Det Norske Videnskaps-Akademi i Oslo

Geophysica Norvegica

GEOFYSISKE PUBLIKASJONER

JON HELGE KNUTSEN

Approximate Analytical Solutions to the Non-Divergent Barotropic Vorticity Equation in Spectral Form

VOL. 30 NO. 1 . 1973
UNIVERSITETSFORLAGET

Geophysica Norvegica

is a journal of geophysics, issued under the auspices of the Norwegian Academy of Science and Letters in Oslo

PUBLISHER

Universitetsforlaget: P. O. Box 307, Blindern, Oslo 3, Norway.
P. O. Box 142, Boston, Mass. 02113, U.S.A.

SUBSCRIPTION

Geophysica Norvegica (Geofysiske Publikasjoner) is published at irregular intervals. Order from the Publisher, Universitetsforlaget.

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Approximate Analytical Solutions to the Non–Divergent Barotropic Vorticity Equation in Spectral Form

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Knudsen, J. H. Approximate Analytical Solutions to the Non-Divergent Barotropic Vorticity Equation in Spectral Form. *Geophysica Norvegica*, Vol. 30, No. 1, 1972.

The non-divergent barotropic vorticity equation in spectral form has been investigated and problems related to the precision attainable in predictions with such a model on a short time basis discussed. It has been shown that, in a system containing 45 variables, individual 'natural modes' (i.e. waves with constant amplitude and phase speeds) have low predictive value, but may be used as a substitute for more exact methods for time spans up to 24 hours. Algebraic approximate solutions to the deviations from the natural modes are also valid for time spans up to 24 hours; hence approximate analytic solutions valid up to 24 hours seem possible.

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INTRODUCTION

The non-divergent barotropic vorticity equation, long a popular model for numerical predictions, gained renewed interest in the early sixties with works by Platzman (1960), Baer & Platzman (1961), and others. Because of the non-divergent property of the flow an equation for the change of the stream function could be found; this stream function could be represented by an expansion in spherical (surface) harmonic functions (Haurwitz 1940, Blinova 1943 – see I. A. Kibel' (1963) p. 103ff. – Silberman 1954). Such a transformation had many virtues; it was possible to find analytical solutions to some severely truncated expansions (Lorentz 1965). It was also shown that kinetic energy was preserved exactly in the equation, regardless of the degree of truncation (Elsaesser 1966). The model has been used by Eliasen & Machenhauer (1965) to explain important observational evidence concerning the fluctuations of planetary flow. Platzman (1960) found that hemispheric spectral predictions could be done with expansions containing only terms with a zero on equator. These odd expansions make hemispheric spectral predictions possible, and Elsaesser (1966) showed that such predictions compared favorably with barotropic predictions using grid point methods.

In this paper some simple results are presented on the discrepancies between the solution of the non-divergent barotropic vorticity equation in spectral form and an expansion which attains the same initial value but propagates the waves with constant amplitude and phase speed. Such an expansion seems a natural starting point for approximations of the non-linear solution. This approximate solution is improved by making use of the predicted deviations from the trial solution. The form of the differential equations for these

A. Eliassen submitted this paper to the Norwegian Academy of Science and Letters in Oslo, 15th September 1972.

deviations suggests a certain structure of the influence from the other waves through interaction. This structure is investigated for two wave vectors and the results compared.

In a forthcoming paper (Knudsen 1972) the writer will present results on how uncertainties of the variables in a hemispheric model propagate in time. The present paper uses the same technique and may be considered an introduction to the latter.

1. MATHEMATICAL MODEL

If the atmosphere is considered to be a two-dimensional incompressible, horizontally homogeneous fluid, the governing equations for atmospheric flow reduce to the two-dimensional vorticity equation

$$\frac{\partial}{\partial t}(\zeta+f) = -\mathbf{V}\cdot\nabla(\zeta+f)$$

(see, e.g. Thompson 1961, p. 60). Here ζ is the vertical component of the absolute vorticity and f the coriolis parameter. This equation may be rewritten if the velocity is represented by means of a stream function in spherical harmonic functions. If the stream function is expressed by the orthonormal representation

$$\psi(\theta,\lambda,t) = a^2\omega \sum_{m,n\geq m} K_n^m(t)Y_n^m(\theta,\lambda) \qquad (1)$$

where a is the radius and ω the rate of rotation of the earth, θ the colatitude, λ the logitude and Y_n^m the spherical harmonic function

$$Y_n^m(\theta,\lambda) = \exp(im\lambda)P_n^m(\theta)\,,$$

the time derivatives of each expansion coefficient $K_n^m(t)$ may be expressed by the equation

$$\frac{dK_n^m}{dt} = \frac{2mi\omega}{n(n+1)}K_n^m + \frac{i\omega}{2}\sum\sum\sum\sum K_s^r K_l^j\, H_{snl}^{rmj} \qquad (2)$$

