_1 g}\left(1-\frac{c^2}{c_0{}^2}\right). \tag{3.5}$$

The free waves in the elastic bottom layer are found by putting $p(\zeta_0)=0$, the frequency equation is $B=0$, or from (2.14)

$$D_2=\frac{g}{kc_2{}^2}D_1+\frac{2p^0(0)}{\lambda+2\mu}D_3\,. \tag{3.6}$$

By means of (2.10) we find the approximate form of (3.6) for $kH\gg 1$ and $c<c_2$

$$(1+\beta_2{}^2)^2-4\beta_1\beta_2=\frac{g}{kc_2{}^2}\beta_1(1-\beta_2{}^2)+\frac{2p^0(0)}{\lambda+2\mu}(1-2\beta_1\beta_2+\beta_2{}^2)\,. \tag{3.7}$$

When gravity and the mean pressure are neglected, (3.7) reduces to the RAYLEIGH's equation [4], the solution of which varies between $c=0{,}8741c_2$ and $c=0{,}9554c_2$ as Poisson's ratio varies between 0 and 0,5. This implies that the solution of (3.7) depends chiefly on the rigidity μ in the medium and only to a small extent on the compressibility. Both gravity and the mean pressure lead to an increase of the wave velocity since the left hand side increases with c. The gravity effect is of order $(kH)^{-1}$, and vanishes in the limit $kH\rightarrow\infty$, expressing the fact that the shorter the wave length is, the larger the relative displacements are and the more dominating are the elastic stresses.

In the limit $kH\rightarrow 0$, B tends to the finite value $\frac{\rho_2}{\rho_1}\left(1+\frac{c_1^2}{gH}\right)$ and hence there can be no finite wave velocities for infinitely long waves. If we assume c to tend to infinity as $(kH)^p$, we find for the group velocity $c_g=(1+p)c$. To avoid an infinite value of c_g we have to put $p=-1$. β_1 and β_2 defined by (2.6), become imaginary for $c>c_1$ and $c>c_2$, We then put $\beta_1=i\alpha_1$ and $\beta_2=i\alpha_2$, i.e.,

$$\alpha_1^2=\frac{c^2}{c_1^2}-1\,,\quad \alpha_2^2=\frac{c^2}{c_2^2}-1\,. \tag{3.8}$$

The solutions of (3.6) for $kH\rightarrow 0$ should therefore be characterized by $c\rightarrow\infty$, while $kH\alpha_1$ and $kH\alpha_2$ approach finite values.

To examine the solutions of (3.6) for $c>c_2$ we first consider $B=\pm\infty$, i.e., $D_1=0$ or

$$\mathrm{Cos}\,kH\beta_1\sin kH\alpha_2-\beta_1\alpha_2\,\mathrm{Sin}\,kH\beta_1\cos kH\alpha_2=0\,. \tag{3.9}$$

Putting

$$\beta_1\alpha_2\,\mathrm{Tan}\,kH\beta_1=\tan\gamma \tag{3.10}$$

we find from (3.9)

$$kH\alpha_2=\gamma+n\pi\,.\quad n=0\,,\pm1\,,\pm2\,,\ldots\ldots \tag{3.11}$$

In the interval $c_2<c<c_1$, $\tan\gamma$ does not change its sign; for $c>c_1$ it changes its sign whenever $\tan kH\alpha_1$ does. For the variation of kH and c along a curve for $D_1=0$ we have therefore

$$\begin{aligned} &n\pi<kH\alpha_2<(n+\tfrac{1}{2})\pi && c_2<c<c_1\\ &(n-\tfrac{1}{2})\pi<kH(\alpha_1+\alpha_2)<(n+\tfrac{1}{2})\pi && c>c_1 \end{aligned} \tag{3.12}$$

For $c>c_2$ there will also be an infinite series of curves for $B=0$, each of them lying between two neighbouring curves for $D_1=0$. Thus, as a fairly good approximation for this series of free waves we may write $kH\alpha_2=a_n$, or

$$\frac{c}{c_2}=\pm\left(1+\frac{a_n^2}{(kH)^2}\right)^{1/2}, \tag{3.13}$$

with $a_{n+1} - a_n \approx \pi$. The group velocity c_g based on this approximation is found to be

$$\frac{c_g}{c_2} = \pm\left(1+\frac{a_n^{\,2}}{(kH)^2}\right)^{-1/2} = \left(\frac{c}{c_2}\right)^{-1}. \tag{3.14}$$

These waves are virtually unaffected by external effects such as gravity and mean pressure. As will be shown below, they are also mostly independent of the fluid motion when the surface of the bottom layer is not free.

