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Part. I. Theory and analysis of observations

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INTERNAL WAVES OF TIDAL ORIGIN

BY JONAS EKMAN FJELDSTAD

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PART I. THEORY AND ANALYSIS OF OBSERVATIONS

Summary. In the first chapter we give a deduction of the basic equation of internal waves when the influence of the compressibility of sea water is taken into account. The result is that the kinetic energy depends on the density in situ, $\varrho_p = 1 + 10^{-3}\sigma_{t,p}$, while the potential energy depends on $\varrho_0 = 1 + 10^{-3}\sigma_t$. If the depth is not too large, it will not be necessary to make this distinction.

The different types of waves which are possible under the influence of the Earth's rotation are deduced by a uniform procedure. Formulae for potential and kinetic energy are given, and the influence of eddy viscosity on internal waves is investigated. Waves of higher order are very effectively damped.

Methods for numerical integration of the basic differential equation are given and methods for the detection and analysis of internal waves are deduced.

In the second chapter the observations of internal waves in Herdlefjord 1934 and 1949 are discussed and the results are compared with the theory. It is found that the waves are mainly of progressive character. A comparison of the results from the two years show that the waves appear with the same phase angles when the hydrographic conditions are similar.

In the appendix a more elaborate treatment of the frictional influence on internal waves is given.

Part II contains tables of the observations.

CONTENTS

CHAPTER I. THEORY

1. Introduction. The system of large ocean currents which are more or less stationary have always attracted the attention of the oceanographers, and it is clear that the investigation and explanation of these phenomena will allways be a major aim in oceanography. For this purpose observations of temperature, salinity, oxygen and other characteristic properties of the sea water are collected, in order to get an indication of the prevailing currents.

Strictly, the observations ought to be simultaneous, but this condition can not be fulfilled at present. The question then arises: How far can observations which are not simultaneous be combined? Or otherwise expressed, can a single observation give a sufficiently close representation of the prevailing conditions at the point considered. Most oceanographers have tacitly assumed that, apart from the surface layers, the conditions are sufficiently stationary to allow the combinations of observations as if they were simultaneous.

Different expeditions have, however, made observations which show that the conditions may alter in a very short time.

Repeated observations during 24 hours show that the changes often are more or less periodic and that the tidal periods are predominant. The changes are in some way connected with the tidal currents.

The observed variations of temperature and salinity can only be explained by assuming large vertical oscillations of the deeper water layers, and a full explanation is only possible by making use of a theory of internal waves.

A theory of internal waves was first given by STOKES in 1847. [11]

He considered the oscillations of two superposed liquids of different density. It is then possible to have large vertical displacement of the common boundary, while the surface is nearly undisturbed. The horisontal velocity is discontinuous at the boundary and the velocity of propagation will depend on the difference of density between the two layers, being:

$$c = \sqrt{\frac{g h_1 h_2 (\rho_1 - \rho_2)}{(h_1 + h_2)\rho_1}}$$

where h_1 and h_2 are the depths of the two layers, the total depth being h. ρ_1 and ρ_2 are the densities of the two liquids. Such waves are often called boundary waves and it was the theory of boundary waves which was used by oceanographers when trying to explain the oscillations in the deeper layers of the sea.

The theory of Stokes is based upon the assumption that there is a sharp discontinuity in density, but that the two layers are otherwise homogeneous.

Near the coast one may sometimes find a sudden increase in density which approximates to a discontinuity and then the theory of boundary waves may be applied, but in the open sea real discontinuities are practically never found. Sometimes one may find a layer with rapid variation of the density, but the layers above and below are never homogeneous unless in great depths.

Observations of internal waves were made by Otto Pettersson in 1907 in Storebelt, and in 1910, observations in the Faroe-Shetland channel were made by a Danish, a Scottish and a Norwegian research vessel. [7]

Large variations in temperature and salinity were found which could only be attributed to vertical oscillations of tidal period. But here the variation of density with depth could not even approximately be represented by a discontinuity layer.

On the Meteor-Expedition vertical oscillations of tidal period were found in the south Atlantic, but the discussions of these observations by Defant [1] were still unpublished when I published my paper "Interne Wellen" [4].

The current measurements which were made by Helland-Hansen and Ekman in 1930 showed that the tidal currents varied greatly with depth in a manner which could not be explained by the classical theory of tidal currents.

The theory of boundary waves may explain a certain phase difference between the tidal currents in a top layer and a bottom layer, but the observed variations were of a more complicated character.

The theory at that time also suffered from another defect. When the influence of the Earth's rotation is taken into account, one may find a solution of the Kelvin type, which gives a wave where the amplitude has an exponential decrease in the direction perpendicular to the direction of propagation. This decrease depends on the velocity of propagation and is more marked when the wave propagation is slow. The consequence is that for internal waves the amplitude would be infinitely small at a short distance from the coast. Defant was hereby led to the conclusion that real internal tidal waves would be impossible in the open ocean.

The observations had shown that relatively large variations of temperature and salinity of tidal period were present in the ocean, but the current theory was unable to explain them. Helland-Hansen has drawn attention to the possibility that variations might be observed where the temperature and salinity varied rapidly in a horisontal direction, but a closer examination shows that the observed variations in most cases were too large to be explained in this way.

Such was the state of affairs when I started my theoretical investigation in 1931.

I saw that a new theory must take into account the real density distribution such as it was found from actual observations. It was also necessary to see if other types of waves were possible, than the Kelvin type. I succeeded in finding a new type of waves where the amplitude showed a periodic variation in the direction perpendicular to the direction of propagation.

For the amplitude of the vertical oscillation I found a differential equation, and this equation with the appropriate boundary conditions admitted an infinity of waves with the same period but different velocities of propagation, and consequently different wave lengths.

The velocities of propagation of the different waves were as a rule incommensurable and as a consequence the whole wave phenomenon would not be periodic in space.

In order to test the theory some observations were made in Nordfjord 1933, and it was found, that tidal variations of temperature and salinity occurred, indicating that internal waves were present, but owing to unfavourable weather conditions the observations had to be discontinued after 42 hours.

The next summer observations were made in Herdlefjord, near Bergen, and some preliminary results of the expedition were presented at the meeting of the International Union of Geodecy and Geophysics in Edinburgh 1936 and at the congress of Geography in Amsterdam 1938.

In 1949 I had an opportunity to make a new series of observations with two ships at the same place, and it is the results of all these observations which will be presented here.

Before going into the treatment of the observations we shall give som theoretical considerations on internal waves.

2. Influence of compressibility on internal waves. In the previous paper "Interne Wellen" [4] it was assumed that the water could be treated as incompressible, but in most oceanographical research the influence of the compressibility is taken into account, and we shall in this chapter investigate what modifications of the theory are necessary when the compressibility is taken into account.

The equations of motion are unaltered, and since we are here concerned with long waves, the vertical acceleration may be neglected. Consequently the equations of motion may be written in the form:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \lambda v + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0\,, \\ &\frac{\partial v}{\partial t} + \lambda u + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0\,, \\ &g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0\,. \end{aligned} \tag{2,1}$$

The equation of continuity is

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{dx}+\frac{\partial v}{dy}+\frac{\partial w}{dz}\right)=0\,. \tag{2,2}$$

To these equations we must add an equation giving the influence of pressure on the density. This equation may be given in the form:

$$\frac{dp}{d\rho}=\kappa^2\,,$$

where κ is the velocity of sound. More explicitely we write

$$\kappa^2\left(\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right)=\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z} \tag{2.3}$$

and we assume that κ may be regarded as a function of the depth only.

To simplify the equations we put

$$w=\frac{d\zeta}{dt}\,,\quad \rho=\rho_0(z)+\rho_1(x,y,z,t)$$

$$p=p_0+g\int_z^h\rho_0 dz+p_1$$

where ρ_1 and p_1 are assumed to be small quantities. The equations then take the form

$$\frac{\partial u}{\partial t}-\lambda v+\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}=0\,,$$

$$\frac{\partial v}{\partial t}+\lambda u+\frac{1}{\rho_0}\frac{\partial p_1}{\partial y}=0\,,$$

$$g\rho_1+\quad\frac{\partial p_1}{\partial z}=0\,, \tag{2,4}$$

$$\frac{\partial\rho_1}{\partial t}+\frac{\partial\zeta}{\partial t}\cdot\frac{d\rho_0}{dz}+\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial^2\zeta}{\partial z\partial t}\right)=0\,,$$

$$\kappa^2\left(\frac{\partial\rho_1}{\partial t}+\frac{\partial\zeta}{\partial t}\frac{d\rho_0}{dz}\right)=\frac{\partial p_1}{\partial t}-g\rho_0\frac{\partial\zeta}{\partial t}.$$

When the depth is constant it will be possible to isolate a factor depending on z only. We therefore assume

$$u\;=U(x,y,t)cF(z)\,,$$
$$v\;=V(x,y,t)cF(z)\,,$$
$$p_1=Z(x,y,t)c^2\rho_0F(z)\,,$$
$$\zeta\;=Zw(z)\,,$$
$$\rho_1=Z\phi(z)\,.$$

Dropping the common factor cF, the two first equations (2,4) take the form

$$\frac{\partial U}{\partial t} - \lambda V + c\frac{\partial Z}{\partial x} = 0\,,$$
$$\frac{\partial V}{\partial t} + \lambda U + c\frac{\partial Z}{\partial y} = 0\,, \tag{2,5}$$

and to these equations we add the following, which is suggested by the form of the equation of continuity

$$\frac{\partial Z}{\partial t} + c\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0\,. \tag{2,6}$$

In the equation of continuity all terms will then contain the factor

$$\frac{\partial Z}{\partial t}\,.$$

Dropping this factor, the equation takes the form

$$\phi + w\frac{d\rho_0}{dz} - \rho_0 F + \rho_0\frac{dw}{dz} = 0 \tag{2,7}$$

and the physical equation is then

$$\kappa^2\left(\phi + w\frac{d\rho_0}{dz}\right) - c^2\rho_0 F + g\rho_0 w = 0\,. \tag{2,8}$$

The equations (2,7) and (2,8) may be solved with respect to ϕ and F giving

$$F = \frac{\kappa^2\dfrac{dw}{dz} - gw}{\kappa^2 - c^2}\,,$$

and

$$\phi + w\frac{d\rho_0}{dz} = \rho_0 \cdot \frac{c^2\dfrac{dw}{dz} - gw}{\kappa^2 - c^2}\,.$$

The remaining equation

$$g\rho_1 + \frac{\partial p_1}{dz} = 0$$

is in the same manner reduced to

$$g\phi + c^2\frac{d}{dz}(\rho_0 F) = 0\,.$$

or introducing the values above

$$c^2\frac{d}{dz}\left(\rho_0\frac{\kappa^2\dfrac{dw}{dz} - gw}{\kappa^2 - c^2}\right) + g\rho_0\frac{c^2\dfrac{dw}{dz} - gw}{\kappa^2 - c^2} - gw\frac{d\rho_0}{dz} = 0\,.$$

This equation may be written in the form

$$\frac{d}{dz}\left(\frac{\rho_0}{1-\frac{c^2}{\kappa^2}}\frac{dw}{dz}\right)-\frac{g}{c^2}\left[\frac{d}{dz}\left(\frac{\rho_0}{1-\frac{c^2}{\kappa^2}}\right)+\frac{g}{\kappa^2}\frac{\rho_0}{1-\frac{c^2}{\kappa^2}}\right]w=0\,. \tag{2.9}$$

The surface condition is

$$\int_{h+\xi}^{h}\rho_0 dz+p_1=0\,,$$

or

$$-g\rho_0\zeta+p_1=0$$

which again reduces to

$$gw-\frac{c^2}{\kappa^2-c^2}\left(\kappa^2\frac{dw}{dz}-gw\right)=0$$

giving

$$c^2\frac{dw}{dz}-gw=0\;;\quad z=h\,.$$

The surface condition is thus unchanged. The bottom condition is

$$w=0\;;\quad z=0\,.$$

In the equation (2,9) c/κ is the ratio of the wave velocity to the velocity of sound, and this ratio is extremely small and may safely be dropped. The equation is finally

$$\frac{d}{dz}\left(\rho_0\frac{dw}{dz}\right)-\frac{g}{c^2}\left(\frac{d\rho_0}{dz}+\frac{g}{\kappa^2}\rho_0\right)w=0\,. \tag{2,10}$$

The density is now a function of temperature, salinity and pressure.

Salinity and temperature are again assumed to be functions of the depth. Accordingly we may write

$$\frac{d\rho_0}{dz}=\frac{\partial\rho_0}{\partial z}+\frac{\partial\rho_0}{\partial p}\frac{\partial p}{\partial z}=\frac{\partial\rho_0}{\partial z}-\frac{g\rho_0}{\kappa^2}$$

and consequently

$$\frac{d\rho_0}{dz}+\frac{g\rho_0}{\kappa^2}=\frac{\partial\rho_0}{\partial z}\,.$$

If we put

$$\rho_0=1+10^{-3}\sigma_{t,p}$$

and

$$\rho_0{}^1=1+10^{-3}\sigma_t$$

we have

$$\frac{\partial\rho_0}{\partial z}=10^{-3}\frac{\partial\sigma_t}{\partial z}$$

The only difference between the equation (2,10) and the corresponding equation in "Interne Wellen" is that the density in the first term of (2,10) is the density in situ while the equation in "Interne Wellen" only used the density $1+10^{-3}\sigma_t$ in both terms. When the depth is not too great it will not be necessary to take the influence of compressibility into account.

3. Different solutions of the wave equations. The equations:

$$\frac{\partial U}{\partial t}-\lambda V+c\frac{\partial Z}{\partial x} \qquad =0\,,$$

$$\frac{\partial V}{\partial t}+\lambda U+c\frac{\partial Z}{\partial y} \qquad =0\,,$$

$$\frac{\partial Z}{\partial t}+c\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)=0\,,$$

may be solved by writing

$$U=\lambda\frac{\partial \phi}{\partial t}+c^2\frac{\partial^2 \phi}{\partial x \partial y}\,,$$

$$V=\frac{\partial^2 \phi}{\partial t^2}-c^2\frac{\partial^2 \phi}{\partial x^2}\,, \qquad (3,1)$$

$$Z=-c\lambda\frac{\partial \phi}{\partial x}-c\frac{\partial^2 \phi}{\partial y \partial t}\,,$$

where ϕ is a solution of the partial differential equation:

$$\frac{\partial^2 \phi}{\partial t^2}+\lambda^2\phi-c^2\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=0\,. \qquad (3,2)$$

In an infinite uniform straight canal we get a solution of the Kelvin type by taking

$$\phi=Ae^{-\lambda y/c}\sin\sigma\left(t-\frac{x}{c}\right)\,.$$

Then

$$U=2\sigma\lambda Ae^{-\lambda y/c}\cos\sigma\left(t-\frac{x}{c}\right)$$

$$V=0$$

$$Z=2c\lambda Ae^{-\lambda y/c}\cos\sigma\left(t-\frac{x}{c}\right)$$

The complete solution is then with

$$2\sigma\lambda A = C\,.$$

$$u = Ce^{-\lambda y/c}\cos\sigma\left(t-\frac{x}{c}\right)c\frac{dw}{dz}$$

$$v = 0 \tag{3,3}$$

$$\zeta = Ce^{-\lambda y/c}\cos\sigma\left(t-\frac{x}{c}\right)w(z)\,.$$

For the internal waves the velocity of propagation c is very small and consequently the exponential factor $e^{-\lambda y/c}$ will be of greater influence than the corresponding factor for ordinary tidal waves. The amplitude of the internal wave will tend rapidly to zero with increasing value of y. In the open sea far from the coast waves of this type are impossible.

If we disregard the limitation of the sea altogether we may take

$$\phi = A\sin(\sigma t - kx)$$

where σ and k are connected by the equation

$$\sigma^2 - \lambda^2 = c^2k^2$$

the velocity of propagation $\sigma/k = \kappa$ is then given by

$$\kappa = \frac{c}{\sqrt{1-\dfrac{\lambda^2}{\sigma^2}}}$$

and we find

$$U = \sigma\lambda A\cos(\sigma t - kx)$$
$$V = -\lambda^2 A\sin(\sigma t - kx)$$
$$Z = c\lambda k A\cos(\sigma t - kx)$$

The complete solution is then with $\sigma\lambda A = C$

$$u = C\cos(\sigma t - kx)c\frac{dw}{dz}\,,$$

$$v = -\frac{\lambda}{\sigma}C\sin(\sigma t - kx)c\frac{dw}{dz}\,, \tag{3,4}$$

$$\zeta = \sqrt{1-\frac{\lambda^2}{\sigma^2}}C\cos(\sigma t - kx)w(z)\,.$$

In a very broad channel we may also have a solution of another type. If we take

$$\phi = A\sin(\sigma t - kx)\sin\gamma y$$

we have

$$\sigma^2 - \lambda^2 = c^2(k^2+\gamma^2)$$

and

$$U = A(\sigma\lambda \sin\gamma y - c^2 k\gamma \cos\gamma y)\cos(\sigma t - kx)$$
$$V = -A(\lambda^2 + c^2\gamma^2)\sin\gamma y \sin(\sigma t - kx)$$
$$Z = A(c\lambda k \sin\gamma y - c\sigma\gamma\cos\gamma y)\cos(\sigma t - kx)$$

or if we take

$$c^2 k\gamma A = -C$$

$$u = C\left(\cos\gamma y - \frac{\sigma\lambda}{c^2 k\gamma}\sin\gamma y\right)\cos(\sigma t - kx)c\frac{dw}{dz},$$
$$v = C\frac{\lambda^2 + c^2\gamma^2}{c^2 k\gamma}\sin\gamma y \sin(\sigma t - kx)c\frac{dw}{dz}, \qquad (3,5)$$
$$\zeta = \frac{\sigma}{ck}C\left(\cos\gamma y - \frac{\lambda k}{\sigma\gamma}\sin\gamma y\right)\cos(\sigma t - k\lambda)w(z).$$

The formulae give

$$v = 0 \quad \text{for} \quad y = 0$$

and $\gamma y = m\pi$, where m is an integer.

The condition of possibility of these waves is

$$\sigma^2 - \lambda^2 - c^2\gamma^2 > 0$$

and a fortiori

$$\sigma^2 - \lambda^2 > 0.$$

It may however be remarked that if we take the influence of eddy viscosity into account it may be possible to get wave propagation even if this condition is not fulfilled.

4. Energy of internal waves. The potensial energy of an ordinary tidal wave is given by

$$E_p = g\rho \int_h^{h+z} (z - h)dz = \frac{1}{2}g\rho\zeta^2.$$

In a two layer ocean the corresponding expressions will be

$$E_p = \frac{1}{2}g(\rho_1 - \rho_2)\zeta_1^2 + \frac{1}{2}g\rho_2\zeta_2^2$$

where ζ_1 is the amplitude of the wave at the internal boundary and ζ_2 the amplitude at the surface.

In an n-layer ocean the corresponding value will be

$$E_p = \frac{1}{2}g\Sigma(\rho_i - \rho_{i+1})\zeta_i^2 + \frac{1}{2}g\rho_n\zeta_n^2.$$

If the layers have all the same thickness Δz we may write

$$E_p = \frac{1}{2} g \Sigma \frac{\rho_i - \rho_{i+1}}{\Delta z} \Delta z \zeta_i^2 + \frac{1}{2} g \rho_n \zeta_n^2 .$$

For an ocean with continuously varying density we then get the expression

$$E_p = -\frac{1}{2} g \int_0^h \frac{d\rho}{dz} \zeta^2 dz + \frac{1}{2} g \rho_h \zeta_h^2 .$$

Introducing

$$\zeta = Zw(z)$$

we finally get the expression for the potensial energy of the internal wave

$$E_p = \frac{1}{2} g Z^2 \left[\rho_h w_h^2 - \int_0^h \frac{d\rho}{dz} w^2 dz \right] .$$

The kinetic energy is expressed by the integral

$$E_k = \frac{1}{2} \int_0^h \rho (u^2 + v^2) dz$$

or if we introduce

$$u = Uc \frac{dw}{dz} , \quad v = Vc \frac{dw}{dz} ,$$

we get

$$E_k = \frac{1}{2} (U^2 + V^2) \int_0^h c^2 \rho \left(\frac{dw}{dz} \right)^2 dz$$

Remembering that w is an integral of the equation

$$c^2 \frac{d}{dz} \left(\rho \frac{dw}{dz} \right) - g \frac{d\rho}{dz} w = 0$$

we get

$$c^2 \int_0^h w \frac{d}{dz} \left(\rho \frac{dw}{dz} \right) dz = g \int_0^h \frac{d\rho}{dz} w^2 dz$$

or integrating the first integral by parts, we get

$$c^2 \rho \left(w \frac{dw}{dz} \right)_h - c^2 \int_0^h \rho \left(\frac{dw}{dz} \right)^2 dz = g \int_0^h \frac{d\rho}{dz} w^2 dz .$$

The boundary conditions

$$w = 0 \ ; \quad z = 0$$

$$c^2 \frac{dw}{dz} - gw = 0 \ ; \quad z = h ,$$

then give

$$c^2 \int_0^h \rho \left(\frac{dw}{dz}\right)^2 dz = g(\rho w^2)_h - g \int_0^h \frac{d\rho}{dz} w^2 dz .$$

The two integrals involved in the energy expressions are thus equal.