(Silberman 1954). Here $P_n^m(\theta)$ are associated Legendre polynomials of degree n and order m. When $m = 0$ the polynomials are called zonal. If $\psi(\theta,\lambda)$ is any sufficiently smooth function of longitude and colatitude, the expansion coefficients for an orthogonal representation of ψ are given by

$$K_n^m = \frac{1}{4\pi}\int \psi Y_n^{m*}\delta S$$

where the asterisk denotes the complex conjugate function, δS is a surface element on the sphere, and the integration is carried out over the unit sphere. If ψ is real, only the coefficients for $m \geq 0$ need to be determined since

$$K_n^{-m} = (-1)^m K_n^{m*} \quad (m \geq 0)\,. \qquad (3)$$

The functions H_{kns}^{jmr} are a set of constants defined by

$$H_{kns}^{jmr} = \frac{s(s+1)-k(k+1)}{n(n+1)}\int_0^\pi P_n^m\left(jP_k^j\frac{dP_s^r}{d\theta} - rP_s^r\frac{dP_k^j}{d\theta}\right)d\theta$$

in the non-zero case. These interaction coefficients can be shown to be zero unless the following relations are fulfilled:

$$\begin{aligned}
&j^2+r^2 \neq 0\\
&snk \neq 0\\
&(m,n) \neq (-j,k),\ (m,n) \neq (-r,s) \qquad (4)\\
&(|j|-k)^2+(|r|-s)^2 \neq 0\\
&|k-s|<n<k+s\\
&n+k+s = \text{odd integer}
\end{aligned}$$

(Elsaesser 1966). These so-called *selection rules* specify which components may contribute to the time derivatives of the expansion coefficients. Expressed in words, (4) states that any two components may interact provided neither is of zero degree, and they are not zonal, sectional ($j = k$ or $-j = k$), or identical. Through mutual interaction they may contribute to any component whose order is such as to form with their degree a triangle of non-zero area and odd perimeter. A wave vector may contribute to itself by interacting with a zonal component of odd degree. In the non-divergent case K_1^0 remains invariant. It is also seen that if all terms, where the sum of the super- and subscripts is even, initially have vanishing coefficients they will remain zero throughout the subsequent history of the flow (Platzman 1960). Such representation using only odd spherical harmonics may be used in 'hemispheric predictions' since the equator in that case will act as a fixed boundary. Predictions done with only odd spherical harmonics have proved to compare favourably with predictions using grid point methods in physical space (Elsaesser 1966). In such hemispheric predictions initial data are only required on, say, the Northern Hemisphere – this in spite of the fact that the functions in the orthonormal set are defined over the whole sphere.

Because of (3) one may wish to remove the redundancies from (2) by writing (1) in the following manner:

$$\psi(\theta,\lambda,t) = a^2\omega \sum_M \sum_N (A_N^M(t)\cos M\lambda + B_N^M(t)\sin M\lambda)P_N^M(\theta)\,.$$

This process leads to two equations of the same type as (2) but with a more complicated interaction coefficient, viz.

$$\frac{d}{dt}A_N^M = \frac{2M\omega}{N(N+1)}B_N^M + \frac{\omega}{4}\sum_{R,S,J,L} A_S^R\, B_L^J\, K_{SNL}^{RMJ} \tag{5a}$$

$$\frac{d}{dt}B_N^M = -\frac{2M\omega}{N(N+1)}A_N^M + \frac{\omega}{4}\sum_{R,S,J,L} A_S^R\, A_L^J\, L_{SNL}^{RMJ} + \frac{\omega}{4}\sum_{R,S,J,L} B_S^R\, B_L^J\, M_{SNL}^{RMJ} \tag{5b}$$

where

$$K_{LNS}^{JMR} = \begin{cases} H_{SNL}^{RMJ} + H_{SNL}^{-RMJ}(-1)^R - H_{SNL}^{RM-J}(-1)^J \\ \qquad \text{if } R>0,\ J>0,\ M>0 \\ 4H_{LNS}^{JMR} \text{ if } J=0,\ R\geq 0,\ M>0 \end{cases}$$

$$L_{SNL}^{RMJ} = -H_{SNL}^{RMJ} - (-1)^J H_{SNL}^{RM-J} - (-1)^R H_{SNL}^{-RMJ}$$

$$M_{SNL}^{RMJ} = H_{SNL}^{RMJ} - (-1)^J H_{SNL}^{RM-J} - (-1)^R H_{SNL}^{-RMJ}$$

(Knudsen 1971). In K the first pair of the super and subscripts refers to the cosine-coefficient; in all three the middle index refers to the expansion coefficient whose time derivative is sought.