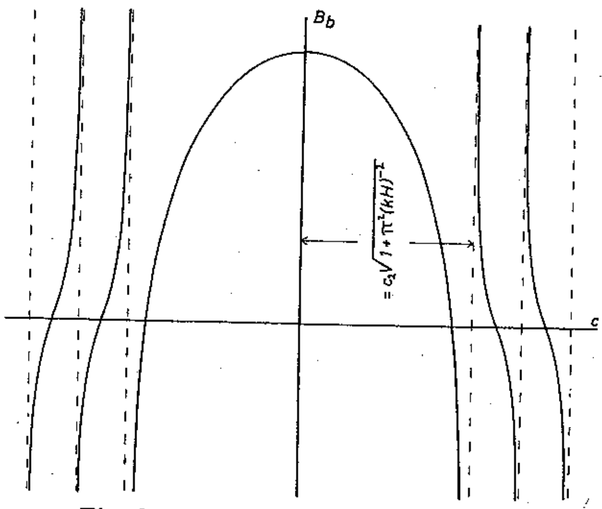

Fig. 2. The bottom response function.

In the following sections the function B, defined by (2.14), will be called the bottom response function and will be written $B=B_b(c, k)$. Another expression will be found for B. This expression will be determined from the fluid flow and will be called a fluid response function $B=B_f(c, k)$. The frequency equation may then be written $B_b(c, k)=B_f(c, k)$. We shall find it convenient to discuss the solutions of the frequency equation by considering the intersections of the curves for B_b and B_f for given values of k.

In Fig. 2 a sketch of B_b is given. For $kH \gg 1$, the first zero will occur for a value of c near the Rayleigh's velocity and the distances between the asymptotes are about $c_2(kH)^{-2}$. For $kH \ll 1$ the first zero and the asymptote distances are of order $c_2(kH)^{-1}$. Here, and in the rest of the paper, the mean pressure term $\dfrac{2p^0(0)}{\lambda+2\mu}$ will be omitted. The dependence of $B_b(c=0)$ upon kH may be of some interest. It is found to be

$$B_b(c=0) = \frac{\rho_2}{\rho_1}\left(1+\frac{c_2^{\,2}}{gH}f(kH)\right), \tag{3.15}$$

where $f(kH)$ is given by

$$f(kH) = kH\frac{2\mathrm{Cos}2kH+(2kH)^2+2(3-4\upsilon)}{(3-4\upsilon)\mathrm{Sin}2kH-2kH}. \tag{3.16}$$

ν is the Poisson's ratio.

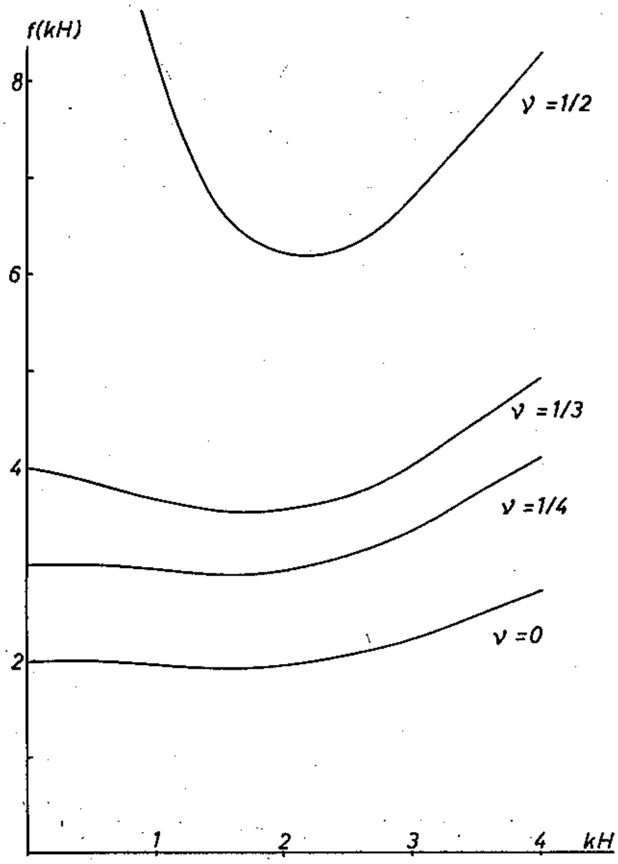

Fig. 3. The function $f(kH)$ defined by (3.16).