The eigen functions w and $c(dw/dz)$ may be normalised for instance by the condition

$$c^2 \int_0^h \rho \left(\frac{dw}{dz}\right)^2 dz = 1$$

or a similar condition.

For the functions tabulated as a result of the numerical integrations, we have chosen the value

$$10^3 .$$

This was convenient because the numerical integrations were carried out with a unit of depth equal to 10 m.

For waves of the Kelvin type we get

$$E_\rho = E_k$$

For the other types of waves we get.

$$E_\rho < E_k .$$

5. Influence of eddy viscosity on internal waves. For simplicity we only treat the case when the influence of the Earth's rotation is disregarded.

The linearized equations are then

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial}{dz}\left(\eta \frac{\partial u}{\partial z}\right) \\ &g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \\ &\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \quad = 0 \\ &\frac{\partial \rho}{\partial t} + w\frac{\partial \rho}{dz} \quad = 0 . \end{aligned} \qquad (5,1)$$

As before we put

$$\rho = \rho_0(z) + \rho_1 ,$$

$$p = p_0 + g \int_z^h \rho_0 dz + p_1 ,$$

$$w = \frac{d\zeta}{dt} .$$

The equations may then be written in the form

$$\begin{aligned}
\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} &= \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right), \\
\frac{\partial u}{\partial x}+\frac{\partial^2 \zeta}{\partial z \partial t} &= 0, \\
g\rho_1+\frac{\partial p_1}{\partial z} &= 0, \qquad (5,1) \\
\frac{\partial \rho_1}{\partial t}+\frac{\partial \zeta}{\partial t}\frac{d\rho_0}{dz} &= 0.
\end{aligned}$$

When the depth is constant we may try to isolate a factor containing only z, and consequently we assume

$$\begin{aligned}
u &= U(x_1 t)c\frac{dw}{dz} \\
p_1 &= Z(x_1 t)c^2\rho_0\frac{dw}{dz} \\
\zeta &= Zw(z) \qquad (5,3) \\
\rho_1 &= -\frac{d\rho_0}{dz}Zw(z).
\end{aligned}$$

The equations may then be written

$$\begin{aligned}
\left(\frac{\partial U}{\partial t}+c\frac{\partial Z}{\partial x}\right)\frac{dw}{dz} &= U\frac{1}{\rho_0}\frac{d}{dz}\left(\eta\frac{d^2 w}{dz^2}\right) \\
\left(c\frac{\partial U}{\partial x}+\frac{\partial Z}{\partial t}\right)\frac{dw}{dz} &= 0 \qquad (5,4) \\
gZw\frac{d\rho_0}{dz} &= c^2 Z\frac{d}{dz}\left(\rho_0\frac{dw}{dz}\right).
\end{aligned}$$

It will then be necessary that

$$\frac{1}{\rho_0}\frac{d}{dz}\left(\eta\frac{d^2 w}{dz^2}\right) = -f\frac{dw}{dz}$$

where f is a constant,
or

$$\frac{d}{dz}\left(\eta\frac{d^2 w}{dz^2}\right)+f\rho_0\frac{dw}{dz}=0 \qquad (5,5)$$

and

$$\frac{d}{dz}\left(\rho_0\frac{dw}{dz}\right)-\frac{g}{c^2}\frac{d\rho_0}{dz}w=0. \qquad (5,6)$$

The remaining equations then take the form

$$\frac{\partial U}{\partial t} + fU + c\frac{\partial Z}{\partial x} = 0,$$

$$\frac{\partial Z}{\partial t} + c\frac{\partial U}{\partial x} \qquad = 0.$$

It will now be shown that the equations (5,5) and (5,6) are equivalent if

$$\eta = -\frac{fc^2 \rho_0^2}{g\frac{d\rho_0}{dz}}. \tag{5,7}$$

Equation (5,6) may be written

$$\frac{\rho_0}{\frac{d\rho}{dz}} \frac{d^2 w}{dz^2} + \frac{dw}{dz} - \frac{g}{c^2} w = 0.$$

If we put

$$\frac{\rho_0}{\frac{d\rho_0}{dz}} = -\frac{g\eta}{fc^2 \rho}$$

the equation will be

$$\frac{-g}{fc^2 \rho_0} \eta \frac{d^2 w}{dz^2} + \frac{dw}{dz} - \frac{g}{c^2} w = 0.$$

If we differentiate with respect to z, we get

$$-\frac{g}{fc^2 \rho_0} \cdot \frac{d}{dz}\left(\eta \frac{d^2 w}{dz^2}\right) + \frac{g}{fc^2 \rho_0^2} \frac{d\rho}{dz} \eta \frac{d^2 w}{dz^2} + \frac{d^2 w}{dz^2} - \frac{g}{c^2} \frac{dw}{dz} = 0.$$

But

$$\frac{g}{fc^2 \rho_0^2} \frac{d\rho}{dz} \eta + 1 = 0$$

and the equation reduces to

$$\frac{d}{dz}\left(\eta \frac{d^2 w}{dz^2}\right) + f\rho_0 \frac{dw}{dz} = 0$$

which is equation (5,5).

With this solution it will not be possible to satisfy the condition $u = 0$ at the bottom, but this is not so serious, because the bottom friction will probably be small in comparison to the friction caused by the great shear in the upper layers where the velocity changes rapidly with depth.

The expression (5,7) for the eddy viscosity gives as a result that it is inversely proportional to the stability, and this is in good accordance with experience.

In greater depth where the stability is small the expression (5,7) will probably give too large values of the eddy viscosity, but this will to some degree be compensated by allowing slipping motion at the bottom.

It might be mentioned that W. WERENSKIOLD [12] has deduced a similar formula from quite different assumptions.

In order that the coefficient of eddy viscosity shall be independent of the order of the internal wave, it is necessary that

$$fc^2 = \text{constant}.$$

This means that the friction coefficient f will increase rapidly with the order of the wave, and this must be expected since the velocity will change more rapidly with depth, and thus the shear will be greater. If we put

$$u, \zeta \sim e^{i(\sigma t - kx)}$$

we get

$$(i\sigma + f)U - cikZ = 0$$

$$-cikU \quad + i\sigma Z = 0$$

giving

$$-\sigma^2 + i\sigma f + c^2k^2 = 0$$

and if we put

$$k = \frac{\sigma}{\kappa} - i\mu$$

we find

$$\frac{\sigma^2}{\kappa^2} - \mu^2 = \frac{\sigma^2}{c^2}$$

$$2\mu\frac{\sigma}{\kappa} = \frac{\sigma f}{c^2}.$$

An approximate solution is then

$$\kappa \sim c,$$

$$\mu \sim \frac{f}{2c},$$

but more exact values may be found by solving the equations. We then get damped waves with the exponential factor

$$e^{-fx/2c}.$$

A more complete treatment of this problem is given in the appendix.

6. Numerical Integration Methods. The density ρ is determined by observations at selected depths and the coefficients of the differential equation are given in tabular form only. Hence the integration has to be performed by numerical methods. The horisontal velocity is given by

$$u = c\frac{dw}{dz}$$

and consequently it will be necessary to calculate both w and dw/dz. Introducing $u = c(dw/dz)$ into the differential equation, we have

$$c\frac{d}{dz}(\rho u) = g\left[\frac{d}{dz}(\rho w) - \rho\frac{dw}{dz}\right]$$

or

$$c\frac{d}{dz}\left[\rho u - \frac{g}{c}\rho w\right] = -g\rho\frac{dw}{dz}.$$

Integrating between z and h, we get

$$c\left(\rho u - \frac{g}{c}\rho w\right)_z^h = -\frac{g}{c}\int_z^h \rho u dz.$$

or since $cu - gw = 0$ at the upper limit, it follows that

$$c\rho u = g\rho w + \frac{g}{c}\cdot\int_z^h \rho u dz.$$

Here we may write

$$w = \frac{1}{c}\int_0^z u dz$$

and finally we get the integral equation

$$\rho u = \frac{g}{c^2}\left[\rho\int_0^z u dz + \int_z^h \rho u dz\right]. \qquad (6,1)$$

This equation may also be used when the density is discontinuous.

If ρ is constant we may take u = constant and the equation gives simply

$$1 = \frac{gh}{c^2}$$

which is the well known formula of Lagrange.

In a two layer ocean with density ρ_1 and velocity u_1 in the bottom layer and similarly ρ_2 and u_2 in the top layer, we get one equation if $z < h_1$

$$\rho_1\mu_1 = \frac{gh_1}{c^2}\rho_1\mu_1 + \frac{gh_2}{c^2}\rho_2 u_2$$

and if $z > h_1$ we get

$$\rho_2 u_2 = \frac{gh_1}{c^2}\rho_2 u_1 + \frac{gh_2}{c^2}\rho_2 u_2$$

or

$$\left(1-\frac{gh_1}{c^2}\right)u_1-\frac{gh_2}{c^2}u_2\frac{\rho_2}{\rho_1}=0$$

$$-\frac{gh_1}{c^2}u_1+\left(1-\frac{gh_2}{c^2}\right)u_2=0$$

giving

$$c^4-c^2(gh_1+gh_2)+gh_1\cdot gh_2\frac{\rho_1-\rho_2}{\rho_1}=0$$

which gives the velocity of the boundary wave.

The integral equation may easily be transformed to an equation with symmetrical kernel. If we put $u\sqrt{\rho}=\phi$, the equation may be written in the form

$$\phi=\mu\int_0^h K(z,s)\phi(s)ds$$

where

$$\mu=\frac{g}{c^2}$$

and

$$K(z,s)=\sqrt{\frac{\rho(z)}{\rho(s)}}\,;\quad s\leq z$$

$$=\sqrt{\frac{\rho(s)}{\rho(z)}}\,;\quad s>z\,.$$

From the general theory of integral equations with symmetrical kernel, we know that the kernel may be expressed by the bilinear expansion

$$K(z,s)=\Sigma\frac{\phi_n(z)\ \phi_n(s)}{\mu_n}\,,$$

where μ_n are the eigen values and the functions ϕ_n form a normal orthogonal system. If we put $z=s$ and integrate between 0 and h we get

$$h=\Sigma\frac{1}{\mu_n}$$

or since

$$\mu_n=\frac{g}{c_n^{\ 2}}\,,$$

$$gh=\Sigma c_n^{\ 2}\,. \qquad (6,2)$$

This may be regarded as a generalization of Lagrange's formula.

For the actual calculation of u and w the integral equation is not convenient.

The integration is best performed by a step by step method. One such method has been given in "Interne Wellen" and we shall here give details of two such methods.

Ordinarily it will be convenient to introduce other units in the equation, mostly I have used $x = 10^{-3}z$ as a unit of depth, and with $\rho = 1 + 10^{-3}\sigma_t$ the equation is then

$$\frac{d}{dx}\left(\rho\frac{dw}{dx}\right) = \frac{g}{c^2}w \cdot \frac{d\sigma_t}{dx}.$$

We also denote dw/dx by u, since it only differs from the function u defined above by a numerical factor which can be restored after the integration.

We divide the interval of integration in equal steps and denote the abscissae by

$$x_1 = \tilde{\omega} \ ; x_2 = 2\tilde{\omega}, \ldots \qquad \text{etc.}$$

If we integrate the equation between x_{n-1} and x_{n+1}, we get

$$(\rho u)_{n+1} - (\rho u)_{n-1} = \frac{g}{c^2}(\sigma w)_{n+1} - (\sigma w)_{n-1} - \int_{x_{n-1}}^{x_{n+1}} \sigma u dx$$

and at the same time we have

$$w_{n+1} - w_{n-1} = \int_{x_{n-1}}^{x_{r+1}} u dx .$$

From the two equations we may eliminate w_{n+1}, giving

$$(\rho u)_{n+1} - (\rho u)_{n-1} = \frac{g}{c^2}\left[(\sigma_{n+1} - \sigma_{n-1})w_{n-1} + \int_{x_{n-1}}^{x_{n+1}} (\sigma_{n+1} - \sigma)u dx\right].$$

Expressing the last term by Simpson's formula, the equations will be

$$(\rho u)_{n+1} - (\rho u)_{n-1} = \frac{g}{c^2}\left[(\sigma_{n+1} - \sigma_{n-1})\left(w_{n-1} + \frac{\tilde{\omega}}{3}u_{n-1}\right) + (\sigma_{n+1} - \sigma_n)\frac{4\tilde{\omega}}{3}u_n\right] \qquad (6,3)$$

$$w_{n+1} - w_{n-1} = \frac{\tilde{\omega}}{3}[u_{n+1} + 4u_n + u_{n-1}] . \qquad (6,4)$$

By means of these two formulae the step by step integration may be performed, if only the two first values are known.

Starting from the surface, we may chose an arbitrary value of u_0, and w_0 is so chosen that the surface condition is satisfied. With the unit used for the integration, this will be

$$u_0 - 10^3\frac{g}{c^2}w_0 = 0 .$$

w_1 and u_1 may then be calculated by Taylors formula, and remembering that

$$\frac{d^2w}{dz^2} = 0$$

at the surface, we may chose the interval so small that sufficient accuracy is obtained when we stop with the terms of third order. The formulae then takes the form

$$w_1 = w_0 + u_0\tilde{\omega}\left[1 + \frac{g}{c^2}\frac{\rho_1 - \rho_0}{6\rho_1}\tilde{\omega}\right]$$

$$u_1 = u_0\left(1 + \frac{g}{c^2}\frac{\rho_1 - \rho_0}{2\rho_1}\tilde{\omega}\right).$$

Since we start from the surface, the increment $\tilde{\omega}$ is taken negative.

The integration formulae supposes that the parameter g/c^2 is known. This is not the case, and we have to perform the numerical integration with an assumed value of the parameter, and surface values of u and w, which satisfy the surface condition. If the bottom condition is also satisfied, the parameter value is correct. If not, we have to chose another value and make a new integration. If the bottom condition is still violated, a linear interpolation will ordinarily give a nearly correct value of the parameter.

To find a preliminary value of g/c^2 we compute the integral

$$a = \int_0^h \sqrt{-\frac{1}{\rho}\frac{d\rho}{dz}}\, dz$$

and approximate value of g/c^2 is then given by

$$\frac{g}{c^2} = \left(\frac{m\pi}{a}\right)^2$$

where m is an integer.

Another method of numerical integration is set forth in the paper mentioned above. (3)

Let us write the equation in the form

$$\frac{d^2w}{dz^2} = \frac{1}{\rho}\frac{d\rho}{dz}\left[\frac{g}{c^2}w - \frac{dw}{dz}\right]$$

let

$$\xi = \frac{d^2w}{dz^2} \cdot \tilde{\omega}^2 = \frac{\tilde{\omega}^2}{\rho}\frac{d\rho}{dz}\left[\frac{g}{c^2}w - \frac{dw}{dz}\right]$$

and

$$\overline{w} = w - \frac{1}{12}\xi$$

then

$$\Delta^2 \overline{w}_{n-1} = \xi_n$$

which is a very convenient formula for the integration process.

The equation may also be solved by means of a differential analyzer.

Theoretically we have an infinite number of solutions corresponding to waves of different velocities of propagation, but in practice a finite set of eigenfunctions and eigenvalues will be sufficient. As will be seen 5 such solutions are sufficient to express the observed oscillations.

7. Detection and analysis of internal waves. The marked discontinuity of density which is supposed in the theory of boundary waves are practically never found in the open sea, but near the coast one may sometimes find variations of density which approximates more or less to the ideal discontinuity.

The motion of the boundary layer may then be registered by means of a buoy, whose weight is adjusted so as to keep floating in the discontinuity surface. The motion of the buoy may then be registered by a pressure gauge.

Such methods have been applied especially by Hans Petterson in Kattegat and Gullmarefiord, but when there is a more gradual change of the density the only method is to take regularly repeated observations of temperature and salinity. Experience shows that when the density increases continously with depth there will generally be a number of internal waves present, and to get a sufficient picture of the motion it is necessary to have observations from as many depths as possible.

The vertical oscillations may then be detected by observing any conservative property of the sea water. Let S be such a property. Then

$$\frac{dS}{dt}=0$$

or if the change of S in a horizontal direction is small, the equation is approximately

$$\frac{\partial S}{\partial t}+w\frac{\partial S}{\partial z}=0$$

or if we put

$$S=S_0(z)+S_1\,,\quad w=\frac{d\zeta}{dt}\,,$$

The vertical alevation of a water particle, ζ, from its equilibrium position may then be determined by making regular observations of S.

S_0 is then determined by taking means over one period and S_1 is the deviation of the single measurement from the mean. ζ may then be calculated, either from the single observations or by harmonic analysis of the S_1 values, and then calculate the corresponding harmonic constants of ζ.

When we have regular measurements of temperature and salinity, we may compute the corresponding value of the density which may then be harmonically analysed.

Suppose that we have found

$$\rho_1=A_\rho\cos\sigma t+B_\rho\sin\sigma t\,,\quad \sigma=\frac{2\pi}{T}$$

The harmonic constants A_ρ and B_ρ are functions of the depth and we may express them by the series

$$A_\rho=-\frac{d\rho_0}{dz}\Sigma a_n w_n(z)$$

$$B_\rho=-\frac{d\rho_0}{dz}\Sigma b_n w_n(z)$$

and

$$\zeta_A = \Sigma a_n w_n , \quad \zeta_B = \Sigma b_n w_n .$$

If A_ρ and B_ρ were known as functions of z, from the surface to the bottom, the determination of the coefficients might be done by the Fourier method since the eigenfunctions w_n have orthogonal properties expressed by

$$(\rho w_m w_n)_h - \int_0^h \frac{d\rho}{dz} w_n w_m dz = 0 \ ; \quad n \neq m .$$

The first term is ordinarily negligible. Multiplying the series by w_m and integrating, we get

$$\int_0^h A_\rho w_m dz = -\Sigma a_n \int_0^h \frac{d\rho}{dz} w_n w_m dz ,$$

$$\rho w_m \zeta_h = \Sigma a_n (\rho w_n w_m)_h ,$$

and adding, and taking the condition of orthogonality into account, we get

$$\int_0^h A_\rho w_m dz + (\rho w_m \zeta)_h = a_m \left[(\rho w_m{}^2)_h - \int_0^h \frac{d\rho}{dz} w_m{}^2 dz \right]$$

from which the coefficient a_m may be determined.

In the practical computation the eigenfunctions have been normalised by the condition

$$c_n{}^2 \int_0^h \rho \left(\frac{dw_n}{dz} \right)^2 dz = g(\rho w_n{}^2)_h - g \int_0^h \frac{d\rho}{dz} w_n{}^2 dz = 10^3 ,$$

or

$$a_m = 10^{-3} g \cdot \left[\int_0^h A_\rho w_m dz + (\rho w_m \zeta)_n \right] .$$

The coefficients b_m are determined similarly.

The theory supposes that the repeated observations have been done at a constant value of z but in reality the observations have been taken at a constant depth below the surface which itself exhibits tidal variations.

Let the direct observed value of the density be ρ' which is

$$\rho' = \rho(z + \zeta_h) = \rho(z) + \frac{d\rho}{dz} \cdot \zeta_h .$$

Let

$$\rho' = \rho_0 + \rho_1' , \quad \rho = \rho_0 + \rho_1 ,$$

then

$$\rho_1' = \rho_1 + \frac{d\rho_0}{dz} \zeta_h ,$$

and

$$\int_0^h \rho_1' w_m dz = \int_0^h \rho_1 w_m dz + \zeta_h \int_0^h \frac{d\rho_0}{dz} w_m dz$$

$$= \int_0^h \rho_1 w_m dz + (\zeta \rho w_m)_h - \zeta_h \frac{c^2}{g}\left(\rho \frac{dw}{dz}\right)_0$$

or

$$10^{-3} g \int_0^h \rho_1' w_m dz = a_m - \zeta_h \cdot 10^{-3} c (\rho u)_0 .$$

The last term is ordinarily small and may be neglected.

The coefficients a_m may then be determined by the direct observed values of the density.

If the surface elevation is known, a correction may be applied, but this will seldom be possible.

Generally the observations will be restricted to a small number of depths and then the general Fourier method can not be applied. The only possibility is then to try to represent the observations by a finite number of eigenfunctions and determine the coefficients by the method of least squares.

When it is possible to integrate over an interval which does not cover the whole depth, the eigenfunctions are no longer orthogonal and we then get a set of equations from which the values of the coefficients may be found. This is the method which we are going to use in the following chapters.

The current measurement may be treated in a similar manner.

Let one component of the current be expressed by

$$u = F\cos\sigma t + G\sin\sigma t .$$

F and G are then functions of the depth and may be represented by a series of eigenfunctions

$$F = \Sigma f_n u_n(z) , \quad G = \Sigma g_n u_n(z) .$$

The eigenfunctions form an orthogonal system and have been normalised by the condition

$$\int_0^h \rho u^2 dz = 10^3 .$$

It will here be necessary to introduce also the zero order function, which to a sufficient approximation may be taken to be a constant.

$$F = f_0 + \Sigma f_n u_n , \quad G = g_0 + \Sigma g_n u_n .$$

The coefficients may be found by the method of the least squares, but often we then meet the difficulty that the determinant of the equations is nearly zero.

This may be understood since it will also be possible to represent a discontinuous function by a sufficient number of eigenfunctions.

If we are able to draw curves representing F and G from the surface to the bottom, the Fourier method may be used, and the coefficients are then determined without ambiguity.

If the waves are progressive we should get the same results from the analysis of the vertical elevations of the water layers and from the analysis of the current measurements

$$f_n = a_n\,, \quad g_n = b_n\,.$$

For standing waves the results are more complicated.

8. Treatment of observations. The current measurements were taken at irregular intervals and the single measurements show quite large fluctuations both in direction and velocity. The variations might be characterized as some sort of turbulence but may partly be due to standing lateral oscillations in the fjord.

To eliminate these short period oscillations and get values which are more readily analysed for tidal currents, we have smoothed the curves by taking means of four hours by means of a planimeter. But the currents also include more or less irregular wind currents and currents set up by pressure gradients caused by the distribution of density.

We were mainly interested in the tidal currents, and so the non tidal currents had to be eliminated.

Suppose that the observed current could be represented by a formula of the kind

$$u = A + Bt + Ct^2 + R\cos(\sigma t - \kappa) + \delta\,.$$

where the polynomium represent the non tidal current and δ the random variations.

By taking means over four hours, we get

$$u = A + Bt + C\left(t^2 + \frac{4}{3}\right) + R\frac{\sin 2\sigma}{2\sigma}\cos(\sigma t - \kappa) + \bar{\delta}\,\cdot$$

Since δ represents random variations we may assume that $\bar{\delta}$ may be neglected.

We now form the difference

$$\Delta u = u(t-3) - u(t+3) = -6B - 12Ct$$

$$+\frac{\sin 2\sigma}{2\sigma} 2\sin 3\sigma R \sin(\sigma t - \kappa)\,.$$

Repeating this process, we get

$$\Delta^2 u = 72c - \frac{\sin 2\sigma}{2\sigma} 4\sin^2 3\sigma R\cos(\sigma t - \kappa)\,.$$

In this manner the non cyclic correction is eliminated, and the amplitude has been multiplied by a factor which is easily evaluated.

For diurnal waves it is better to take differences with 12 hours interval. The whole calculation together with the harmonic analysis of the resulting values is easily included in a computation form.

Ordinarily the measured current vectors are decomposed in a North and an East component and the two resulting curves are treated separately.

As a result of the analysis we get two components of current represented by the formulae

$$E: \; u = M\cos\sigma t + N\sin\sigma t$$

$$N: \; v = P\cos\sigma t + Q\sin\sigma t$$

From the coefficients, M, N, P and Q we have to find the axes and the orientation of the current ellipse and also the time when the current attains its maximum value.

The necessary formulae are easily deduced in the following way:

In a coordinate system we imagine two vectors of constant magnitude rotating in opposite direction

$$V_1 = Ae^{i(\alpha+\sigma t)}, \quad V_2 = Be^{i(\beta-\sigma t)} .$$

The resulting vector has the components

$$u = A\cos(\alpha+\sigma t) + B\cos(\beta-\sigma t)$$

$$v = A\sin(\alpha+\sigma t) + B\sin(\beta-\sigma t) .$$

The maximum velocity is obtained when both vectors point in the same direction

$$\alpha + \sigma t = \beta - \sigma t$$

or

$$\sigma t = \frac{\beta-\alpha}{2} = \tau .$$

At this moment the vector forms an angle χ with the x-axis determined by

$$\mathrm{tg}\chi = \left(\frac{v}{u}\right)_{\sigma t = \tau}$$

or

$$\chi = \frac{\beta+\alpha}{2} .$$

The maximum velocity is $A+B$, and the minimum velocity is $A-B$. To find α, β, A and B we have only to identify the coefficients of $\cos\sigma r$ and $\sin\sigma t$ in the equations:

$$A\cos(\alpha+\sigma t) + B\cos(\beta-\sigma t) = M\cos\sigma t + N\sin\sigma t$$

$$A\sin(\alpha+\sigma t) + B\sin(\beta-\sigma t) = P\cos\sigma t + Q\sin\sigma t$$

We then easily deduce the equations

$$A\cos\alpha = \frac{M+Q}{2} \qquad B\cos\beta = \frac{M-Q}{2}$$

$$A\sin\alpha = \frac{P-N}{2} \qquad B\sin\beta = \frac{P+N}{2}$$

or

$$\text{tg}\alpha = \frac{P-N}{M+Q}, \quad \text{tg}\beta = \frac{P+N}{M-Q}$$

$$A = \frac{1}{2}\sqrt{(M+Q)^2+(P-N)^2}\,, \quad B = \frac{1}{2}\sqrt{(M-Q)^2+(P+N)^2}\,.$$

The temperatures and salinities are smoothed and analysed in the same manner, such that the results of the analyses are given by formulae

$$T_1 = R_T \cos(\sigma t - \gamma_T)$$

$$S_1 = R_S \cos(\sigma t - \gamma_S)$$

To obtain phase angles which are more easily compared we have computed the κ-angles

$$\kappa = \gamma + v_0 + u$$

where $v_2 + u$ is the initial phase of the tidal component M_2. $v_0 + u$ is computed by the formulae of DOODSON [2].

CHAPTER II

ANALYSIS OF OBSERVATIONS

9. **Observations in Herdlefjord 1934.** The tide generating forces of the Moon and the Sun can not create internal waves of notisable magnitude. But when the tidal currents are distorted by the bottom configuration, this will set up internal waves such that the superposition of the ordinary tidal wave and the internal waves will give a variation of the tidal currents with depth which is necessary to satisfy the boundary conditions of the current.

When searching for a suitable locality for the measurement of internal waves, we were guided by the following considerations: There should be no discontinuity, but a gradual increase of the density from the surface to the bottom. The waves should be of a comparably simple type, preferably of the Kelvin type. This suggested that the observations should be made in a fjord, and the bottom configuration should be favorable for the formation of internal waves.

In the summer 1933 some preliminary investigations were made in Nordfjord. The ship was anchored at a place where the depth is quite uniform, about 580 m. Determination of temperature and salinity were made hourly at different depths, and at the same time current measurements were taken to determine the tidal currents. The temperature readings proved at once that internal waves of tidal origin were present, but after 36 hours the observations were broken off owing to unfavorable weather conditions.

In July 1934 a new expedition was undertaken and measurements were made in Herdlefjord near Bergen. A map of the fjord is shown on Fig. 1. At the south-easterly end there is free communication with a larger fjord system, while the north-westerly end is nearly closed by the small island Herdla, and the communication to the fjord system outside is only through narrow and comparatively shallow openings. The tidal currents then have free acces at the south easterly end, but in the other end the currents are cut off in the deeper layers. This will favor the formation of internal waves.

The measurements were taken at three different stations. At station I, which is near the northern end of the fjord, the observations were started on July 13th 20^h 40^m. The depth was 231 m. The first three series were taken to find the detailed distribution of temperature and salinity from the surface to the bottom. The first series D comprised the depths 40, 50, 60, 75, 100, 150, and 200 m. Series E at 21^h 20^m was taken at the depths 6, 8, 10, 12, 15 and 20 m. At 21^h 50^m a series F with the depths 0, 1, 3, 5, 7, 9, 11, 13 and 15 m were taken. Then the series 32 to 45 were taken hourly. It was then decided to take temperature observations every half hour, while salinity samples were taken only every hour.

The observations at this station were continued for 80 hours. The temperature and salinity values are given in table I (Part II). A graphical representation of the temperatures is given in Fig. 2 and the salinity in Fig. 3. The points representing the individual observations are connected with straight lines, and the curves have been smoothed by taking means over four hours by means of a planimeter.

In the following table we give the results of the harmonic analysis of the observations.

Table 1.

Depth	R_T °C	κ_T	R_S	κ_S	R_σ	κ_σ
5	0.265	200°	0.376	14°	0.344	14°
10	0.399	227	0.302	48	0.306	50
15	0.354	228	0.281	38	0.272	42
20	0.142	243	0.197	38	0.158	43
30	0.0132	65	0.0784	42	0.057	44
100	0.0145	226	0.0097	14	0.0087	18

The mean values of temperature, salinity and density are given in the following table:

Table 2.

Depth	T_m°C	S_m‰	$\sigma_{t,m}$
5	11.254	30.575	23.310
10	9.546	32.466	25.076
15	8.197	33.331	25.950
20	7.706	33.793	26.399
30	7.665	34.320	26.809
100	7.540	34.799	27.212

We have also made harmonic analyses for single days to see how the conditions change from day to day. The results are given in the following table.

Table 3. *Results for single days.*

St. I, 1934.

Depth	R_T	κ_T	R_S	κ_S	R_σ	κ_σ	Depth	$v$cm/sec.	κ
	.245	213°	.382	16°	.344	20°		10.4	265°
5 m	.241	219°	.236	31°	.238	34°	5 m	5.8	228°
	.298	167°	.511	356°	.450	359°		9.9	233°
	.448	239°	.365	59°	.361	58°			
10 m	.438	229°	.293	46°	.313	46°	10 m	10.2	244°
	.312	213°	.247	41°	.243	41°		12.1	239°
	.401	238°	.360	42°	.354	50°		8,1	215°
15 m	.358	253°	.239	51°	.219	57°	15 m	7.8	276°
	.304	195°	.244	19°	.242	17°		6.0	215°
	.189	244°	.254	39°	.215	42°		4.0	264°
20 m	.161	257°	.192	58°	.148	62°	35 m	3.5	255°
	.075	225°	.144	17°	.111	19°		3.0	212°
	.0146	44°	.0966	48°	.0709	49°		0.91	212°
30 m	.0150	96°	.0753	50°	.0518	50°	50 m	.46	223°
	.0099	54°	0.632	29°	.0481	31°		1.01	221°
	.0130	236°	.0118	27°	.0099	31°			
100 m	.0155	248°	.0084	17°	.0077	22°	100 m	1.20	121°
	.0150	194°	.0089	358°	.0085	0°			

Results of current measurements 1934.

Current measurements were made at the following depths, surface, 5, 10, 15, 35. 50 and 100 m. For the surface observations and the measurements at 35 m depth electric current meters of Sverdrup and Dahl's construction were used. All other current measurements were made with Ekman current meters. The single measurements show irregular variations and the shots are mostly scattered in many compartments in the compass box,

The main direction of the fjord is SE-NW. We have chosen the direction towards SE as the positive direction since it seems as if the internal waves are advancing in this direction.

The following table gives the mean values for 3 lunar days.

Table 4.

	v_1	κ	χ
5	8.68	245°	−56°
10	11.11	241°	−45°
15	6.02	241°	−45°
35	3.51	243°	−48°
50	0.88	216°	
100	1.20	121°	

The surface observations were so much disturbed by non-periodic currents that no results could be found for the tidal current.

To give an indication of the stability of the tidal currents we also give the results of the analyses of single days (Table 3). In 100 meter the results are from 1 day only.

Results from station II. 1934.

The observations at this station covered a time interval of 30 hours and fell approximately at neap tide, such that the waves were small and irregular.

In the following table we give the results of the harmonic analyses of temperature, salinity and density.

Table 5.

	R_T	κ_T	R_S	κ_S	R_σ	κ_σ
5	0.1753	226°	0.2687	28°	0.1612	45°
10	0.0682	40°	0.0492	150°	0.0475	141°
15	0.0610	21°	0.0470	193°	0.0420	197°
20	0.0129	162°	0.0835	145°	0.0635	144°
30	0.0117	179°	0.0543	160°	0.0406	159°
50	0.0135	331°	0.0225	171°	0.0197	178°

The mean values of temperature, salinity and density are given in the following table:

Table 6.

	T	S	σ_t
5	11.26	30.973	23.632
10	9.36	32.543	25.177
15	8.06	33.266	25.929
20	7.62	33.813	26.419
30	7.77	34.335	26.801
50	7.46	34.702	27.097

It will be seen from this table that a temperature inversion is found at 20—30 meters depth. This is also evident from the comparison of the phase angles at 10, 15, 20 and 30 m depth.

The current measurements at this station give the following results.

Table 7.

	v	κ
5	5.4	316°
10	5.1	329°
15	2.9	352°
50	0.5	319°

The measurements with the electric current meter at 35 m depth gave no satisfactory results. The only safe conclusion was that the current was very weak.

The measurements at station III covered a time interval of 36 hours.

The analyses of the temperature and salinity measurements gave the following results.

Table 8.

	R_T	κ_T	R_S	κ_S	R_σ	κ_σ
5	0.3085	10°6	0.4599	172°5	0.4235	174°2
10	0.3823	21°7	0.2442	201°6	0.2488	200°5
15	0.1951	14°9	0.1375	183°2	0.1243	181°0
20	0.0260	22°4	0.0784	208°7	0.0640	209°6
30			0.0196	250°0	0.0145	257°0
50						

The currents at this station were very weak. The analyses give the following results.

Table 9.

	v	κ
5	2.8	13°
10	2.8	21°
15	2.1	6°

From the mean distribution of density the theoretical internal waves may be calculated. In the numerical integration we have used a vertical unit of 10 meters, and the eigenfunctions have been adjusted such that

$$\int_0^h \rho u^2 dz = 10^3$$

or with $z = 10^3 x$

$$\int_0^h \rho u^2 dx = 1 .$$

The eigenfunctions are tabulated in tab. 25 and 26 for the 5 first orders of internal waves.

From the density amplitudes the coefficients of the internal waves have been calculated according to the method set out above.

In the present case we have only observations of σ_t for $d=5$, 10, 15, 20, 30 and 100 m. We have made a graph and interpolated values of σ_t, such that the integration from the surface down to 100 m can be carried through, but in this interval the eigenfunctions are not orthogonal, and the integrals

$$\int_{h_1}^h w_n w_m \frac{d\rho}{dz} dz$$

are different from zero. The result is that we get equations which have to be solved with respect to the coefficients a_n and b_n.

In the present case we get the following equations:

Coefficients of normal equations st. I. 1934.

1	2	3	4	5	*A*	*B*
1,01611	−0,01487	−0,01798	0,02300	−0,03148	9,0003	7,0935
−0,01487	0,97360	0,03168	−0,03597	0,07648	−1,6700	−2,4043
−0,01798	0,03168	0,97203	0,06636	−0,08825	1,2419	−0,9420
0,02300	−0,03597	0,06636	0,96289	0,18231	−0,5137	−0,2706
−0,03148	0,07648	−0,08825	0,18231	0,84442	0,7059	0,5830

Solving these equations, we find:

$$\begin{array}{llll}
a_1 = 8,9464 & b_1 = 6,9906 & R_1 = 11,3502 & \kappa_1 = 38^\circ \\
a_2 = -1,8237 & b_2 = -2,4708 & R_2 = 3,0708 & \kappa_2 = 234^\circ \\
a_3 = 1,7524 & b_3 = -0,5932 & R_3 = 1,8501 & \kappa_3 = -19^\circ \\
a_4 = -1,2756 & b_4 = -0,7404 & R_4 = 1,4749 & \kappa_4 = 210^\circ \\
a_5 = 1,7932 & b_5 = 1,2727 & R_5 = 2,1989 & \kappa_5 = 35^\circ
\end{array}$$

With these values of the coefficients we find the amplitudes of ζ and σ_t, which are given in the following table:

Table 10.

	ζ	κ	R_σ	κobs.	$R_\sigma$obs.
5	50.7	19°	0.294	14°	0.344
10	146.1	40°	0.324	50°	0.306
15	209.6	45°	0.258	42°	0.272
20	220.2	46°	0.164	43°	0.158
30	228.3	44°	0.056	44°	0.057
100	514.4	32°	0.004	18°	0.009

In the same manner we may express the observed tidal current by means of the eigenfunctions $u_n(z)$. In this case it seems reasonable to include the zero order wave, i.e. the ordinary tidal current.

Let the current in the main direction of the fjord be

$$u = F\cos\sigma t + G\sin\sigma t$$

and

$$F = f_0 + \Sigma f_n u_n(z)$$

where $u_0(z) = 1$.
The coefficients are then determined by the equations

$$\int_{h_1}^{h} \rho u_i F dz = \Sigma f_n \int_{h_1}^{h} \rho u_i u_n dz .$$

The normal equations obtained are

0	1	2	3	4	5	F	G
10.2646	−1.06452	1.02638	−.97553	.74426	−.69993	14.5653	−20.5970
−1.06452	.91451	.09048	−.08866	.05831	−.06112	4.13866	9.79710
1.02638	.09048	.87055	.05918	−.05491	.07575	−2.71485	−3.58017
−0.97553	−.08866	.05818	.91486	.05076	−.08875	−0.13741	−0.94397
0.74426	.05831	−.05491	.05076	.91188	.09774	−0.28994	0.53971
−0.69993	−.06112	.07575	−.08875	.09774	.84233	0.50054	−0.18950

and the solution of these equations give the following values of the coeﬃcients.

$f_0 = -0.702$ $g_0 = -0.261$ $R_0 = 0.750$ $\kappa_0 = 200°$
$f_1 = 4.005$ $g_1 = 11.005$ $R_1 = 11.716$ $\kappa_1 = 70°$
$f_2 = -2.750$ $g_2 = -5.056$ $R_2 = 5.756$ $\kappa_2 = 241°$
$f_3 = -0.269$ $g_3 = 0.185$ $R_3 = 0.325$ $\kappa_3 = 146°$
$f_4 = -0.210$ $g_4 = -0.306$ $R_4 = 0.312$ $\kappa_4 = 236°$
$f_5 = 0.545$ $g_5 = 0.866$ $R_5 = 1.024$ $\kappa_5 = 58°$

As will be seen only the waves of orders 1 and 2 have significant values. It is also remarkable that the zero order current is very small such that the corresponding phase angle is unreliable.

The waves of first and second order have phase angles which are not very different from the angles which were found for the vertical oscillations. This is an indication that the waves are mainly of a progressive character.

If we compute the currents which correspond to progressive waves, using the coefficients a, b instead of f and g we get the following values:

Table 11.

	V	κ	Vobs	κobs
0	11.7	203°		
5	8.3	228°	8.7	245°
10	7.9	227°	11.1	241°
15	4.6	227°	6.0	241°
35	1.8	206°	3.5	243°
50	1.3	203°	0.9	216°
100	0.9	39°	1.2	121°

As will be seen, the computed velocities are on the whole somewhat smaller than the observed ones, and the phase angles also show a systematic deviation from the observed values. From the coefficients f and g we calculate the following values of the velocities.

Table 12.

	V	κ	Vobs	κobs
0	7.4	257°		
5	9.5	246°	8.7	245°
10	9.5	243°	11.1	241°
15	7.2	242°	6.0	241°
35	2.4	234°	3.5	243°
50	1.4	222°	0.9	216°
100	0.8	112°	1.2	121°

The agreement is quite satisfactory.

For station 2 we have only observations between the surface and 50 m. The oscillations are much smaller and consequently the results are affected with larger errors. For the vertical oscillations the normal equations are

1	2	3	4	5	A	B
0.94336	.19117	−.14988	.11311	−.06362	−2.44892	1.45253
.19117	.77715	.25011	−.18365	.13545	2.60647	.10956
−.14988	.25011	.72643	.24152	−.16580	.56993	−.02407
.11311	−.18365	.24152	.83012	.25680	−1.52329	.31286
−.06362	.13545	−.16580	.25680	.77239	.89994	−.75844

and the solution of the equations results in the following values for the different internal waves.

$$
\begin{array}{llll}
a_1=-4.5699 & b_1=1.2111 & R_1=4.7276 & \kappa_1=165^\circ.2 \\
a_2=6.0403 & b_2=0.4900 & R_2=6.0601 & \kappa_2=4^\circ.7 \\
a_3=-3.0370 & b_3=-0.5772 & R_3=3.0914 & \kappa_3=190^\circ.8 \\
a_4=1.4263 & b_4=0.9207 & R_4=1.6977 & \kappa_4=32^\circ.4 \\
a_5=-1.3976 & b_5=-1.3981 & R_5=1.9762 & \kappa_5=225^\circ
\end{array}
$$

The distance from station I to station II is 8.2 km. The first order wave gives the phase difference $165^\circ - 38^\circ = 127^\circ$ corresponding to $4^h\ 23^m$.

By assuming a progressive wave proceeding from station I to station II the velocity of propagation would be 52 cm/sec. This is somewhat less than the theoretical value which is 61,9 cm/sec.

From the current measurements we find the following values of the coefficients, f and g

$$
\begin{array}{llll}
f_0=0.457 & g_0=0.014 & R_0=0.457 & \kappa_0=2^\circ \\
f_1=-4.354 & g_1=4.050 & R_1=5.946 & \kappa_1=137^\circ \\
f_2=1.681 & g_2=-0.157 & R_2=1.688 & \kappa_2=355^\circ \\
f_3=1.207 & g_3=-0.046 & R_3=1.208 & \kappa_3=358^\circ \\
f_4=-0.420 & g_4=-1.034 & R_4=1.116 & \kappa_4=248^\circ \\
f_5=-0.309 & g_5=0.717 & R_5=0.780 & \kappa_5=113^\circ
\end{array}
$$

Most of the coefficients are so small that no value can be attached to them.