The function $f(kH)$ is shown in Fig. 3 for some selected values of ν. Unless ν is near 0,5, the minimum of $f(kH)$ will be slightly less than $f(0)$. The wave length giving this minimum is of same order as H. For $kH \to 0$ we find $f(0) = c_1^2/c_2^2$, and for large $kH, f(kH)$ tends to $\dfrac{2}{3-4\nu} kH$.

4. Surface waves on a uniform stream. We now consider the fluid to have a free surface at $z=h$, and a uniform basic flow with velocity U. A periodic and irrotational disturbance given to this flow yields surfaces waves superposed on the uniform stream. The fluid layer thickness h and the bottom layer thickness H are of the same order of magnitude. Waves which are long compared with h will also be long compared with H, i.e., both of the approximations $kh \ll 1$ and $kH \ll 1$ must be used. Analogously we have both $kh \gg 1$ and $kH \gg 1$ for the shortest waves.

The free surface is given by

$$z = h + \zeta_h(x, t) = h + a_h e^{ik(x-ct)}, \tag{4.1}$$

and the boundary conditions $\dfrac{D\zeta_h}{dt} = w$ and $p^0 + p = \text{const.}$ at $z = h + \zeta_h$ become, in terms of the stream function,

$$\psi'(h)-\frac{g}{(c-U)^2}\psi(h)=0\,. \tag{4.2}$$

Since the stream function is a solution of Laplace's equation $\nabla^2\Psi=0$, we write

$$\psi(z)=C_1\mathrm{Cos}kz+C_2\mathrm{Sin}kz\,, \tag{4.3}$$

and the elimination of C_1 and C_2 from (4.2) and (2.13) (with $U'(0)=0$) gives the frequency equation

$$\frac{k(c-U)^2\mathrm{Tan}kh-g}{k(c-U)^2-g\mathrm{Tan}kh}=\frac{g}{k(c-U)^2}(B-1)\,. \tag{4.4}$$

The fluid response function for this motion is found from (4.4),

$$B_f(c,k)=\frac{k}{g}\mathrm{Tan}kh\frac{(c-U)^4-\left(\frac{g}{k}\right)^2}{(c-U)^2-\frac{g}{k}\mathrm{Tan}kh}\,. \tag{4.5}$$

Its variation with c for a given kh is given in Fig. 4. For $(c-U)^2<\frac{g}{k}\mathrm{Tan}kh$ it is

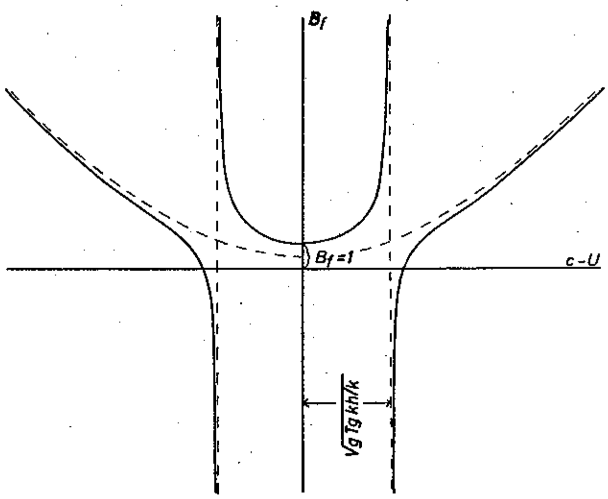

Fig. 4. The fluid response function for the surface waves.

restricted to $B_f\geq 1$, while for $(c-U)^2>\frac{g}{k}\mathrm{Tan}kh$ and $B_f\to+\infty$ the curve tends asymptotically towards the parabola $B_f=\left(\frac{k}{g}\mathrm{Tan}kh\right)(c-U)^2+\mathrm{Tan}^2kh$. As mentioned above, the frequency equation may be written $B_f(c,k)=B_b(c,k)$. If the curves in Fig. 2 and Fig. 3 was drawn on a single diagram for a common value of k, the points of intersection would directly give the real solutions for c.