The normal equations for station III give the following values of the coefficients a and b.

$$
\begin{array}{llll}
a_1=-5.6838 & b_1=-1.6408 & R_1=5.9159 & \kappa_1=196^\circ.1 \\
a_2=-1.6322 & b_2=1.3192 & R_2=2.0986 & \kappa_2=141^\circ.1 \\
a_3=0.1922 & b_3=-0.0638 & R_3=0.2024 & \kappa_3=341^\circ.7 \\
a_4=1.9661 & b_4=0.1737 & R_4=1.9737 & \kappa_4=5^\circ.0 \\
a_5=-0.4265 & b_5=-0.1698 & R_5=0.4590 & \kappa_5=201^\circ.2
\end{array}
$$

The distance from this station to station I is 12.2 km. We then find the following phase differences.

I	$196°.1 - 38° = 158° - 5^h.45$
II	$141°.1 - 232° = 269° - 9^h.28$
III	$342° + 18° = 360° - 12^h.42$
IV	$365° - 210° = 155° + 360° = 515° - 17^h.77$
V	$201° - 36° = 165° + 360° = 525° - 18^h.11$

The corresponding velocities of propagation are

$$c_1 = \frac{9822}{158} = 62.2 \quad (61.9)$$

$$c_2 = \frac{9822}{269} = 36.5 \quad (34.0)$$

$$c_3 = \frac{9822}{360} = 27.2 \quad (23.1)$$

$$c_4 = \frac{9822}{515} = 19.1 \quad (17.2)$$

$$c_5 = \frac{9822}{525} = 18.7 \quad (13.7)$$

The theoretical values are given in parenthesis. The agreement is excellent, but since the waves of higher order have rather small amplitudes, no very high confidence can be attached to the results from these.

A comparison of the amplitudes at the different stations show that the amplitudes on station II and III are much smaller than those found at station I. For the first order we have had the values

11.72 4.72 5.92

respectively. To some degree the smaller values found at station II and III are due to the different phase of the moon, the observations at station II being taken approximately at neap tide. From the harmonic constants M_2, S_2, K_2 and N_2 we may compute an amplitude of 49 cm on the 14th of July, while the amplitude on the 18th when the observations were taken at station II, we find an amplitude of only 33 cm. Another cause is the geographic features of the fjord.

The distance accross the fjord at station II and III are nearly the double of the distance at station I. The waves will be slightly diverging and consequently the amplitude will diminish approximately inversely as the distance between the stations.

If we assume that the amplitudes are proportional to the surface tide, the values at station I has to be multiplied by a factor: .67 and a similar factor for the effect of the widening of the fjord which will approximately reduce the amplitude to the value found at station III. It will therefore not be necessary to take frictional resistance into account to explain the diminution of the amplitudes. The result of these measurements may therefore be taken as an indication that the observed oscillations of temperature and salinity and also the tidal currents can be explained by the theory.

The waves are mainly of a progressive character.

10. Observations in 1949. The observations from 1934 showed that the oscillations had very different amplitudes at the three different stations, the amplitudes at station I being more than twice as large as those at station III. Since the observations were not simultaneous it was difficult to compare the results from the three stations. Particularly it would be interesting to see if the diminution of the amplitudes might be explained by the different phase of the moon and the configuration of the fjord, or if it would be necessary to take frictional influences into account.

It was therefore fortunate that in 1949 I had the opportunity to repeat the observations, and this time I had two vessels at my disposal, the 'Armauer Hansen" belonging to the Geophysical Institute, Bergen and the "Johan Hjort" belonging to the Directorate of Fisheries, Bergen. I take this opportunity to express my best thanks to the institutions for the generosity with which they placed the ships and observers to my disposal. I also wish to express my gratitude to the "Norsk Varekrigsforsikrings Fond" for financial support of the expeditions.

The "Armauer Hansen" was anchored at the same place as St. 1 in 1934, while the "Johan Hjort" was anchored at a distance of some 11 km somewhere between St. 2 and 3 in 1934. The measurements started on Aug. 8th 15^h and ended on Aug. 13th 11^h.

The program was quite similar to that carried out in 1934. Observations of temperature and salinity were taken at the following depths: 0, 5, 10, 20, 30, 50, and 100 m. The temperature observations being taken every half hour, while water samples for the determinations of salinity were taken only hourly. On board the "Armauer Hansen" the current near the surface was measured by means of an electric registering current meter of Sverdrup-Dahl type. The apparatus did not work satisfactorily such that only the results of two days registrations may be used. Two Ekman current meters of the well known type were used, one alternatively in 5 and 10 m depth and the other in 15 and 30 m. Two mechanical registering current meters of Fjeldstad-Dahl's construction were used alternatively in 20, 40 and 60 m depth. From these current measurements we can get four days of observations. On board the other vessel 'Johan Hjort' temperature and salinity measurements were taken after the same program as on board 'Armauer Hansen'. Current measurements were taken with two Ekman current meters and one Fjeldstad-Dahl current meter in 5, 10, 15, and 20 m.

The weather conditions during the whole time were very unfavorable with heavy rain and strong winds. The surface conditions were much influenced by this. This may particularly be seen on the surface temperature which one day drops to the value ordinarily found at 5 m depth, indicating that the surface layer had been swept away by the gale. Stationary conditions are consequently not obtained in the surface layers, but in the deeper layers the oscillations are very little disturbed by the changing surface conditions.

The temperature variations at the different depths are shown in Figs. (9) and (11). The single measurements show quite irregular variations, but the curve obtained by smoothing over 4 hours show quite regular tidal oscillations, especially in the sub surface layers.

The variation in salinity at the surface is also irregular, but in the deeper layers we find the same regular tidal oscillations of salinity as shown on the temperature curves.

The hourly ordinates of the smoothed curves have been measured after lunar time, and by taking differences of six hours twice, we can analyse 3 lunar days. The results are given in the following table for st. I, "Armauer Hansen".

Table 13.

	R_T	κ_T	R_S	κ_S	σ_t	κ_σ
5	0.1132	254°	0.4881	79°	0.4230	78°
10	0.2500	249°	0.3334	53°	0.3038	56°
20	0.3438	255°	0.1789	70°	0.1887	74°
30	0.2574	261°	0.1051	77°	0.1381	80°
50	0.0578	243°	0.0446	64°	0.0502	58°
100					0.0064	61°

The phase differences between temperature and salinity oscillations are approximately 180° as may be expected.

We have also made analyses for single days, to show how the oscillations vary with time. The results of these analyses are given in the following table:

Table 14. *Results for single days.*

St. I 1949.

Depth		R_T	κ_T	R_S	κ_S	R_σ	κ_σ
5 m	1	.2444	282°	.7813	113°	.6439	112°
	2	.2209	220°	1.0323	57°	.8277	56°
	3			.1621	345°	.1247	345°
	4	.1557	202°	.7681	30°	.6131	29°
10 m	1	.2473	263°	.2465	83°	.2313	75°
	2	.3388	243°	.5942	45°	.5201	46°
	3	.2274	232°	.2251	50°	.2090	50°
	4	.0929	193°	.2199	38°	.1816	37°
20 m	1	.2191	253°	.1209	76°	.1424	78°
	2	.3141	254°	.1767	67°	.1886	68°
	3	.4873	260°	.2293	75°	.2616	76°
	4	.4933	226°	.2265	53°	.2660	51°

Table 14. (cont.)

Depth		R_T	κ_T	R_S	κ_S	R_σ	κ_σ
30 m	1	.1810	274°	.1008	96°	.1076	97°
	2	.2115	251°	.0978	71°	.1044	71°
	3	.3881	259°	.1810	77°	.1960	78°
	4	.3294	217°	.1706	38°	.1868	39°
50 m	1	.0588	274°	.0250	103°	.0309	94°
	2	.0696	234°	.0509	65°	.0501	61°
	3	.0830	235°	.0601	59°	.0598	56°
	4	.0666	204°	.0472	26°	.0453	26°
100 m	1	.0235	237°	.0068	83°	.0070	75°
	2	.0144	213°	.0071	64°	.0055	54°
	3	.0136	218°				
	4	.0239	213°				

The results of the harmonic analysis of the observations from St. 2 "Johan Hjort" are given in the following table:

Table 15.

	R_T	κ_T	R_S	κ_S	R_σ	κ_σ
5	0.1205	11°	0.4510	196°	0.3331	210°
10	0.1270	34°	0.1229	207°	0.1162	213°
20	0.1877	13°	0.0651	190°	0.0942	191°
30	0.0603	11°	0.0452	180°	0.0414	185°
50	0.0121	46°	0.0200	212°	0.0205	211°
100					0.0022	168°

We also give the results of analyses for single days in the following table:

Table 16. *Results for single days.*

St. II. 1949.

Depth		R_T	κ_T	R_S	κ_S	R_σ	κ_σ
5 m	1	.1575	13°	.2999	211°	.2625	210°
	2	.1419	19°	.2520	212°	.2270	211°
	3	.1004	358°	.4366	209°	.3643	208°
	4	.1991	3°	1.0091	182°	.8179	183°
10 m	1	.2454	47°	.1549	225°	.1481	227°
	2	.1961	14°	.2417	198°	.2211	197°
	3	.0561	29°	.6657	198°	.0586	207°
	4	.0564	41°	.0965	227°	.0820	225°

Table 16. (Cont.)

Depth		R_T	κ_T	R_S	κ_S	R_σ	κ_σ
20 m	1	.1225	29°	.0407	226°	.0411	218°
	2	.2790	8°	.0860	186°	.1073	184°
	3	.2362	16°	.0932	187°	.1114	192°
	4	.3352	10°	.1384	177°	.1579	181°
30 m	1	.0428	26°	.0260	209°	.0258	210°
	2	.0819	353°	.0737	157°	.0681	157°
	3	.0863	21°	.0577	196°	.0534	195°
	4	.1289	0°	.0749	172°	.0734	173°
50 m	1	.0117	64°	.0130	222°	.0121	229°
	2	.0133	37°	.0270	207°	.0245	208°
	3	.0115	35°	.0286	214°	.0248	211°
	4	.0127	17°	.0208	153°	.0187	152°
100 m	1					.0024	81°
	2					.0038	193°

A graphical representation of the current measurements is given in the Fig. 10 and 12. The single measurements give results which show irregular variations. Some of these are probably due to lateral oscillations in the fjord. The change in the surface conditions which is demonstrated by the temperature curve for the surface, has also influenced the current distribution. This is particularly seen from the curve giving the current in 20 m depth. While the amplitude of the tidal current is about 6 cm/sec. for the 3 first days, the last 24 hours give an amplitude of about 12 cm/sec.

The observations have been smoothed by taking means over 4 hours by planimeter. By harmonic analysis of the north-south and east-west components it was found that the currents were nearly alternating, and we therefore only give the velocities in the longitudinal direction of the fjord.

Table 17.

St. I.

Depth	V	κ
0 m.......	13.9 cm	261°
5 m.......	14.5 cm	280°
10 m.......	13.2 cm	275°
15 m	12.4 cm	265°
20 m.......	5.5 cm	254°
30 m.......	3.4 cm	204°
40 m.......	2.6 cm	237°
60 m.......	1.3 cm	173°

Results for single days (37 hourly values):

Table 18.

Depth	I		II		III		IV	
	V	κ	V	κ	V	κ	V	κ
5 m	16.7	276°	12.3	286°	14.9	281°	12.4	271°
10 m	14.2	276°	11.2	276°	12.6	273°	12.9	262°
15 m			13.2	267°	13.2	262°	13.5	255°
20 m	4.4	261°	5.7	236°	6.3	261°	10.8	252°
30 m	3.1	203°	3.6	220°	3.7	193°	4.6	203°
40 m	3.5	225°	2.2	231°	2.4	254°		

At st. II, "Johan Hjort", the current velocities are much smaller. The analyses of the current measurements in 40 and 60 m depth gave no satisfactory results, and the only conclusion is that the tidal currents in these depths are very weak. In the following table we give the results for 5, 10, 15 and 20 m depth.

Table 19.

Depth	V	κ
5 m.......	6.6	29°
10 m.......	6.6	50°
15 m.......	6.6	65°
20 m.......	5.3	78°

The results of analyses for single days are:

Table 20.

Depth	I		II		III		IV	
	V	κ	V	κ	V	κ	V	κ
5 m	7.4	28°	4.9	3°	7.9	42°	7.7	1°
10 m	6.3	47°	7.2	36°	7.7	66°	6.6	18°
15 m	5.0	75°	6.4	41°	8.2	76°	8.6	29°
20 m	6.1	114°	4.6	55°	7.7	81°	6.5	41°

If we compare the corresponding phase angles from the harmonic analyses for single days, we may form 59 phase differences; 19 from temperature observations and 20 from observations of salinity and density. The mean value of these differences is:

$$139°,7 \pm 3°,8 .$$

From the analyses of current observations we may in the same manner form 15 phase differences with the mean value:

$$138°.9 \pm 9°.4 .$$

By assuming a propagation from st. I to st. II, we find a time difference of $4^h\ 49^m$. This corresponds to a velocity of propagation of some 63 cm/sec, which is near the theoretical value for the first order internal wave. The phase difference for the tidal current give a similar value.

We may also compare the phase differences for the different depths. The results are shown in the following table.

Table 21.

Depth	$\Delta\kappa$
5 m.......	151°.8±13°.5
10 m.......	160°.0±6°.6
20 m.......	126°.4±3°.8
30 m.......	115°.1±5°.1
50 m.......	146°.0±4°.7

As we see there is a definite variation of the phase angles with depth. This indicates that internal waves of different order must be present.

From the mean density distribution we have by numerical integration determined the eigenfunctions $u_i(z)$ and $w_i(z)$ for 5 internal waves. The functions have been normated such that

$$\int_0^h \rho u^2 dz = 10^3$$

or with the variable $z_i = z10^{-3}$ which has been used for the numerical integration

$$\int_0^h \rho u^2 dz_1 = 1$$

The functions $w_i(z)$ and $u_i(z)$ are given in the tables 27 and 28. The theoretical velocities of propagation are:

$c_1 = 59.8$ cm/sec. $c_2 = 33.9$ cm/sec. $c_3 = 21.7$ cm/sec.
$c_4 = 17.1$ cm/sec. $c_5 = 13.6$ cm/sec.

As we see the velocity of propagation of the first order wave corresponds well with value found from the direct comparison of the observed phase angle differences. It is to be remarked, however, that owing to the wind conditions there was a non periodic surface current, which to some degree might influence the propagation of the internal waves. A current in the direction of the wave propagation would increase the velocity of propagation of the waves. A tentative numerical integration, assuming a current of some 20 cm/sec at the surface and decreasing to nearly zero at 20 m depth, indicated that the velocity of propagation of the first order wave might increase by some 10 cm/sec.

The coefficients of the different internal waves have been determined in the same manner as has been done with the observations from 1934, and the results for st. I ("Armauer Hansen") are the following:

$$\begin{array}{llll} a_1 = 5.471 & b_1 = 14.941 & R_1 = 15.912 & \kappa_1 = 70° \\ a_2 = -4.095 & b_2 = -8.528 & R_2 = 9.460 & \kappa_2 = 244° \\ a_3 = 1.885 & b_3 = 1.405 & R_3 = 2.352 & \kappa_3 = 37° \\ a_4 = -3.682 & b_4 = -0.266 & R_4 = 3.692 & \kappa_4 = 184° \\ a_5 = 0.017 & b_5 = 0.790 & R_5 = 0.790 & \kappa_5 = 89° \end{array}$$

The vertical elevation ζ is then expressed by the formula:

$$\begin{aligned} \zeta = {} & 15.912 w_1(z)\cos(\sigma t - 70°) + 9.460 w_2(z)\cos(\sigma t - 244°) \\ & + 2.352 w_3(z)\cos(\sigma t - 37°) + 3.692 w_4(z)\cos(\sigma t - 184°) \\ & + 0.790 w_5(z)\cos(\sigma t - 89°)\,. \end{aligned}$$

The results of the current measurements have been analysed in a similar manner, and we have also included the ordinary tidal current u_0. We find the following coefficients:

$$\begin{array}{llll} f_0 = 0.314 & g_0 = -1.626 & V_0 = 1.656 & \kappa_0 = 281° \\ f_1 = 0.416 & g_1 = 15.068 & V_1 = 15.074 & \kappa_1 = 88° \\ f_2 = -4.050 & g_2 = -4.842 & V_2 = 6.312 & \kappa_2 = 230° \\ f_3 = 5.155 & g_3 = -3.631 & V_3 = 6.273 & \kappa_3 = 325° \\ f_4 = -0.044 & g_4 = -1.264 & V_4 = 1.264 & \kappa_4 = 268° \\ f_5 = 0.241 & g_5 = 2.568 & V_5 = 2.579 & \kappa_5 = 85° \end{array}$$

In the following table we have for comparison compiled the amplitudes and phase angles for the vertical oscillation ζ and the horizontal current u.

Table 22.

	R	κ_S	V	κ-
1	15.91	70°	15.07	88°
2	9.46	244°	6.31	230°
3	2.35	37°	6.27	325°
4	3.69	184°	1.26	268°
5	0.79	89°	2.58	85°

From the table we see that the waves of first and second order give results which are in fair accordance with the assumption that the waves are progressive. The waves of higher order give results which are so uncertain that no weight should be attached to them.

The vertical oscillation, ζ, may also be found directly from the analysis of temperature and salinity, using the equations:

$$T_1 + \zeta \frac{dT_0}{dz} = 0 ,$$

and

$$S_1 + \zeta \frac{dS_0}{dz} = 0 .$$

From the temperature observations in 5, 10, 20, 30, 50 and 100 m we may determine the coefficients of five internal waves by the method of least squares, and find:

$$\begin{aligned} R_1 &= 16.99 & \kappa_1 &= 59° \\ R_2 &= 4.79 & \kappa_2 &= 245° \\ R_3 &= 3.23 & \kappa_3 &= 19° \\ R_4 &= 0.58 & \kappa_4 &= 123° \\ R_5 &= 1.01 & \kappa_5 &= 24° \end{aligned}$$

From the salinity observations in 5, 10, 20, 30, 50 and 100 m we find in the same manner:

$$\begin{aligned} R_1 &= 15.57 & \kappa_1 &= 64° \\ R_2 &= 4.90 & \kappa_2 &= 245° \\ R_3 &= 3.17 & \kappa_3 &= 77° \\ R_4 &= 1.41 & \kappa_4 &= 92° \\ R_5 &= 4.47 & \kappa_5 &= 87° \end{aligned}$$

These results do not correspond exactly to the results of the analysis of density given above, since the density values were given as the results of harmonic analysis of 3 days after lunar time, while the results of the temperature and salinity observations are found as mean values for four single days. This explains the different values of the kappa numbers for the first order wave. The results of the temperature and salinity analyses give two independent determinations of the first and second order waves which are in good agreement. The higher order waves differ much and cannot be trusted.

Station II, "Johan Hjort".

The internal waves at this station are much smaller and the results of the analysis are consequently affected with relatively larger errors. The salinity observations from 100 m are incomplete owing to the lack of bottles for storing the water samples.

By graphical interpolation and integration from the surface to 100 m, and taking five internal waves into account, we get the following values:

$$\begin{aligned} R_1 &= 7.10 & \kappa_1 &= 201° \\ R_2 &= 3.10 & \kappa_2 &= 20° \\ R_3 &= 1.76 & \kappa_3 &= 224° \\ R_4 &= 1.45 & \kappa_4 &= 49° \\ R_5 &= 0.98 & \kappa_5 &= 176° \end{aligned}$$

A direct determination from the observations of temperature and salinity gives for T:

$$R_1 = 8.36 \qquad \kappa_1 = 195°$$

and for S:

$$R_1 = 8.48 \qquad \kappa_1 = 192°$$

These values are found from the mean values of the results given in the table (St. II) for four days, while the results above are from the analysis of three lunar days.

The current measurements gave reliable values only for the four depths 5, 10, 15 and 20 m. The depths are too few to make a determination of the coefficients of the different internal waves possible, but the current amplitudes are of the same magnitude as that found for the first order internal wave.

11. Comparison between the results from 1934 and 1949. St. I 1934 and st. I 1949 were taken at the same place, and it will be of interest to compare the results. In the preceding analysis the kappa numbers have been related to the prinsipal lunar component M_2, since this was the simplest method. It would have been better to take also the other larger components S_2, N_2 and K_2 into account. But in this case we have the possibility of comparing the results with the tide in Bergen harbour, from which complete registrations are available. If we make analyses for single days after the same scheme which has been used for the present observations, we get the following results:

Table 23. *July 1934.*

	R	κ
14th	49.6 cm	290°.3
15th	48.2 cm	279°.2
16th	46.3 cm	266°.6
17th	43.2 cm	262°.5

Table 24. *August 1948.*

	R	κ
9th	56.0 cm	324.°0
10th	55.1 cm	304°.2
11th	54.4 cm	287°.2
12th	54.6 cm	284°.5

The kappa number of M_2 in Bergen is 298°. This means that to get comparable results the kappa numbers given in the table of results for single days in 1934 should be increased by 8°, 19° and 21°, and correspondingly the kappa numbers in the table giving results for single days in 1949 should be corrected by −26°, −6°, +11° and 13° respectively.

By making an analysis of mean values for three days of the results from 1934 and four days in 1949 with the kappa numbers corrected in this manner, we get the following results for the two first order waves:

1934		1949	
$R_1 = 12.13$	$\kappa_1 = 56°.8$	$R_1 = 17.36$	$\kappa_1 = 56°.2$
$R_2 = 4.00$	$\kappa_2 = 245°.9$	$R_2 = 6.23$	$\kappa_2 = 236°.1$

This means that the internal waves appear in the same manner every summer when the hydrographic conditions are the same.

12. Energy of the internal waves in Herdlefjord. In chapter 4 we have deduced the formulae:

$$E_p = \frac{1}{2} g \left[(\rho\zeta^2)_h - \int_0^h \zeta^2 \frac{d\rho}{dz} dz \right],$$

and

$$E_k = \frac{1}{2} \int_0^h \rho u^2 dz \,.$$

If we put

$$\zeta = \Sigma Z_n w_n(z) \cos(\sigma t - \kappa_n)$$

and

$$u = \Sigma V_n u_n(z) \cos(\sigma t - \kappa_n) \,,$$

we get

$$E_p = \frac{1}{2} g \Sigma Z_n^2 \cos^2(\sigma t - \kappa_n) \left((\rho w_n^2)_h - \int_0^h w_n^2 \frac{d\rho}{dz} dz \right)$$

and

$$E_k = \frac{1}{2} \Sigma V_n^2 \cos^2(\sigma t - \kappa_n) \int_0^h \rho u_n^2 dz \,.$$

We have:

$$\int_0^h \rho u_n^2 dz = g \left[(\rho w_n^2)_h - \int_0^h w_n^2 \frac{d\rho}{dz} dz \right] = 10^3$$

and taking the mean over one period, we find:

$$E_p = 250 \Sigma Z_n^2$$

and

$$E_k = 250 \Sigma V_n^2 \,.$$

The potential energy of the ordinary surface tide is:

$$\frac{1}{4} g \rho Z_0^2 = 250 Z_0^2 \,,$$

where Z_0 is the amplitude.

From the analysis of the observations from 1934 we have $Z_1 = 11.473$ and this gives $E_p = 32900$ ergs/cm². The total energy of five internal waves is 38500 ergs/cm².

The simultaneous registrations of the tide in Bergen harbour give an amplitude of 48 cm, and the potential energy is then 5.76 10^5 ergs/cm². The potential energy of the internal waves is thus only 1/15 of the surface tide.

The kinetic energy of the first order internal wave is: $E_k = 250x11.716^2 = 34300$ ergs/cm², or about the same as that found for the potential energy.

From the observations in 1949 we have found $Z_1 = 15.912$, which gives $E_p =$ 63300 ergs/cm², which is about twice the value from 1934. At the same time the tide registration from Bergen give an amplitude of 56 cm, and the potential energy is then 7.84 10^5 ergs/cm².

13. Concluding remarks. Comparably little has been done to investigate the occurrence of internal waves in the open ocean. The theory as given in "Interne Wellen" has been applied to observations taken by the "Snellius" expedition in the East Indian waters and treated by L. Lek [8], and observations by "Atlantis" in the southern North Atlantic treated by H. Seiwel [10]. Both found that the observed oscillations could be very satisfactorily represented by the theoretical internal waves.

But most oceanographers still try to apply the boundary wave theory to a two layer ocean. In my opinion this is not satisfactory. Even if the thermocline is rather sharp, and approximates to a discontinuity, the distribution with depth of the density will be very poorly represented by a two layer ocean.

It is hoped that the study of internal waves will be given greater attention in the future. The task is by no means simple. The research vessel has to be anchored, because the wavelength of the internal waves is so short that a displacement of a few miles will affect the phase of the waves.

To obtain satisfactory results a cooperation of several ships is highly desirable. Current measurements and repeated series of temperature and salinity observations are necessary.

An international cooperation might be planned and set into effect under the auspices of SCOR or IOC.

Table 25. *Theoretical internal waves w(z). 1934.*

	w_1	w_2	w_3	w_4	w_5
0	−0.075	−0.047	−0.033	−0.026	−0.019
1	1.898	3.964	5.363	7.212	6.917
2	3.671	7.045	8.199	8.012	5.308
3	5.295	9.096	8.599	4.575	−0.947
4	6.778	10.279	7.318	−0.564	−6.924
5	8.131	10.776	5.010	−5.635	−10.348
6	9.367	10.703	2.171	−9.573	−10.826
7	10.502	10.233	−0.847	−12.194	−8.948
8	11.547	9.453	−3.825	−13.467	−5.506
9	12.508	8.433	−6.613	−13.532	−1.253
10	13.391	7.226	−9.106	−12.494	3.172
10	13.391	7.226	−9.106	−12.494	3.172
12	14.941	4.429	−12.988	−7.958	10.636
14	16,237	1.361	−15.307	−1.627	14.584
15	16,800	−0.209	−15.898	1.733	15.051
16	17.312	−1.777	−16.138	5.029	14.582
18	18.193	−4.848	−15.679	10.994	11.304
20	18.907	−7.766	−14.190	15.685	5.920
22	19.481	−10.484	−11.942	18.879	−0.382
24	19.942	−12.990	−9.194	20.663	−6.670
26	20.315	−15.295	−6.149	21.263	−12.384
28	20.616	−17.411	−2.947	20.894	−17.233
30	20.859	−19.352	0.318	19.744	−21.087
30	20.859	−19.352	0.318	19.744	−21.087
35	21.302	−23.505	8.386	14.421	−26.265
40	21.492	−26.811	15.828	7.127	−26.035
45	21.525	−29.454	22.477	−0.875	−21.972
50	21.429	−31.477	28.137	−8.827	−15.236
55	21.222	−32.934	32.788	−16.208	−6.908
60	20.922	−33 881	36.404	−22.670	2.055
60	20.922	−33.881	36.404	−22.670	2.055
70	20.101	−34.511	40.823	−32.227	19.045
80	19.079	−33.932	42.352	−37.779	32.495
90	17.952	−32.281	42.244	−40.722	42.497
100	16.774	−30.729	41.268	−42.161	50.092
100	16.774	−30.729	41.268	−42.161	50.092
110	15.565	−28.984	39.767	−42.631	55.903
120	14.338	−27.120	37.933	−42.456	60.404
130	13.097	−25.161	35.824	−41.726	63.672
140	11.834	−23.113	33.460	−40.456	65.652
150	10.577	−20.983	30.854	−38.657	66.292
160	9.299	−18.770	27.997	−36.285	65.403
170	8.007	−16.458	24.840	−33.212	62.538
180	6.701	−14.036	21.359	−29.348	57.347
190	5.381	−11.521	17.551	−24.671	49.680
200	4.047	−8.870	13.444	−19.235	39.625
210	2.704	−6.204	9.098	−13.179	27.582
220	1.354	−3.454	4.585	−6.698	14.148
230	0.000	−0.673	0.000	0.000	0.000

Table 26. *Theoretical internal waves, u(z), 1934.*

	u_1	u_2	u_3	u_4	u_5
0	−1.1934	−1.4080	−1.3802	−1.5014	−1.3537
1	−1.1435	−1.1999	−0.9717	−0.7045	−0.3762
2	−1.0540	−0.8461	−0.3402	0.2840	0.6800
3	−0.9621	−0.5304	0.1360	0.7842	0.9191
4	−0.8790	−0.2724	0.4432	0.8974	0.6692
5	−0.8016	−0.0682	0.6169	0.7852	0.2604
6	−0.7347	0.0877	0.6935	0.5738	−0.1154
7	−0.6761	0.2043	0.7052	0.3401	−0.3846
8	−0.6225	0.2938	0.6715	0.1150	−0.5434
9	−0.5728	0.3642	0.6176	−0.0888	−0.6111
10	−0.5252	0.4176	0.5366	−0.2625	−0.5932
10	−0.5252	0.4176	0.5366	−0.2625	−0.5932
12	−0.4405	0.4841	0.3586	−0.4885	−0.4048
14	−0.3671	0.5101	0.1790	−0.5758	−0.1313
15	−0.3341	0.5129	0.0987	−0.5756	0.0011
16	−0.3031	0.5089	0.0172	−0.5538	0.1250
18	−0.2470	0.4895	−0.1174	−0.4627	0.3116
20	−0.1998	0.4593	−0.2210	−0.3393	0.4144
22	−0.1602	0.4244	−0.2929	−0.2110	0.4407
24	−0.1297	0.3899	−0.3378	−0.0994	0.4171
26	−0.1049	0.3579	−0.3629	−0.0070	0.3649
28	−0.0854	0.3278	−0.3752	0.0673	0.3000
30	−0.0684	0.3067	−0.3779	0.1276	0.2291
30	−0.0684	0.3067	−0.3779	0.1276	0.2291
35	−0.0353	0.2438	−0.3616	0.2253	0.0605
40	−0.0131	0.1944	−0.3274	0.2680	−0.0642
45	0.0044	0.1521	−0.2872	0.2768	−0.1484
50	0.0190	0.1125	−0.2396	0.2658	−0.2113
55	0.0317	0.0774	−0.1928	0.2385	−0.2376
60	0.0423	0.0451	−0.1445	0.2038	−0.2471
60	0.0423	0.0451	−0.1445	0.2038	−0.2471
70	0.0583	−0.0042	−0.0639	0.1272	−0.2116
80	0.0674	−0.0346	−0.0121	0.0684	−0.1578
90	0.0717	−0.0484	0.0141	0.0359	−0.1198
100	0.0741	−0.0573	0.0297	0.0149	−0.0905
100	0.0741	−0.0573	0.0297	0.0149	−0.0905
110	0.0754	−0.0614	0.0388	0.0021	−0.0706
120	0.0764	−0.0654	0.0456	−0.0079	−0.0534
130	0.0773	−0.0678	0.0517	−0.0172	−0.0364
140	0.0781	−0.0713	0.0575	−0.0264	−0.0181
150	0.0788	−0.0733	0.0629	−0.0355	0.0008
160	0.0796	−0.0769	0.0693	−0.0464	0.0248
170	0.0804	−0.0797	0.0766	−0.0595	0.0548
180	0.0814	−0.0838	0.0843	−0.0734	0.0882
190	0.0822	−0.0866	0.0916	−0.0871	0.1223
200	0.0829	−0.0901	0.0979	−0.0991	0.1531
210	0.0834	−0.0917	0.1026	−0.1082	0.1765
220	0.0838	−0.0937	0.1054	−0.1140	0.1911
230	0.0839	−0.0937	0.1064	−0.1156	0.1961

Table 27. *Internal waves. 1949.*

	W_1	W_2	W_3	W_4	W_5
0	−0.073	−0.036	−0.017	−0.015	−0.015
0,5	0.933	1.516	1.708	2.501	3.813
1	1.932	3.034	3.342	4.805	7.132
1,5	2.917	4.486	4.801	6.702	9.503
2	3.881	5.840	6.010	8.034	10.610
2,5	4,818	7.067	6.905	8.687	10.310
3	5.721	8.141	7.440	8.608	8.642
3,5	6.584	9.039	7.587	7.804	5.830
4	7.401	9.741	7.340	6.344	2.247
4	7.401	9.741	7.340	6.344	2.247
5	8.875	10.504	5.735	1.994	−5.294
6	10.117	10.418	2.998	−3.026	−10.426
7	11.190	9.782	−0.148	−7.512	−12.297
8	12.145	8.823	−3.292	−11.060	−11.648
9	12.997	7.634	−6.238	−13.500	−9.136
10	13.765	6.287	−8.875	−14.857	−5.451
10	13.765	6.287	−8.875	−14.857	−5.451
12,5	15.415	2.580	−14.091	−14.786	5.254
15	16.781	−1.295	−17.374	−11.352	14.365
17,5	17.916	−5.113	−18.892	−6.084	19.802
20	18.871	−8.787	−18.967	0.098	21.276
20	18.871	−8.787	−18.967	0.098	21.276
25	20.324	−15.408	−15.768	11.893	14.769
30	21.286	−20.889	−9.693	20.398	1.358
35	21.856	−25.186	−2.243	24.474	−12.749
40	22.114	−28.339	5.477	23.988	−23.540
45	22.132	−30.568	12.774	20.730	−29.417
50	21.959	−31.947	19.290	14.796	−30.822
50	21.959	−31.947	19.290	14.796	−30.822
60	21.361	−33.584	30.525	1.497	−26.131
70	20.590	−34.374	39.983	−12.119	−17.078
80	19.708	−34.611	47.871	−24.594	−6.064
90	18.719	−34.241	53.782	−35.872	5.562
100	17.650	−33.435	57.978	−44.819	16.700
110	16.495	−32.103	60.052	−51.648	26.435
120	15.281	−30.432	60.504	−55.637	34.149
130	14.006	−28.365	59.157	−57.598	39.573
140	12.696	−26.090	56.653	−57.021	42.554
150	11.338	−23.489	52.553	−54.632	43.134
150	11.338	−24.489	52.553	−54.632	43.134
170	8.551	−17.855	41.866	−44.855	38.233
190	5.699	−11.721	28.763	−30.921	27.778
210	2.819	−5.474	14.573	−15.116	14.664
230	−0.070	0.825	0.048	1.249	0.701

Table 28. *Internal waves u(z), 1949.*

	u_1	u_2	u_3	u_4	u_5
0	−1.2037	−1.0553	−0.7567	−0.8721	−1.0638
0,5	−1.1995	−1.0439	−0.7369	−0.8353	−0.9931
1	−1.1870	−1.0099	−0.6787	−0.7280	−0.7905
1,5	−1.1662	−0.9541	−0.5851	−0.5594	−0.4832
2	−1.1374	−0.8777	−0.4611	−0.3439	−0.1121
2,5	−1.1008	−0.7825	−0.3131	−0.0995	0.2736
3	−1.0566	−0.6704	−0.1489	0.1529	0.6226
3,5	−1,0050	−0.5440	0.0229	0.3923	0.8887
4	−0.9466	−0.4060	0.1933	0.5984	1.0369
4	−0.9466	−0.4060	0.1933	0.5984	1.0369
5	−0.8108	−0.1073	0.4955	0.8460	0.9209
6	−0.6859	0.1378	0.6610	0.8345	0.4624
7	−0.6039	0.2762	0.6948	0.6972	0.0662
8	−0.5423	0.3688	0.6687	0.5142	−0.2333
9	−0.4838	0.4320	0.6125	0.3277	−0.4362
10	−0.4344	0.4775	0.5359	0.1439	−0.5548
10	−0.4344	0.4775	0.5359	0.1439	−0.5548
12,5	−0.3580	0.5202	0.3695	−0.1374	−0.5649
15	−0.2965	0.5243	0.2044	−0.3127	−0.4118
17,5	−0.2489	0.5101	0.0660	−0.4031	−0.1814
20	−0.2068	0.4809	−0.0528	−0.4224	0.0121
20	−0.2068	0.4809	−0.0528	−0.4224	0.0121
25	−0.1424	0.4109	−0.2128	−0.3545	0.2920
30	−0.0897	0.3298	−0.3035	−0.2105	0.3885
35	−0.0479	0.2508	−0.3363	−0.0565	0.3412
40	−0.0155	0.1812	−0.3307	0.0686	0.2194
45	0.0111	0.1183	−0.3004	0.1661	0.0835
50	0.0269	0.0789	−0.2702	0.2088	−0.0183
50	0.0269	0.0789	−0.2702	0.2088	−0.0183
60	0.0430	0.0391	−0.2246	0.2314	−0.1123
70	0.0496	0.0170	−0.1892	0.2253	−0.1491
80	0.0561	−0.0026	−0.1502	0.2045	−0.1672
90	0.0616	−0.0201	−0.1103	0.1741	−0.1663
100	0.0665	−0.0365	−0.0680	0.1356	−0.1453
110	0.0709	−0.0512	−0.0269	0.0921	−0.1288
120	0.0745	−0.0638	0.0109	0.0503	−0.1002
130	0.0773	−0.0736	0.0418	0.0118	−0.0668
140	0.0798	−0.0830	0.0732	−0.0264	−0.0343
150	0.0818	−0.0899	0.0964	−0.0583	−0.0021
150	0.0818	−0.0899	0.0964	−0.0583	−0.0021
170	0.0845	−0.0999	0.1320	−0.1062	0.0469
190	0.0858	−0.1046	0.1494	−0.1299	0.0734
210	0.0863	−0.1064	0.1563	−0.1393	0.0842
230	0.0863	−0.1066	0.1574	−0.1407	0.0861

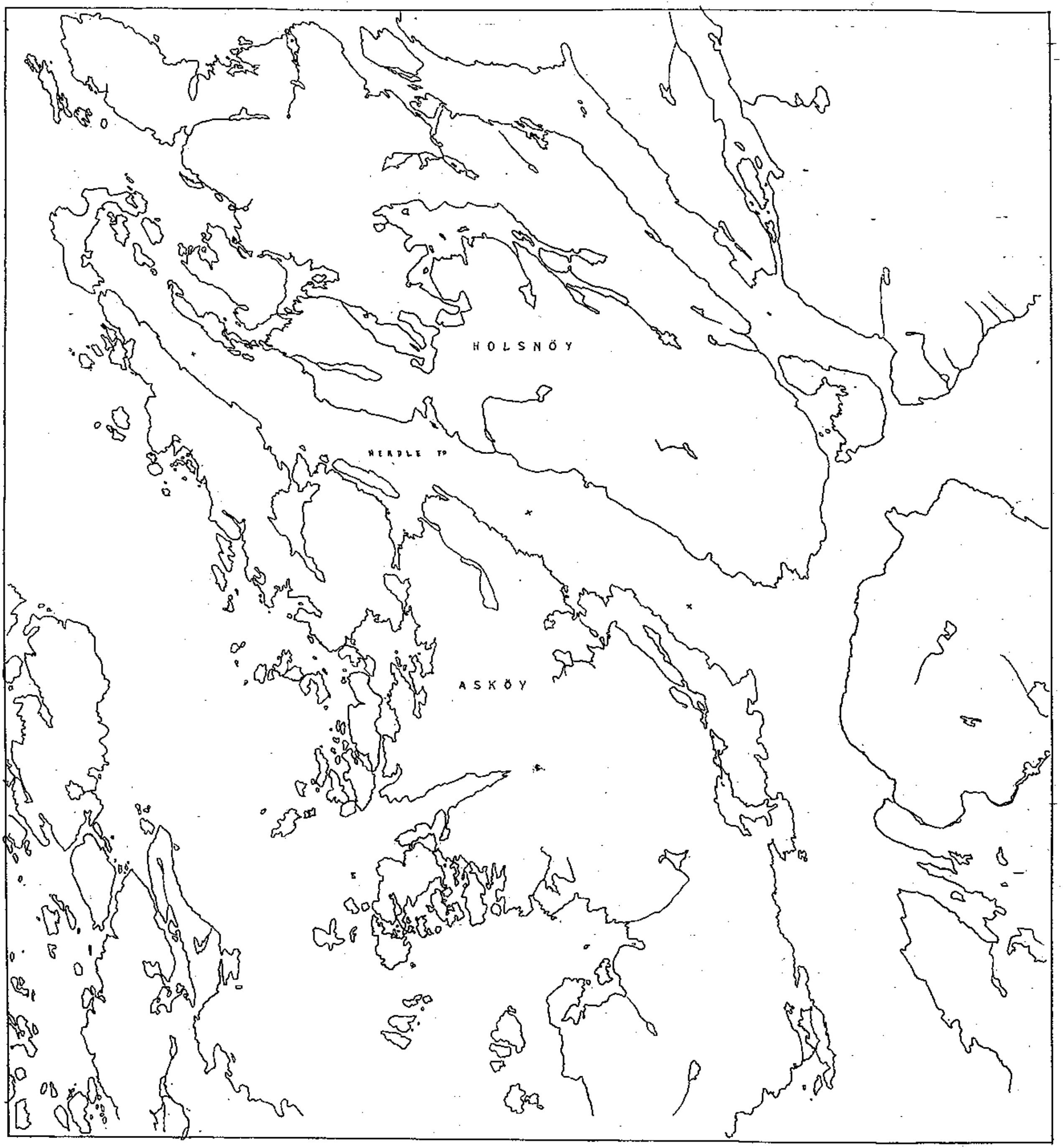

Fig. 1. Map of Herdlefjord.