Let us first assume $U=0$ and $kh \gg 1$. The diagrams show that there will be two intersections between the inner branch of B_b and the inner branch of B_f, corresponding to the free water waves. These two wave modes may be considered as dominated chiefly by the fluid. Further there will be one intersection between the inner branch of B_b and each of the outer branches of B_f, corresponding to the lowest mode of the free elastic waves; and since $c \to \pm\infty$ when $B_f \to +\infty$, there will also be one point of intersection on each of the outer branches of B_b, corresponding to the infinite series of free elastic waves discussed above. The waves corresponding to the free elastic waves will be denoted the elastic dominated waves. For decreasing kH, $B_b(c=0)$ will decrease while $B_f(c=0)=1$. The condition for an intersection between the inner branches of B_b and B_f is $B_b(c=0) \geq B_f(c=0)$, or

$$\frac{\rho_1}{\rho_2} \leq 1+\frac{c_2{}^2}{gH}f(kH)\,. \tag{4.6}$$

Since for a compressible bottom $f(kH)_{\min} \approx f(0)=c_1{}^2/c_2{}^2$ the critical density ratio will be approximately equal to $1+c_1{}^2/gH$. However, the waves which first become unstable for increasing ρ_1/ρ_2 are not the infinitely long ones, but those with wave lengths of the same order as H. For an incompressible bottom the critical value of ρ_1/ρ_2 is about $1+6{,}2c_2{}^2/gH$. The other solutions can not become unstable.

For a free stream velocity U different from zero the diagram for B_f should be shifted a distance U to the right (assuming $U>0$). From the diagrams we may conclude that U will cause a destabilizing effect on the fluid dominated waves since the inner branches of B_f and B_b may now cease to intersect. Another conclusion which may be drawn by examining the diagrams is that whenever the fluid dominated waves are stable, all the elastic dominated waves will also be stable. The destabilizing effect of U can obviously be balanced by making ρ_2/ρ_1 and $c_2{}^2/gH$ large, or by increasing the velocities of the free elastic waves. For increasing wave length the inner branch of B_b will become more oblate, for decreasing wave length it will be steeper while the distances between the asymptotes of B_f decrease. For the longest waves, a small increase in $\dfrac{\rho_2}{\rho_1}\left(1+\dfrac{c_1{}^2}{gH}\right)$ will have an appreciable stabilizing effect, while for the shortest waves the dominating stabilizing effect will be an increase of the free velocities. In summary, the main effects to be taken into account for the long waves are gravity and the compressibility in the bottom layer; for the short waves the important effects are the basic current and the rigidity.

Let us now assume the motion to be stable. The velocities for the fluid dominated waves will be restricted to the inner branches of both B_b and B_f for all wave lengths This restriction means that c is less than the lowest free elastic wave velocity, and further $(c-U)^2 < \dfrac{g}{k}\operatorname{Tan}kh$. Considering infinitely short waves, we find $c \to U$. Hence, a necessary condition for stability is that U must be less than the free elastic wave velocity for infinite wave-number (the Rayleigh's velocity). Further we assume that the frequency equation has no solutions for which $B_f = B_b = \pm\infty$. It means that $c = U + (g/k)^{1/2}$ is less than the lowest solution of $B_b = \pm\infty$, and a sufficient condition for this is $U < c_2 - (gh)^{\frac{1}{2}}$.

By means of (4.5) the frequency equation can be written

$$2\frac{k}{g}\mathrm{Tan}kh(c-U)^2 = B_b \pm [(B_b - 2\mathrm{Tan}^2kh)^2 + 4\mathrm{Tan}^2kh(1-\mathrm{Tan}^2kh)]^{1/2} . \tag{4.7}$$

An examination of the right hand side immediately shows that the solutions when the lower sign is used, are characterized by

$$(c-U)^2 \leq \frac{g}{k}\mathrm{Tan}kh , \quad B_b = B_f \geq 1 . \tag{4.8}$$

By using the lower sign in (4.7) we thus obtain an equation for the fluid dominated waves for all wave lengths. The asymptotic solutions are found to be

$$(c-U)^2 = gh\left[1 - \frac{\rho_1}{\rho_2}\left(1 + \frac{c_1^2}{gH}\right)^{-1}\right] \quad \text{for} \quad kH \to 0 \tag{4.9}$$

and

$$(c-U)^2 = \frac{g}{k} \qquad \text{for} \quad kH \to \infty . \tag{4.10}$$

Using the upper sign we find the right hand side of (4.7) to be positive for all real B_b. Its minimum occurs for $B_b \to -\infty$ and yields the solution

$$(c-U)^2 = \frac{g}{k}\,\mathrm{Tan}kh . \tag{4.11}$$

When $B_b \to +\infty$ the same solution is found when the lower sign is used in the equation. Hence, if there are solutions (4.11) coinciding with $B_b = \pm\infty$, there may be a transition from an upper sign solution to a lower sign solution. According to the assumption made above, there are no points on the dispersion curves where $B_f = B_b = \pm\infty$. The upper sign equation will therefore give a series of dispersion curves, each lying between two neighbouring curves for $B_b = \pm\infty$. For these waves the approximate solutions (3.13) and (3.14) should still be appropriate. The lowest solution, however, will become less than c_2 in the limit $kH \to \infty$. From (4.7) we find the asymptotic equation

$$(c-U)^2 = \frac{g}{k}\,(B_b - 1) , \tag{4.12}$$

and with the asymptotic expression for B_b (4.12) becomes

$$\frac{\rho_1}{\rho_2}\frac{(c-U)^2}{c_2^2} = \frac{4\beta_1\beta_2 - (1+\beta_2^2)^2}{\beta_1(1-\beta_2^2)} \tag{4.13}$$

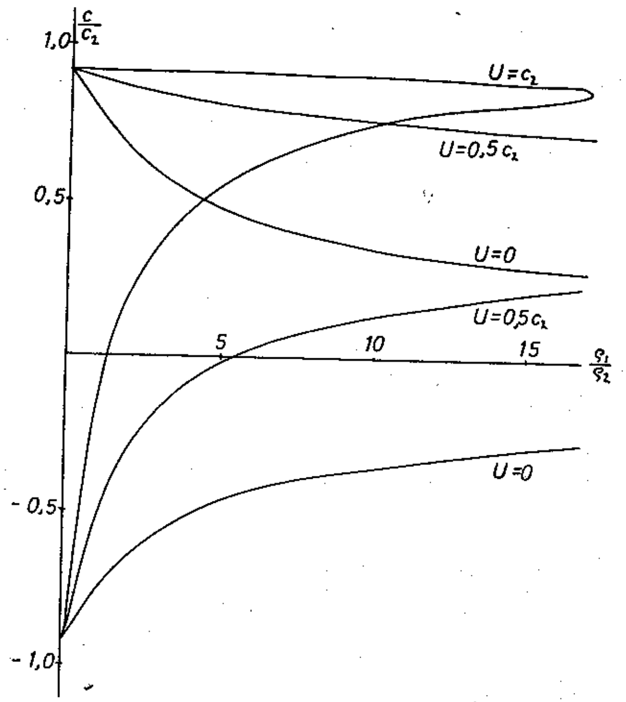

Fig. 5. The solutions of (4.13) for some values of U and for $\nu = {}^1/_4$.

In Fig. 5 the variation of c with ρ_1/ρ_2 is given for some selected values of U. ν is set equal to 1/4 in the construction of the curves. If U is less than the Rayleigh's velocity, the solutions will be real for all values of the density ratio. For larger values of U instability will occur in these elastic dominated waves at some critical value of ρ_1/ρ_2, depending on U/c_2 and ν.

5. Perturbation of a flow with constant shear.

a) *The integral solution.* When the velocity profile in the basic flow is linear, we put

$$U(z) = U_0 + U'z\,, \quad U' \text{ constant}. \tag{5.1}$$

Then the linearized equation for the stream function becomes

$$(c - U(z))(\psi''(z) - k^2\psi(z)) = 0\,. \tag{5.2}$$

The fluid is bounded by a horizontal rigid plane at $z = h$, and the boundary conditions may be written

$$\begin{aligned} \psi'(0) + kF\psi(0) &= 0 \\ \psi(h) &= 0\,, \end{aligned} \tag{5.3}$$

where F is given by (see (2.13))

$$F = \frac{U'}{k(c-U_0)} + \frac{g}{k(c-U_0)^2}(B-1)\,. \tag{5.4}$$

For convenience we put $c = U_0 + U'\eta$. For a given wave number k, F will then be a function of η, $F = F(\eta)$. The solution of (5.2) satisfying (5.3) may be written

$$\psi(z) = A\,\mathrm{Sin}k(h-\eta)(\mathrm{Cos}kz - F(\eta)\mathrm{Sin}kz)\,, \quad 0 \le z \le \eta$$

(5.5)

$$\psi(z) = A\,\mathrm{Sin}k(h-z)(\mathrm{Cos}k\eta - F(\eta)\mathrm{Sin}k\eta)\,, \; \eta \le z \le h\,.$$