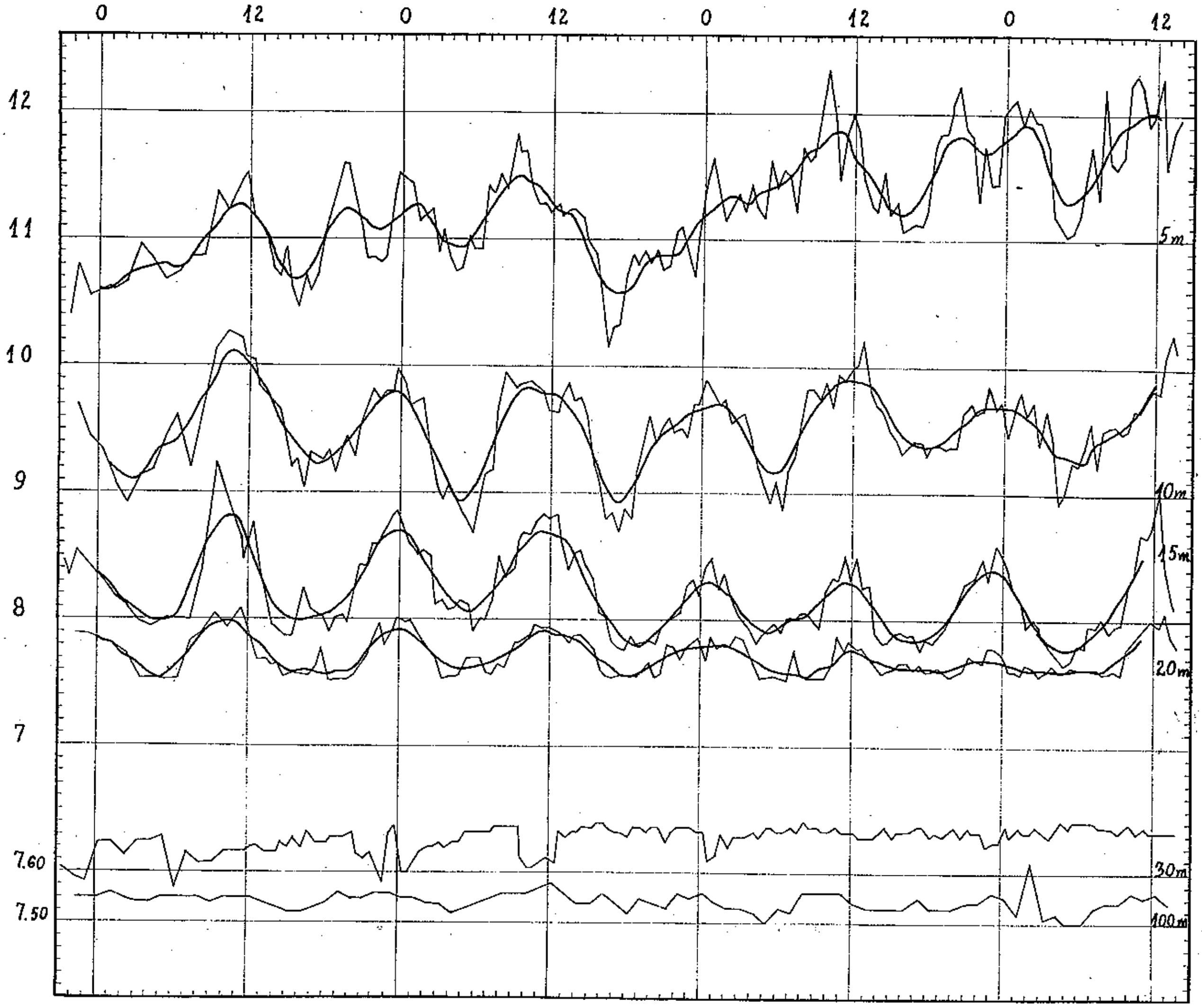

Fig. 2. Temperature oscillations St. I, 1934.

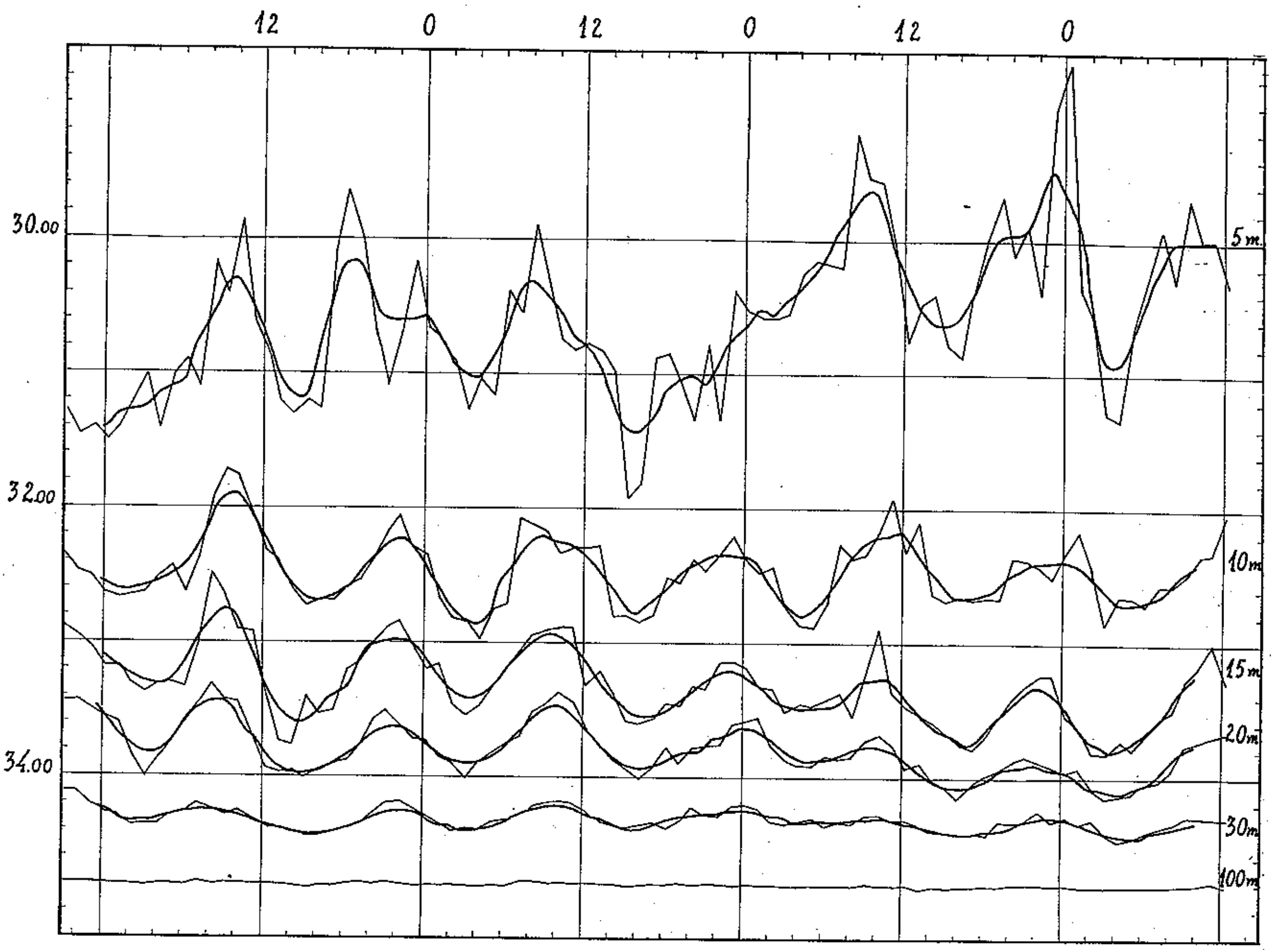

Fig. 3. Salinity oscillations St. I, 1934.

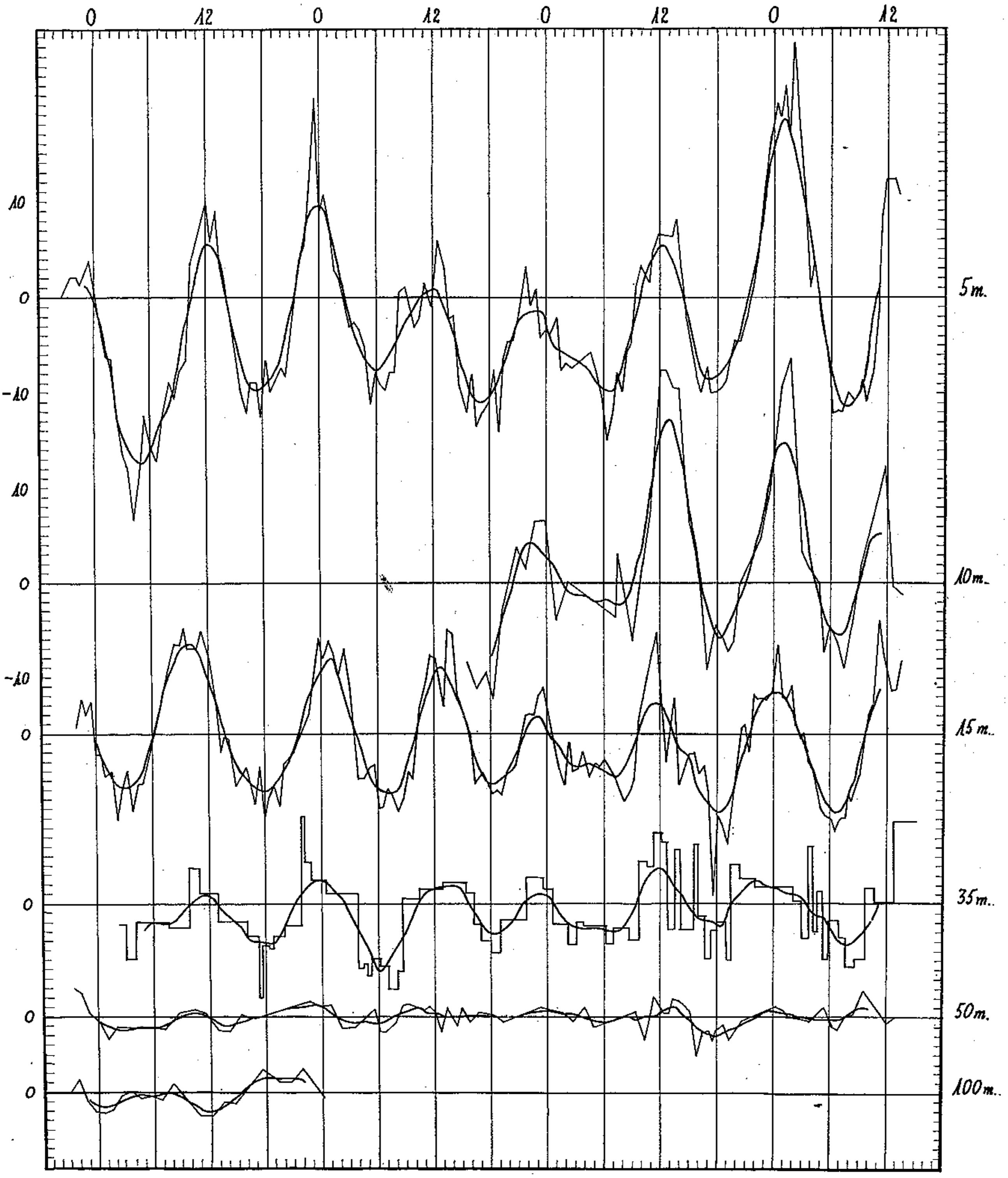

Fig. 4. Longitudinal components of current St. I, 1934.

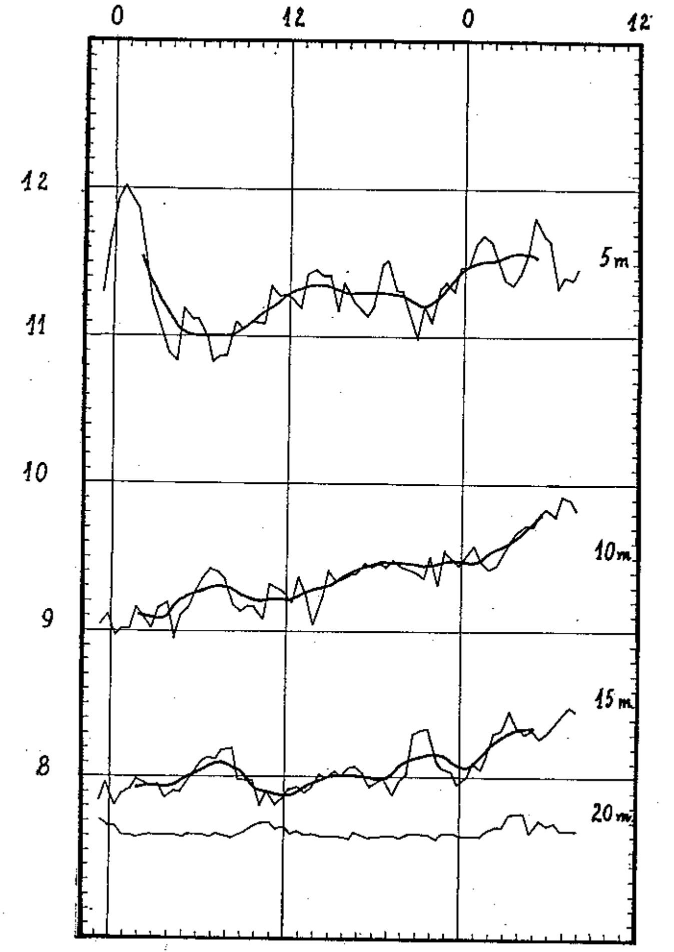

Fig. 5. Temperature oscillations St. II, 1934.

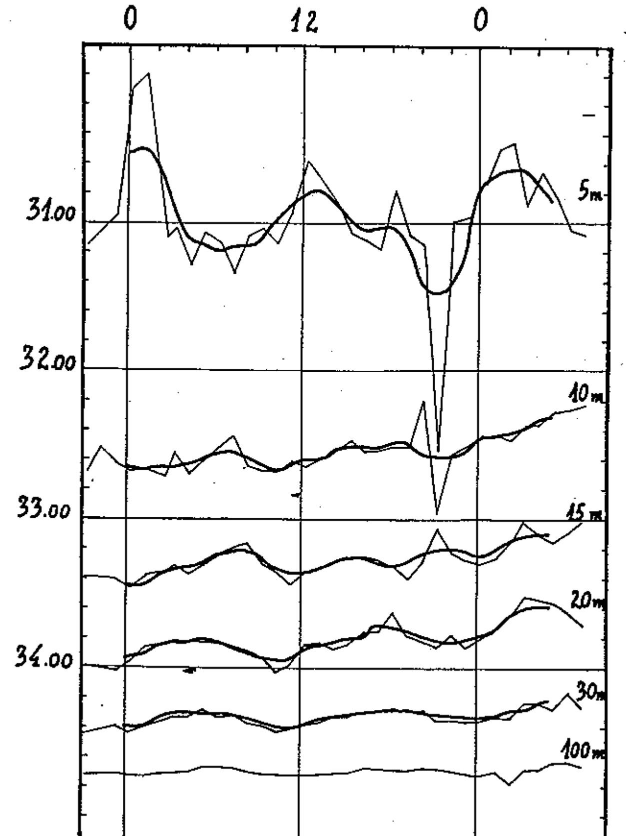

Fig. 6. Salinity oscillations St. II, 1934.

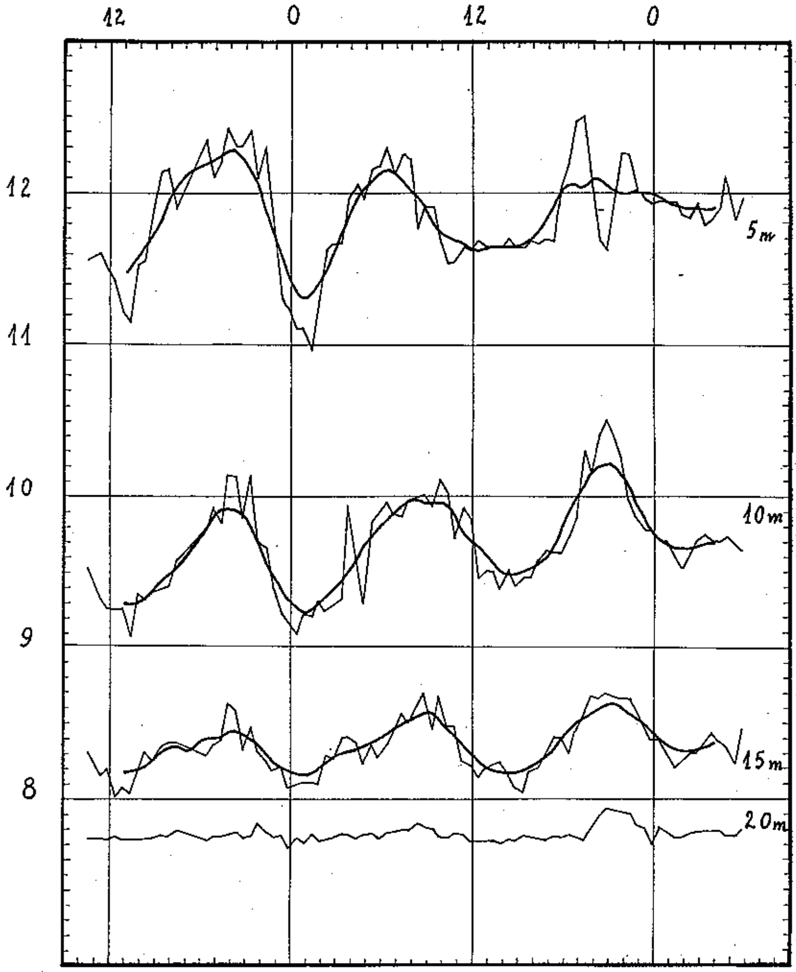

Fig. 7. Temperature oscillations St. III, 1934.

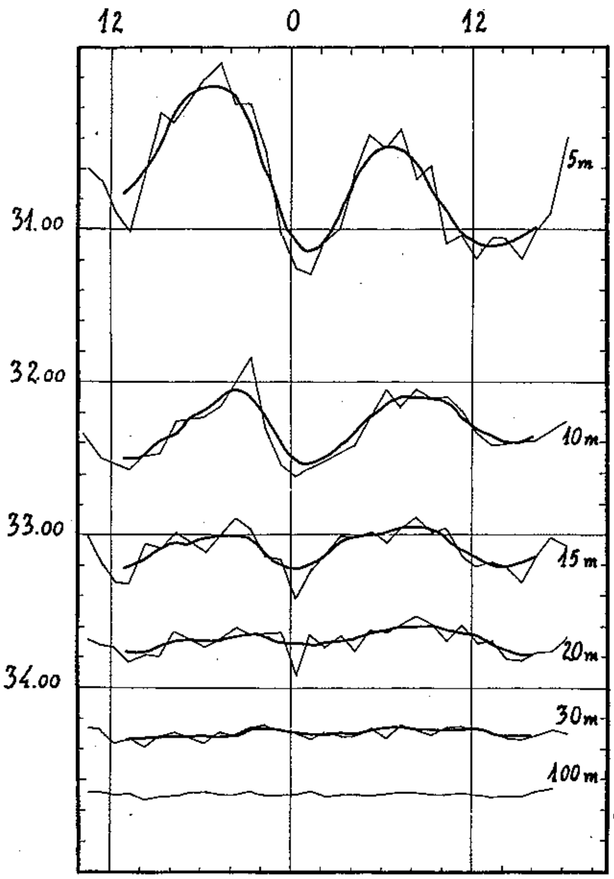

Fig. 8. Salinity oscillations St. III, 1934.

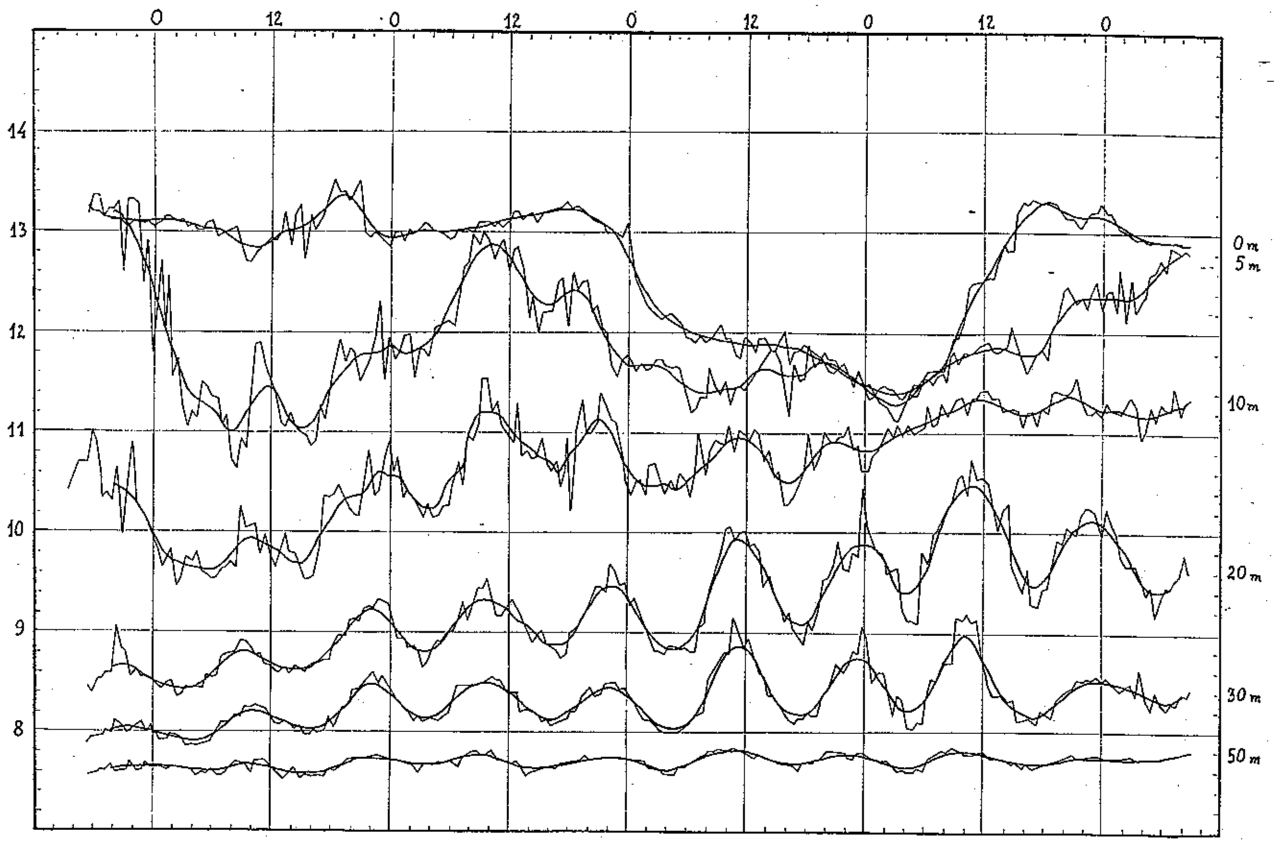

Fig. 9. Temperature oscillations St. I, 1949.