This solution, which is called a singular eigensolution, is continuous at $z = \eta$; but this is in general not the case for $\psi'(z)$, hence there can be infinite vorticity at $z = \eta$. For $F(\eta) = \mathrm{Cot}kh$, the solution becomes

(5.6)
$$\psi(z) = A\frac{\mathrm{Sin}k(h-\eta)}{\mathrm{Sin}kh}\mathrm{Sin}k(h-z)$$

in the whole interval (o, h), and thus $F(\eta) = \mathrm{Cot}kh$ is the frequency equation for the ordinary eigensolutions. These solutions, however, have no vorticity. If we consider a given distribution of vorticity as an initial condition, this condition can not be satisfied by the ordinary eigensolutions alone. To find the complete solution we use the procedure introduced by ELIASSEN, HØILAND and RIIS [5]. Assuming Ψ to be periodic in x but not in t, we put $A = A(\eta)$ in (5.5), and integrate from $\eta = 0$ to $\eta = h$. Thus we obtain

$$\Psi = e^{ik(x-U_0 t)}\left\{\int_0^z A(\eta)\mathrm{Sin}k(h-z)(\mathrm{Cos}k\eta - F(\eta)\mathrm{Sin}k\eta)e^{-ikU'\eta t}d\eta\right.$$

(5.7)

$$\left. + \int_z^h A(\eta)\mathrm{Sin}k(h-\eta)(\mathrm{Cos}kz - F(\eta)\mathrm{Sin}kz)e^{-ikU'\eta t}d\eta\right\}.$$

By differentiating under the integral signs, we find for the vorticity

(5.8)
$$\nabla^2\Psi = -kA(z)(\mathrm{Cos}kh - F(z)\mathrm{Sin}kh)e^{ik(x-U(z)t)}$$

To satisfy an initial condition $(\nabla^2\Psi)_{t=0} = G(z)e^{ikx}$, we have to put

(5.9)
$$A(\eta) = -\frac{1}{k}G(\eta)(\mathrm{Cos}kh - F(\eta)\mathrm{Sin}kh)^{-1}.$$

From this we find that the eigenvalues may give singularities in $A(\eta)$, and when the integration is carried out along paths in the complex η-plane, the integral solution (5.7) may be multivalued. However, since the vorticity $\nabla^2\Psi = G(z)e^{ik(x-U(z)t)}$ is independent of the paths of integration, the difference between two such solutions has no vorticity and must therefore be an eigensolution or a sum of two or more of them. Since the complete solution of (5.2) is the sum of the integral solution and all the ordinary eigensolutions, the paths of integration can be chosen arbitrarily. Obviously they can be chosen in the halfplane where $\mathrm{Im}\eta < 0$ in such a way that both $\frac{\partial\Psi}{\partial x}$ and $\frac{\partial\Psi}{\partial z}$

tend to zero for increasing t. If one of the end points of the paths of integration coincides with an eigenvalue, one or both of the integrals seem to start or to end in a singularity. We must remember, however, that the disturbance is assumed to be small initially. This implies that the initial distribution of vorticity $G(z)$ must be such that the integrals exist at $t=0$. Consequently the integrals exist for all $t>0$, which is easily seen from (5.7). The integral solution can therefore always be considered as a stable one, and the stability of the flow will only depend on whether the frequency equation has complex solutions or not.

Since the integral solution has a continuous spectrum of singular eigensolutions a similar continuous spectrum will occur in the corresponding solution for the motion in the elastic bottom layer. All these singular solutions have the same wave number, while their wave velocities are distributed between $U(0)$ and $U(h)$. Expressions for the displacements and the stresses in the bottom layer will not be given here. We can, however, easily find the form of the interface $z=\zeta_0(x, t)$ by means of the boundary condition

$$\frac{\partial \zeta_0}{\partial t}+U_0\frac{\partial \zeta_0}{\partial x}=\left(\frac{\partial \Psi}{\partial x}\right)_{z=0}. \tag{5.10}$$

Introducing from (5.7) and integrating, we find

$$\zeta_0 = -e^{ik(x-U_0t)}\int_0^h \frac{A(\eta)}{U'\eta}\text{Sin}k(h-\eta)e^{-ikU'\eta t}d\eta\,. \tag{5.11}$$

To find the complete form of the interface we have to add the contributions from the ordinary eigensolutions to the expression in (5.11).