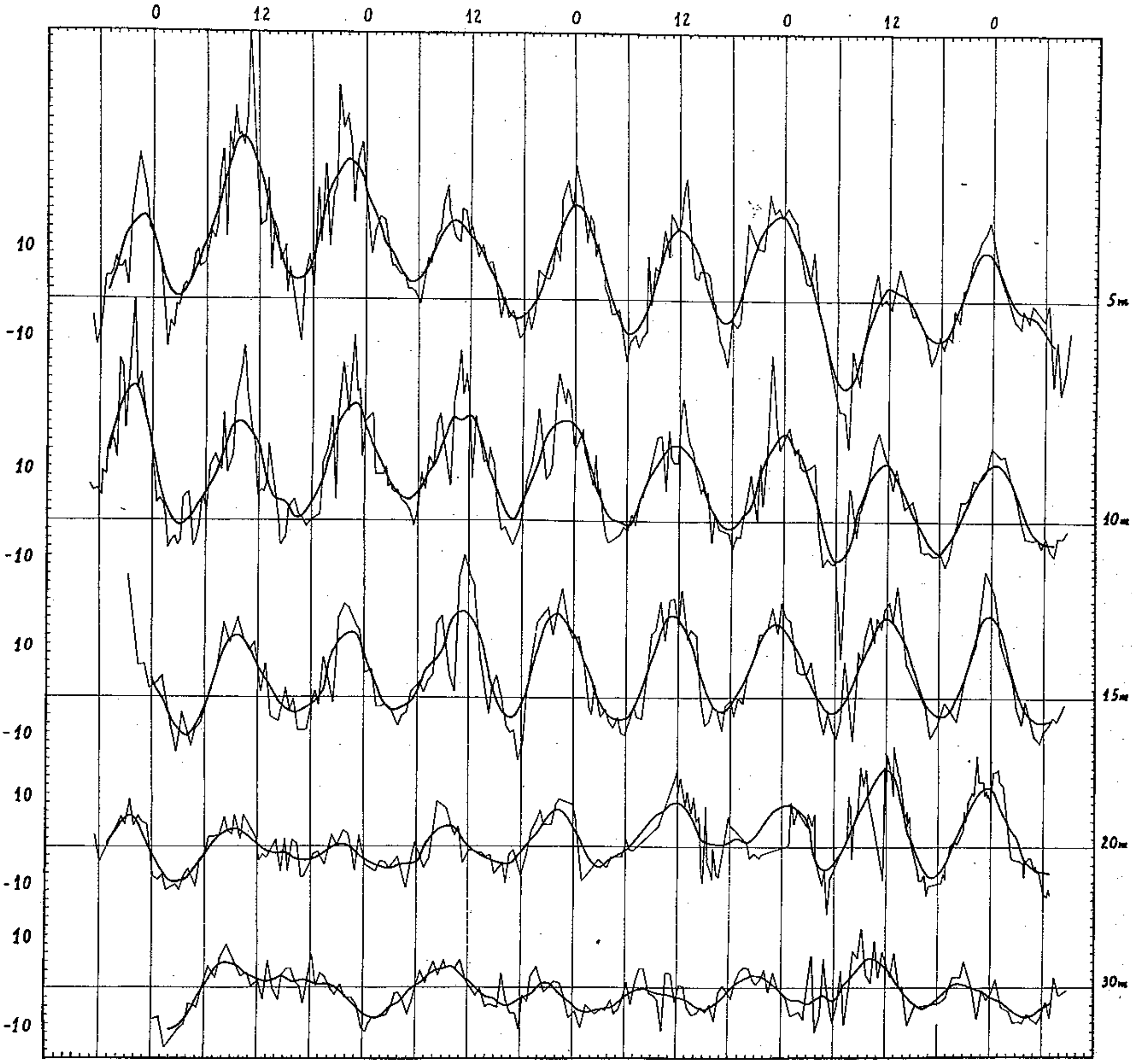

Fig. 10. Longitudinal components of velocity St. I, 1949.

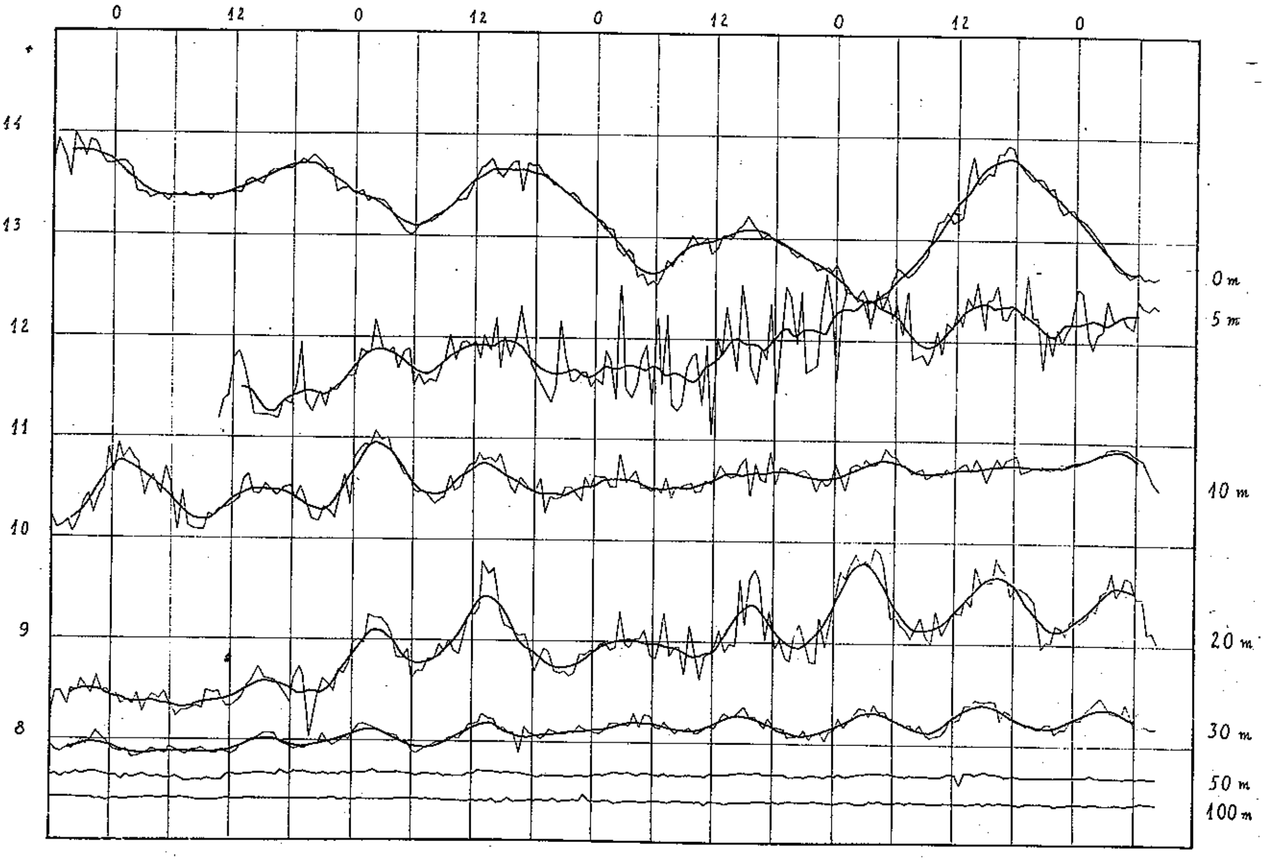

Fig. 11. Temperature oscillations St. II, 1949.

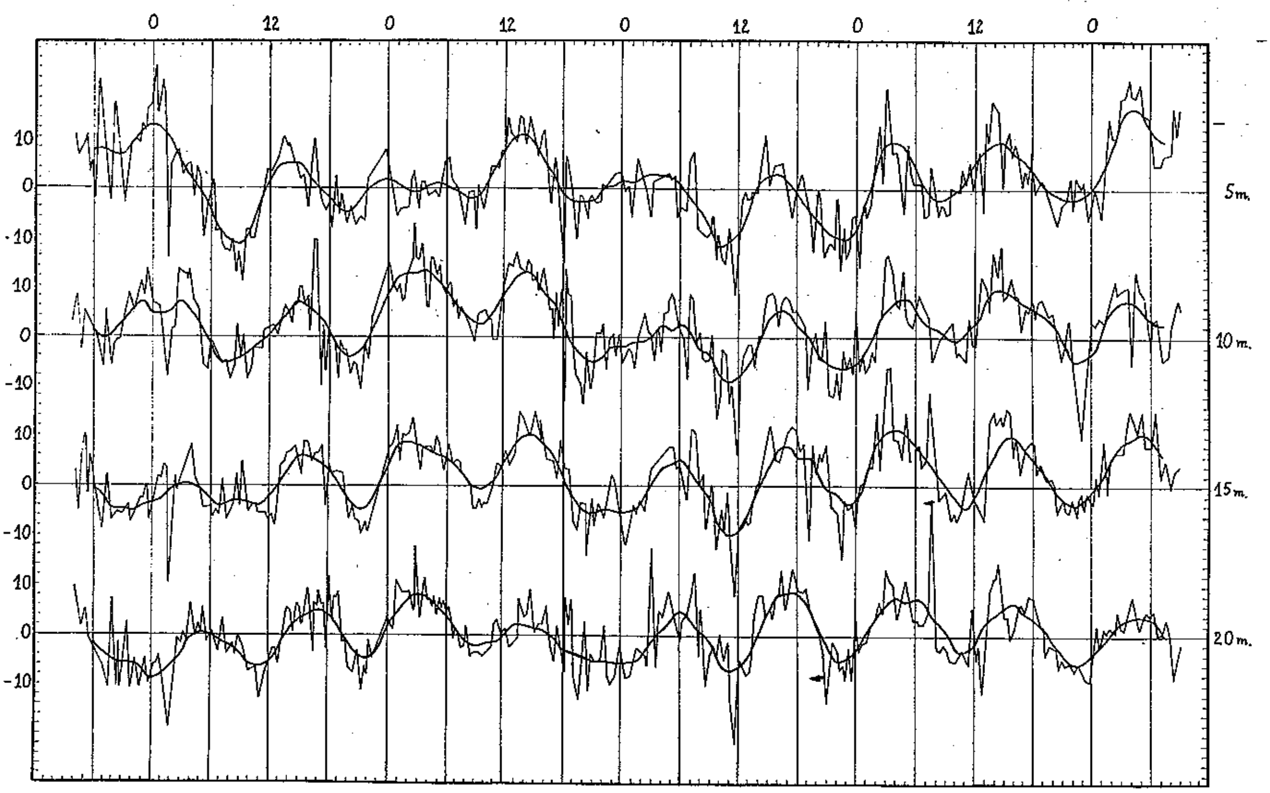

Fig. 12. Longitudinal components of velocity St. II, 1949.

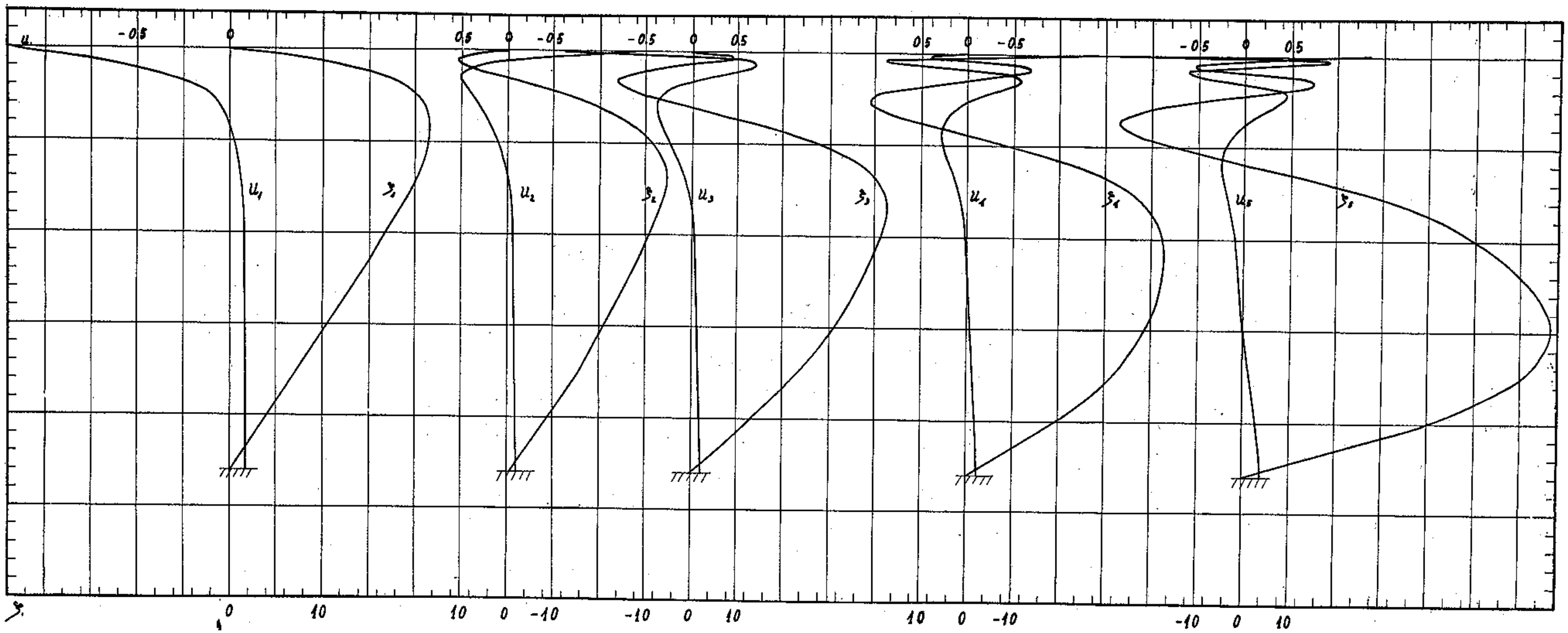

Fig. 13. Theoretical internal waves St. I, 1949.

APPENDIX

Influence of eddy viscosity on internal waves. In internal waves the velocities change rapidly with depth, and this will increase the friction. On the other hand internal waves are intimately connected with the stability, and we know that the stability tend to reduce the coefficient of eddy viscosity. This makes it desirable to investigate the influence of eddy viscosity on internal waves.

In the following we consider the simplest case when the influence of the Earth's rotation is disregarded. We may then write the equations of motion in the following form:

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial z}=\frac{1}{\rho}\frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right),$$

$$g+\frac{1}{\rho}\frac{\partial p}{\partial z}=0,$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

$$\frac{\partial \rho}{\partial t}+w\frac{\partial \rho}{\partial z}=0.$$

Making the same assumptions as earlier, we put

$$\rho=\rho_0(z)+\rho_1,$$

$$p=p_0+g\int_z^h \rho_0 dz+p_1, \quad w=\frac{d\zeta}{dt}$$

where ρ_1 and p_1 are small quantities. The equations then take the form:

$$\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}=\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right),$$

$$\frac{\partial u}{\partial x}+\frac{\partial^2 \zeta}{\partial z\partial t}=0,$$

$$g\rho_1+\frac{\partial p_1}{\partial z}=0, \quad \rho_1+\zeta\frac{d\rho_0}{dz}=0.$$

Assuming now

$$u,\ \xi,\ p,\ \text{and}\ \rho \sim e^{i(\sigma t-kx)}$$

we get

$$i\sigma u-ik\frac{p_1}{\rho_0}=\frac{1}{\rho_0}\frac{d}{dz}\left(\eta\frac{du}{dz}\right),$$

$$-iku+i\sigma\frac{d\zeta}{dz}=0,$$

and eliminating ρ_1, we get the equation

$$g\zeta\frac{d\rho_0}{dz}-\frac{dp_1}{dz}=0$$

From these equations we find

$$u=\frac{\sigma}{k}\frac{d\zeta}{dz},$$

and

$$p_1=\frac{\sigma^2}{k^2}\rho_0\frac{d\zeta}{dz}+\frac{i\sigma}{k^2}\frac{d}{dz}\left(\eta\frac{d^2\zeta}{dz^2}\right)$$

giving

$$\frac{\sigma^2}{k^2}\frac{d}{dz}\left(\rho_0\frac{d\zeta}{dz}\right)+\frac{i\sigma}{k^2}\frac{d^2}{dz^2}\left(\eta\frac{d^2\zeta}{dz^2}\right)=g\zeta\frac{d\rho_0}{dz}.$$

The boundary conditions will be

$$\zeta=0\,,\quad u=\frac{\sigma}{k}\frac{d\zeta}{dz}=0\,;\quad z=0$$

at the bottom. At the surface the shear stress disappears giving

$$\eta\frac{d^2\zeta}{dz^2}=0\,.$$

The pressure condition at the surface $p=p_0$ when $z=h+\zeta$ gives

$$g\rho_0\zeta-p_1=0$$

or

$$\frac{\sigma^2}{k^2}\rho_0\frac{d\zeta}{dz}+\frac{i\sigma}{k^2}\frac{d}{dz}\left(\eta\frac{d^2\zeta}{dz^2}\right)-g\rho_0\zeta=0\,;\quad z=h\,.$$

The problem is thus to find an integral of the differential equation

(1)
$$\frac{d^2}{dz^2}\left(\eta\frac{d^2\zeta}{dz^2}\right)-i\sigma\left[\frac{d}{dz}\left(\rho\frac{d\zeta}{dz}\right)-\frac{gk^2}{\sigma^2}\zeta\frac{d\rho}{dz}\right]=0$$

satisfying the boundary conditions:

(2)
$$\zeta=0\,,\quad \frac{d\zeta}{dz}=0\,;\quad z=0$$

and

$$\eta\frac{d^2\zeta}{dz^2}=0$$

(3)
$$\frac{d}{dz}\left(\eta\frac{d^2\zeta}{dz^2}\right)-i\sigma\left[\rho\frac{d\zeta}{dz}-g\frac{k^2}{\sigma^2}\rho\zeta\right]=0\,;\quad z=h\,.$$

The subscript zero has been dropped, and henceforth we shall use it to designate the value at the bottom, $z=0$.

At the first we shall treat a special case where the density and the coefficient of eddy viscosity are given by the formulae:

$$\rho=\rho_0(1-\alpha z)^{1/2}$$

$$\eta=\eta_0(1-\alpha z)^{3/2}$$

In this case the differential equation can be reduced to an equation with constant coefficients.

We shall prefer to write the equations in the form

$$\rho^2=\rho_0^2-\frac{z}{h}(\rho_0^2-\rho_1^2)$$

and

$$\eta=\eta_0\left(\frac{\rho}{\rho_0}\right)^3.$$

Then

$$\frac{d\rho}{dz}=-\frac{\rho_0^2-\rho_1^2}{2\rho h},$$

and if we introduce as independent variable

$$x=\frac{\rho_0-\rho}{\rho_0-\rho_1}$$

we get

$$\frac{dx}{dz}=\frac{\rho_0+\rho_1}{2h\rho}.$$

The differential equation may then be written in the form:

$$\frac{d^4\zeta}{dz^4}-\frac{i\sigma h'^2}{\nu_0}\left[\frac{d^2\zeta}{dz^2}+\frac{k^2}{\sigma^2}gh'\frac{\rho_0-\rho}{\rho_0}\zeta\right]=0 \tag{5}$$

where

$$h'=\frac{2\rho_0 h}{\rho_0+\rho_1},\quad \nu_0=\frac{\eta_0}{\rho_0}.$$

The boundary conditions are

$$\zeta=0,\ \frac{d\zeta}{dx}=0\ ;\ x=0,$$

and

$$\frac{d^2\zeta}{dx^2}+\frac{\rho_0-\rho_1}{\rho_1}\frac{d\zeta}{dx}=0\,,$$

$$\frac{d^3\zeta}{dx^3}-\frac{i\sigma h'^2}{\nu_0}\left[\frac{d\zeta}{dx}-\frac{k^2}{\sigma^2}gh'\frac{\rho_1}{\rho_0}\zeta\right]=0\;;\quad x=1\,.$$

Since the differential equation is of the fourth order with constant coefficients it will be equivalent with two second order equations of the form

$$\frac{d^2w}{dz^2}-\gamma^2w=0$$

where γ is a root of the equation

(6) $$\gamma^4-\frac{i\sigma h'^2}{\nu_0}\left(\gamma^2+\frac{k^2}{\sigma^2}gh'\frac{\rho_0-\rho_1}{\rho_0}\right)=0$$

The equation and the boundary conditions may then be satisfied if we take two solutions satisfying the surface conditions

$$\frac{\rho_0-\rho_1}{\rho_0}\frac{dw}{dx}+\gamma^2w=0$$

and put

(7) $$\zeta=\frac{w_1(x)}{\left(\dfrac{dw_1}{dx}\right)_0}-\frac{w_2(x)}{\left(\dfrac{dw_2}{dx}\right)_0}\,.$$

The bottom condition $d\zeta/dx=0$, is then automatically satisfied, and the other bottom condition $\zeta=0$, then gives the equation

(8) $$\frac{w_1(0)}{\left(\dfrac{dw_1}{dx}\right)_0}-\frac{w_2(0)}{\left(\dfrac{dw_2}{dx}\right)_0}=0\,.$$

This equation together with equation (6) determines the eigenvalue k.