From the definition (3.2) of the response factor B, it is clear that the solution for a rigid bottom at $z=0$ is obtained by letting B tend to infinity. In this limit (5.7) becomes

$$\begin{aligned}\Psi = -\frac{e^{ik(x-U_0t)}}{k\text{Sin}kh}\Bigg\{&\int_0^z G(\eta)\text{Sin}k(h-z)\text{Sin}k\eta e^{-ikU'\eta t}d\eta\\ &+\int_z^h G(\eta)\text{Sin}k(h-\eta)\text{Sin}kz e^{-ikU'\eta t}d\eta\Bigg\}.\end{aligned} \tag{5.12}$$

Apart from notation, this is the solution given in [5] for a homogenous fluid. Since $F(\eta)\to\infty$ when $B\to\infty$, the frequency equation becomes $\text{Tan}kh=0$. Comparing this with the solution for B finite, we see that the new features introduced by the elastic bottom layer are the singularities in the integrals, and the ordinary eigensolutions.

Finally it should be noted that the solution found above, and the discussion following it, will apply not only to an elastic bottom layer, but to all non-rigid boundaries giving boundary conditions in the form (5.3).

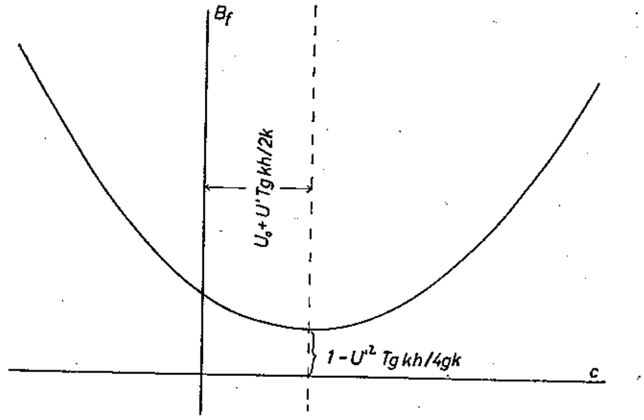

Fig. 6. The fluid response function for the shear flow.

b. *Discussion of the frequency equation.* The frequency equation $F(\eta)=\mathrm{Cot}kh$ is found from (5.4) to be

$$\mathrm{Cot}kh=\frac{U'}{k(c-U_0)}+\frac{g}{k(c-U_0)^2}(B-1)\,. \tag{5.13}$$

Hence, the fluid response function B_f for this motion is the parabola

$$B_f=\frac{k}{g}\mathrm{Cot}kh\left(c-U_0-\frac{1}{2}U'h\frac{\mathrm{Tan}kh}{kh}\right)^2+1-\frac{1}{4}\frac{(U'h)^2}{gh}\frac{\mathrm{Tan}kh}{kh}, \tag{5.14}$$

which is shown in Fig. 6. By comparing Fig. 6 with Fig. 2, some conlucsions can immediately be drawn. The parabola will always intersect each of the outer branches of B_b. As the wave number varies there are no possibility for transition from one branch to another. Hence the approximations (3.13) and (3.14) will apply to these (always real) solutions. For infinitely long waves the minimum of B_f is $1-\frac{1}{4}\frac{(U'h)^2}{gh}$, and the condition for intersection is

$$1-\frac{1}{4}\frac{(U'h)^2}{gh}<B_b(kH=0)\,. \tag{5.15}$$

An increase of the shear U' will cause a stabilizing effect on these waves. For finite wave lengths we find from (5.14) that an increase of the shear U' will displace the parabola towards the right and downwards. The former displacement represents a destabilizing and the latter a stabilizing effect. Considering a small increment $\delta U'$, we find the displacement of the parabola to have a gradient $-U'/g$. If B_b and B_f just meet at $c=c'$, a positive $\delta U'$ will cause a stabilizing effect if $\left|\frac{\partial B_b}{\partial c}\right|<\frac{U'}{g}$ for $c=c'$. Since $\left|\frac{\partial B_b}{\partial c}\right|$ increases for decreasing wave length, we may say that the increase of the shear will make the long waves more stable and the short waves less stable. However, the length of the displacement will be proportional to $\mathrm{Tan}kh/kh$ and will tend to zero for the shortest waves.

From (5.13) we find the asymptotic solutions

$$c-U_0=\frac{1}{2}U'h\pm\left[\frac{1}{4}(U'h)^2+gh(B_b-1)\right]^{1/2} \quad \text{for} \quad kh\to 0$$

(5.16) and

$$(c-U_0)^2=\frac{g}{k}(B_b-1) \quad \text{for} \quad kh\to\infty\,.$$

In the first equation of (5.16) we have to put $B_b=\frac{\rho_2}{\rho_1}\left(1+\frac{c_1^2}{gH}\right)$ and the condition (5.15) becomes

(5.17)
$$\frac{\rho_1}{\rho_2}\left(1-\frac{1}{4}\frac{(U'h)^2}{gh}\right)<1+\frac{c_1^2}{gH}\,.$$

The latter of (5.16) is identical with (4.13).