In the actual case we find

$$w=\frac{a\cosh\gamma(1-x)+\gamma\sinh\gamma(1-x)}{\gamma(a\sinh\gamma+\gamma\cosh\gamma)}$$

where

$$a=\frac{\rho_0-\rho_1}{\rho_1}\,.$$

From equation (6) we get

$$\gamma_1^{\,2}+\gamma_2^{\,2}=\frac{i\sigma h'^2}{\nu_0}$$

and

$$\gamma_1^2\gamma_2^2 = -\frac{i\sigma h'^2}{\nu_0} gh' \frac{\rho_0 - \rho_1}{\rho_0}.$$

Since $\sigma h'^2/\nu_0$ is ordinarily a large quantity, we see that the absolute value of one root, say γ_2, will be large in comparison to the other, and corresponding to this root we have approximately

$$w_2(x) \sim \frac{e^{-\gamma_2 x}}{\gamma_2}.$$

This function will decrease rapidly with x, and therefore the influence will be restricted to the bottom layer.

The function $w_1(x)$ corresponding to the other root γ_1 will then be responsible for the effects of viscosity in the upper layers, where the rapid change of velocity with depth causes shear stresses. This is the reason why the approximate solution in section 5 may give some indication of the influence of eddy viscosity on internal waves.

Proceeding now to numerical examples, we have made two calculations with the values

$$\frac{\sigma h'^2}{\nu_0} = 722 \quad \text{and} \quad \frac{\sigma h'^2}{\nu_0} = 242.$$

For $\sigma h'^2/\nu_0 = 242$ we get the following values of γ_1 and γ_2:

	γ_1	γ_2
0	0.0019+i 0.0792	11 (1+i)
1	0.1512+i 3.1923	11.2237+i10.7364
2	0.2631+i 6.6121	11.9611+i 9.9708
3	0.2871+i 9.9386	13.3151+i 8.8695
5	0.2469+i13.2078	15.2882+i 7.7027

The eigenvalues k are then given in the following table as kc/σ, where c is the velocity of propagation in the frictionless case.

0	1.0228−i0.0245
1	1.0443−i0.0715
2	1.0451−i0.1369
3	1.0547−i0.2439
4	1.0918−i0.3826

Corresponding to the value $\sigma h'^2/\nu_0 = 722$, we get the following values of γ_1 and y_2:

	γ_1	γ_2
0	0.0011+i 0.0785	19 (1+i)
1	0.0862+i 3.2272	19.1301+i18.8562
2	0.1661+i 6.4579	19.5288+i18.4308
3	0.2311+i 9.6991	20.2197+i17.7431
4	0.2720+i12.9482	21.2359+i16.8336

In the following table we then give the values of kc/σ.

0	$1.0132-i0.0138$
1	$1.0261-i0.0348$
2	$1.0258-i0.0561$
3	$1.0264-i0.0915$
4	$1.0279-i0.1410$

In the case just considered the eddy viscosity was chosen in a manner which is in accordance with the hypothesis

$$\eta = -\frac{fc^2\rho^2}{g\dfrac{d\rho}{dz}}.$$

In the general case we shall have to solve the differential equation

$$\frac{d^2}{dz^2}\left(\eta\frac{d^2\zeta}{dz^2}\right) - i\sigma\left[\frac{d}{dz}\left(\rho\frac{d\zeta}{dz}\right) - g\frac{k^2}{\sigma^2}\zeta\frac{d\rho}{dz}\right] = 0 \tag{9}$$

with the boundary conditions

$$\eta\frac{d^2\zeta}{dz^2} = 0\,,$$

$$\frac{d}{dz}\left(\eta\frac{d^2\zeta}{dz^2}\right) - i\sigma\left[\rho\frac{d\zeta}{dz} - \frac{gk^2}{\sigma^2}\rho\zeta\right] = 0\,; \quad z = h$$

and

$$\zeta = 0\,, \quad \frac{d\zeta}{dz} = 0\,; \quad z = 0\,.$$

The solution of this problem is greatly simplified by the introduction of the hypotesis above.

In fact, consider the equation

$$\frac{d}{dz}\left(\rho\frac{dw}{dz}\right) - \lambda w\frac{d\rho}{dz} = 0\,, \tag{10}$$

or

$$\frac{\rho}{\dfrac{d\rho}{dz}}\frac{d^2w}{dz^2} + \frac{dw}{dz} - \lambda w = 0\,.$$

Differentiating this equation, it may be written in the form

$$\frac{1}{\rho}\frac{d}{dz}\left[\frac{\rho^2}{\dfrac{d\rho}{dz}}\frac{d^2w}{dz^2}\right] - \lambda\frac{dw}{dz} = 0\,,$$

or if we put

$$\frac{\rho^2}{\frac{d\rho}{dz}} = -\frac{g\eta}{fc^2},$$

$$\frac{d}{dz}\left(\eta\frac{d^2w}{dz^2}\right) + \frac{fc^2\lambda}{g}\rho\frac{dw}{dz} = 0$$

and

$$\frac{d^2}{dz^2}\left(\eta\frac{d^2w}{dz^2}\right) + \frac{fc^2\lambda}{g}\frac{d}{dz}\left(\rho\frac{dw}{dz}\right) = 0$$

or

$$\frac{d^2}{dz^2}\left(\eta\frac{d^2w}{dz^2}\right) = -\frac{fc^2\lambda}{g}w\frac{d\rho}{dz}.$$

Comparing this with equation (9) above, we see that W is a solution of (9) if:

$$\frac{fc^2\lambda^2}{g} + i\sigma\left(\lambda - \frac{gk^2}{\sigma^2}\right) = 0 \tag{11}$$

The differential equation (9) is then equivalent to two equations of the form (9) if λ is a solution of the second order equation (10). We may then put

$$\zeta = Aw_1(z) + Bw_2(z)\,, \,.$$

and the surface conditions are satisfied if

$$\frac{dw}{dz} - \lambda w = 0\;;\quad z = h\,.$$

The bottom conditions then give

$$Aw_1(0) + Bw_2(0) = 0$$

$$A\left(\frac{dw_1}{dz}\right)_0 + B\left(\frac{dw_2}{dz}\right)_0 = 0$$

which is satisfied if the determinant

$$\begin{vmatrix} w_1(0) & w_2(0) \\ \left(\frac{dw_1}{dz}\right)_0 & \left(\frac{dw_2}{dz}\right)_0 \end{vmatrix} = 0\,.$$

This equation determines the complex eigenvalue k. The problem is thus reduced to the integration of the differential equation

$$\frac{d}{dz}\left(\rho\frac{dw}{dz}\right) - \lambda w\frac{d\rho}{dz} = 0 \tag{12}$$

with the surface condition

$$\frac{dw}{dz} - \lambda w = 0 .$$

Since λ is complex, this cannot be done by a direct numerical integration, but we may express w by a series of eigenfunctions $w_n(z)$, which are solutions of the differential equation

$$\frac{d}{dz}\left(\rho\frac{dw_n}{dz}\right) - \beta_n w_n \frac{d\rho}{dz} = 0 \tag{13}$$

with the boundary conditions

$$\frac{dw_n}{dz} - \beta_n w_n = 0 \ ; \quad z = h \, , \quad \frac{dw_n}{dz} = 0 \ ; \quad z = 0 \, .$$

Let

$$w(z) = A_0 + \Sigma A_n w_n(z)$$

and let

$$L(f) \ = (\rho f)_h - \int_0^h f\frac{d\rho}{dz}dz$$

then

$$L(w) \ = (\rho w)_h - \int_0^h w\frac{d\rho}{dz}dz = (\rho w)_h - \frac{1}{\lambda}\rho\frac{dw}{dz}\Big|_0^h$$

$$= \frac{1}{\lambda}\left(\rho\frac{dw}{dz}\right)_0$$

and

$$L(w_n) \ = \frac{1}{\beta_n}\left(\rho\frac{dw_n}{dz}\right)_0 = 0$$

$$L(w_n{}^2) = \frac{1}{\beta_n}\int_0^h \rho\left(\frac{dw_n}{dz}\right)^2 dz = \frac{\tilde{\omega}_n}{\beta_n} .$$

By a combination of the two equations (12) and (13) we get

$$\rho\left[w_n\frac{dw}{dz} - w\frac{dw_n}{dz}\right]_0^h = (\lambda - \beta_n)\int_0^h ww_n\frac{d\rho}{dz}dz \, ,$$

this gives

$$(\lambda - \beta_n)L(ww_n) = \left(\rho w_n\frac{dw}{dz}\right)_0 .$$

Applying these results to the series we get

$$L(w) = A_0 L(1) \, ,$$

$$L(ww_n) = A_n L(w_n^2) \, ,$$

or

$$L(1) = \rho_0$$

$$L(w) = \frac{1}{\lambda}\left(\rho \frac{dw}{dz}\right)_0$$

which gives

$$A_0 = \frac{1}{\lambda}\left(\frac{dw}{dz}\right)_0$$

$$A_n = \frac{\beta_n}{\tilde{\omega}_n} \frac{\rho w_n \left(\frac{dw}{dz}\right)_0}{\lambda - \beta_n}$$

or

$$w = \left(\frac{dw}{dz}\right)_0 \left[\frac{1}{\lambda} + \Sigma \frac{\beta_n (\rho w_n)_0 w_n(z)}{\tilde{\omega}_n (\lambda - \beta_n)}\right]$$

If we take

$$\left(\frac{dw}{dz}\right)_0 = 1$$

we get

$$w_1(z, \lambda_1) = \frac{1}{\lambda_1} + \Sigma \frac{\beta_n (\rho w_n)_0 w_n(z)}{\tilde{\omega}_n (\lambda_1 - \beta_n)}$$

$$w_2(z, \lambda_2) = \frac{1}{\lambda_2} + \Sigma \frac{\beta_n (\rho w_n)_0 w_n(z)}{\tilde{\omega}_n (\lambda_2 - \beta_n)}$$

and

$$w_1 - w_2 = (\lambda_2 - \lambda_1)\left[\frac{1}{\lambda_1 \lambda_2} + \Sigma \frac{\beta_n (\rho w_n)_0 w_n(z)}{\tilde{\omega}_n (\lambda_1 \lambda_2 - \beta_n(\lambda_1 + \lambda_2) + \beta_n^2)}\right]$$

but since λ_1 and λ_2 are roots of the second order equation

$$\lambda^2 + \frac{i\sigma g}{fc^2}\left(\lambda - \frac{gk^2}{\sigma^2}\right) = 0 \, ,$$

we have

$$\lambda_1 + \lambda_2 = -\frac{i\sigma g}{fc^2} \, , \quad \lambda_1 \lambda_2 = -\frac{i\sigma g}{fc^2} g \frac{k^2}{\sigma^2}$$

and

$$\zeta = \frac{\lambda_2 - \lambda_1}{-i\alpha}\left[\frac{1}{X} + \Sigma \frac{\beta_n (\rho w_n)_0 w_n(z)}{\tilde{\omega}_n\left(X - \beta_n + \frac{i\beta_n^2}{\alpha}\right)}\right]$$

where

$$\alpha = \frac{\sigma g}{fc^2}, \quad X = \frac{gk^2}{\sigma^2}.$$

The bottom condition then gives

$$\frac{1}{X} + \Sigma \frac{\beta_n (\rho w_n^2)_0}{\tilde{\omega}_n\left(X - \beta_n + \frac{i\beta_n^2}{\alpha}\right)} = 0.$$

The equation may be solved by Newtons method.

The theory has been applied to the conditions at St. I. 1949. We have calculated six eigenfunctions corresponding to the eigenvalues 0.222, 0.63, 1·38, 2.87, 4.86 and 7.00.

Since we have only retained these terms in the series, it is inadviseable to compute more than the two first roots of the equation.

Two alternative solutions have been computed, one with the assumption

$$\frac{\sigma g}{fc^2} = \alpha = 6.9 \quad \text{and the other with} \quad \alpha = 27.5.$$

The first value corresponds to an eddy viscosity which has the value $\eta = 2$ in the surface layer, but which increases rapidly with depth. The other assumption gives values of the eddy viscosity which are approximately one fourth of the values in the first case.

With $\alpha = 6{,}9$, the equation which is to be solved is

$$\frac{0.001}{X} + \frac{1.22465}{X - 0.222 + i0.00714} + \frac{3.64895}{X - 0.63 + i0.05752} +$$

$$\frac{5.47883}{X - 1.38 + i0.276} + \frac{2.36629}{X - 2.87 + i1.19375} +$$

$$\frac{2.81725}{X - 4.86 + i3.42313} + \frac{5.49237}{X - 7 + i7.10144} = 0.$$

The zero order root X_0, corresponding to the ordinary tidal wave, is very small, and will not be considered here. For the first order root X_1 we find the value:

$$X_1 = \frac{gk^2}{\sigma^2} = 0.29151 - i0.02076.$$

Since

$$k = \frac{\sigma}{\kappa}(1 - ip)$$

where κ is the velocity of propagation, we find

$$p = 0.03556 \quad \text{and} \quad \kappa = 57.93 \text{ cm/sec.}$$

When friction is disregarded, the corresponding eigenvalue is 0.275 and the velocity of propagation is $c = 59.76$ cm/sec.

The approximate solution indicated in chapter 5 gives

$$X = 0.275 - i\frac{0.275^2}{6.9}$$

from which we find

$$p = 0.01992 \quad \text{and} \quad \kappa = 59.745 \text{ cm/sec.}$$

The second root $X_2 = 0.92042 - i0.15669$, giving

$$p = 0.08450 \quad \text{and} \quad \kappa = 32.85 \text{ cm/sec.}$$

The corresponding eigenvalue without friction is 0.857 and $c = 33.55$ cm/sek. The approximate solution then gives:

$$p = 0.06186 \quad \text{and} \quad \kappa = 33.44 \text{ cm/sec.}$$

The exponent of damping seems to be too large since even the approximate solution would give a reduction of the amplitude of 23% when the distance is 10 km.

The second assumption, $\alpha = 27.5$, gives the equation:

$$\frac{0.001}{X} + \frac{1.22465}{X - 0.222 + i0.00179} + \frac{3.64895}{X - 0.63 + i0.01443}$$
$$+ \frac{5.47883}{X - 1.38 + i0.06925} + \frac{2.36629}{X - 2.87 + i0.29952}$$
$$+ \frac{2.8172}{X - 4.86 + i0.85889} + \frac{5.49237}{X - 7.00 + i1.78182} = 0 .$$

From this equation we find

$$X_1 = 0.28983 - i0.00551$$

giving $$p = 0.00950 \quad \text{and} \quad \kappa = 58.21 \text{ cm/sec.}$$

The approximate solution would give

$$p = 0.00500 \quad \text{and} \quad \kappa = 59.76 \text{ cm/sec.}$$

The second root is

$$X_2 = 0.91516 - i0.04060$$

giving $$p = 0.02217 \quad \text{and} \quad \kappa = 32.75 \text{ cm/sec.}$$

The approximate solution gives here

$$p = 0.01588 \quad \text{and} \quad \kappa = 33.85 \text{ cm/sec.}$$

The approximate solution would give a reduction of the amplitude of some 7.3% over a distance of 10 km.

The hypothesis

$$\eta = -\frac{fc^2\rho^2}{g\frac{d\rho}{dz}}$$

seems to answer well to the conditions in the upper layers where the density gradient is large, but gives too large values in the vicinity of the bottom. In the approximate solution the two surface conditions are satisfied and one of the bottom conditions. The other bottom condition, $u=0$, can not be satisfied, but the bottom friction is probably rather small, and so the error will also be small if we allow slipping motion at the bottom. In this manner there will be some compensation for the assumed large value of the eddy viscosity in the bottom layer.

At last we shall deduce two formulae which give expressions for the energy and the dissipation when friction is included.

The vertical oscillation is a solution of the differential equation

$$\frac{d^2}{dz^2}\left(\eta\frac{d^2\zeta}{dz^2}\right)-i\sigma\frac{d}{dz}\left(\rho\frac{d\zeta}{dz}\right)+i\sigma X\zeta\frac{d\rho}{dz}=0$$

Consider now the conjugate imaginary function $\bar{\zeta}$. It will be a solution of the corresponding equation

$$\frac{d^2}{dz^2}\left(\eta\frac{d^2\bar{\zeta}}{dz^2}\right)+i\sigma\frac{d}{dz}\left(\rho\frac{d\bar{\zeta}}{dz}\right)-i\sigma\bar{X}\bar{\zeta}\frac{d\rho}{dz}=0 .$$

If we multiply the first equation by $\bar{\zeta}$ and the second by ζ and subtract, we get the equation

$$\int_0^h\left\{\bar{\zeta}\left[\frac{d^2}{dz^2}\left(\eta\frac{d^2\zeta}{dz^2}\right)-i\sigma\frac{d}{dz}\left(\rho\frac{d\zeta}{dz}\right)\right]-\zeta\left[\frac{d^2}{dz^2}\left(\eta\frac{d^2\bar{\zeta}}{dz^2}\right)+i\sigma\frac{d}{dz}\left(\rho\frac{d\bar{\zeta}}{dz}\right)\right]\right\}dz$$

$$=-i\sigma(X+\bar{X})\int_0^h\bar{\zeta}\zeta\frac{d\rho}{dz}dz .$$

Integrating the left side by parts and taking the boundary conditions into account, we deduce the equation

$$i\sigma(X+\bar{X})\left[(\rho\zeta\bar{\zeta})_h-\int_0^h\zeta\bar{\zeta}\frac{d\rho}{dz}dz\right]=2i\sigma\int_0^h\rho\frac{d\zeta}{dz}\frac{d\bar{\zeta}}{dz}dz$$

or since $X=\frac{gk^2}{\sigma^2}$, $\bar{X}=\frac{g\bar{k}^2}{\sigma^2}$ and $u=\frac{\sigma}{k}\frac{d\zeta}{dz}$ we get

$$(k^2+\bar{k}^2)g\left[(\rho\zeta\bar{\zeta})_h-\int_0^h\zeta\bar{\zeta}\frac{d\rho}{dz}dz\right]=2k\bar{k}\int_0^h\rho u\bar{u}dz .$$

The expression

$$g\left[(\rho\zeta\bar{\zeta})_h-\int_0^h\zeta\bar{\zeta}\frac{d\rho}{dz}dz\right]$$

represents $2E_p$, where E_p is the potential energy of the wave, while

$$\int_0^h\rho u\bar{u}dz=2E_k$$

where E_k is the kinetic energy.

In a similar manner we get

$$\int_0^h\left\{\bar{\zeta}\left[\frac{d^2}{dz^2}\left(\eta\frac{d^2\zeta}{dz^2}\right)-i\sigma\frac{d}{dz}\left(\rho\frac{d\zeta}{dz}\right)\right]+\zeta\left[\frac{d^2}{dz^2}\left(\eta\frac{d^2\bar{\zeta}}{dz^2}\right)-i\sigma\frac{d}{dz}\left(\rho\frac{d\bar{\zeta}}{dz}\right)\right]\right\}dz$$

$$=i\sigma[X-\bar{X}]\int_0^h\zeta\bar{\zeta}\frac{d\rho}{dz}dz\,.$$

From this equation we deduce the formula

$$i\sigma(X-\bar{X})\left[(\rho\zeta\bar{\zeta})_h-\int_0^h\zeta\bar{\zeta}\frac{d\rho}{dz}dz\right]=2\int_0^h\eta\frac{d^2\zeta}{dz^2}\frac{d^2\bar{\zeta}}{dz^2}dz$$

or

$$2i\sigma(k^2-\bar{k}^2)E_p=2k\bar{k}\int_0^h\eta\frac{du}{dz}\frac{d\bar{u}}{dz}dz\,.$$

If we put $k=\frac{\sigma}{\kappa}(1-ip)$, $\bar{k}=\frac{\sigma}{\kappa}(1+ip)$, we get

$$2\sigma pE_p=(1+p^2)F$$

where

$$2F=\int_0^h\eta\frac{du}{dt}\frac{d\bar{u}}{dz}dz$$

represents the dissipation function, and

$$(1-p^2)E_p=(1+p^2)E_k$$

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