At last we shall assume $U_0=0$ and neglect the effect of gravity. The response functions may be redefined (still dimensionless) by writing

(5.18)
$$B'_f=\frac{g}{k(U'h)^2}B_f$$

$$B'_b=\frac{g}{k(U'h)^2}B_b$$

and putting $g=0$. From the results

$$B'_f=\frac{\mathrm{Cot}kh}{(U'h)^2}\left(c-\frac{1}{2}U'h\frac{\mathrm{Tan}kh}{kh}\right)^2-\frac{1}{4}\frac{\mathrm{Tan}kh}{(kh)^2}$$

(5.19) and

$$B'_b=\frac{\rho_2}{\rho_1}\frac{c_2^2}{(U'h)^2}\frac{f(kH)}{k}>0 \quad \text{for} \quad c=0\,,$$

we conclude at once that the motion is stable for all wave lengths since the parabola pass through the origin for all values of kh.

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REFERENCES

1. Nakamura, K., 1961: *Reports of Tohoku Univ.* **13**, 164.
2. Radok, J. R. M., 1956: *Quart. Appl. Math.* **14**, 289.
3. Brooke, Benjamin T., 1960: *J. Fluid Mech.* **9**, 513.
4. Lord Rayleigh, 1900: *Scientific Papers* II p. 441. Cambridge.
5. Eliassen, A., E. Høiland, and E. Riis, 1953: Two-dimensional perturbation of a stratified fluid. *Inst. for Weather and Climate Research*, Oslo, Publ. no. 1.

First we consider the basic flow given by

$$\mathbf{v}^0 = \mathbf{i}U(z)\,. \tag{1.2}$$

Since the fluid is inviscid, the pressure p^0 depends on z only and the pressure gradient is given by

$$\frac{dp^0}{dz} = -\rho_1 g\,. \tag{1.3}$$

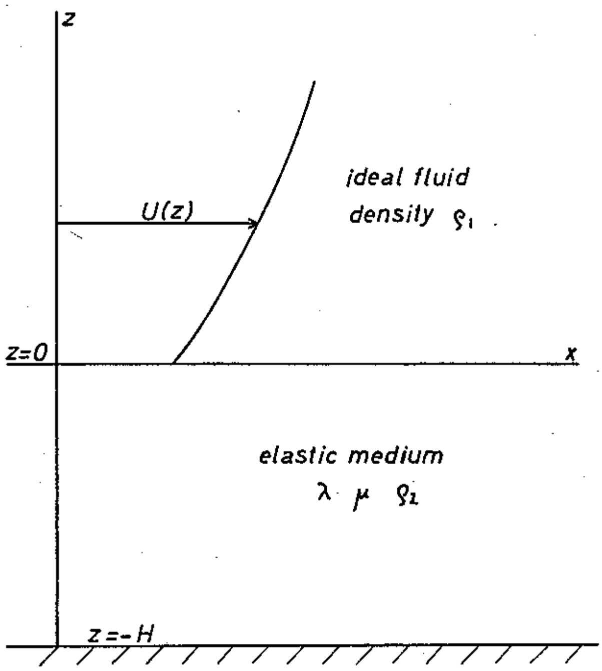

Fig. 1. The basic flow system.

Since there are no tangential tractions on the bottom and the flow is uniform in x, the state of the bottom layer will also be uniform in x, and the displacement relative to the unstressed state is $\boldsymbol{\rho}^0 = \mathbf{k}\zeta^0(z)$. Hence we have for the stresses

$$\sigma_x^0 = \lambda\frac{d\zeta^0}{dz}\,, \quad \sigma_z^0 = (\lambda+2\mu)\frac{d\zeta^0}{dz}\,, \tag{1.4}$$

while τ_{xz}^0 vanishes throughout the medium. The equation of equilibrium gives

$$\frac{d\sigma_z^0}{dz} = \rho_2 g\,, \tag{1.5}$$

and the boundary condition is

$$\sigma_x^0 + p^0 = 0\,, \quad z = 0\,. \tag{1.6}$$

It should be noted that here we are using two coordinate systems, one with its

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