Numerical predictions of the motion of the long waves in the atmosphere using (2) or (5) require the knowledge of the stream function. One possibility is offered by the geostrophic balance equation which expresses a balance between the geometric height field and the stream function. If both the geometric height field and the stream function are represented in spherical harmonics a recursive relationship connects the expansion coefficients from the two fields (Eliasen & Machenhauer 1965). Elsaesser (1966) has suggested a simpler procedure by assuming that the stream function can be found by a simple scaling of the geometric height contours. If f_0 is an average value of the Coriolis parameter, C_N^M, D_N^M expansion coefficients of the geometric height field,

$$\begin{Bmatrix} A_N^M \\ B_N^M \end{Bmatrix} = \frac{g}{a^2\omega f_0}\begin{Bmatrix} C_N^M \\ D_N^M \end{Bmatrix} \tag{6}$$

where g is the constant of gravity. This expression will be used in the following.

2. THE EFFECT OF THE QUADRATIC TERM

If a study of the propagation in time of the effect of initial errors in a non-divergent barotropic model is attempted, two important steps are involved:

1. The selection of a statistical model to represent the 'present state' in the initial state space, and

2. The development of a dynamical model to predict the transformations in time of the uncertain state, and of methods to display the consequences of the uncertainty development.

These two aspects will be dealt with to a greater extent in a forthcoming paper by the present writer. In that paper the importance of the quadratic term in (5) will be shown for the case of statistical dynamical predictions based on the same dynamical model. The importance of the quadratic term in (5) is a legitimate study in itself since it is by no means clear from the outset what orders of magnitude the quadratic terms will have for long wave atmospheric motion. If we introduce in (5)

$$\begin{aligned} A_N^M &= a_N^M \cos\alpha_{M,N}t + b_N^M \sin\alpha_{M,N}t \\ B_N^M &= b_N^M \cos\alpha_{M,N}t - a_N^M \sin\alpha_{M,N}t \end{aligned} \tag{7}$$

where a_N^M, b_N^M are a new set of variables, and

$$\alpha_{M,N} = \frac{2M\omega}{N(N+1)} + \omega H_{1NN}^{OMM} A_1^0\,, \tag{8}$$

we find the following set of equations

$$\begin{aligned}\frac{d}{dt} b_N^M &= \frac{\omega}{4} \sin \alpha_{M,N} t Q_A(M,N) \\ &+ \frac{\omega}{4} \cos \alpha_{M,N} t Q_B(M,N) \\ \frac{d}{dt} a_N^M &= \frac{\omega}{4} \cos \alpha_{M,N} t Q_A(M,N) \\ &- \frac{\omega}{4} \sin \alpha_{M,N} t Q_B(M,N)\end{aligned} \tag{9}$$

Here Q_A and Q_B are the quadratic terms in (5) rewritten in terms of a_N^M, b_N^M. If the right-hand sides of (9) vanish for all M, N in the expansion it means that (5) has a wave solution of constant amplitude and phase speed given by (8). If the right-hand side of (9) is small we may use (7) with constant coefficients as an approximate solution valid over a limited time period and as a basis for second order approximations. Such comparisons and approximations form the main theme of the present paper.

3. RESULTS

Linear versus non-linear prediction.

To investigate the approximation to the solution of (5) offered by (7) with constant coefficients, a comparison was made between the two approaches using data for the geometric height of the 500 mb surface for 12–16 January 1963. Data for the height field at 1200 GMT for the Northern Hemisphere were obtained through the cooperation of Geophysical Fluid Dynamics Laboratories, NOAA, Princeton University. Using least squares an expansion in odd spherical harmonics containing 45 terms was fitted to the geometric height field; the height field expansion coefficients were rescaled to represent the stream field by using (6) (Table I). Comparisons between the solution of (5) and (7) can be made in terms of geometric height by doing the reverse scaling and displaying the height fields on a latitude-longitude grid.

If (5) is integrated numerically with the values in Table I as initial data, one obtains the values in Table II. Here the value of the contributions from the linear and quadratic terms have been listed at initial time, after 24 hours, and after a two-day interval. The constant contribution

$$i\omega K_l^0 H_{lnn}^{0mm} K_n^m$$

has been removed from the quadratic term and added to the linear term. It is seen from Table II that the quadratic contribution is of the same order of magnitude as the linear term and hence one should expect the two solutions to become distinctly different after some time.

A comparison between (5) and (7) in physical space adds more details. If we rescale the solutions to represent geometric height and compute the height at intersections of latitude and longitude circles with 10 degree intervals, the rate at which the two solutions move apart may be measured. Table III shows the root mean squares (RMS) difference between the non-linear and the linear solution (7) at selected epochs throughout the integration; it is shown that after 12 hours the predictions have a RMS difference of 60 metres at 60 degrees latitude. This difference has increased to 111 metres after 24 hours and to 170

Table I. Expansion of stream function for 12 January 1963 (in units of 10^{-5} m^2 sec^{-1})

i	*j*	*A*	*B*
0	1	−1174	
	3	−291	
	5	9	
	7	−25	
	9	−50	
1	2	−8	−124
	4	74	−78
	6	−22	−39
	8	131	−99
	10	−27	53
2	3	30	132
	5	145	69
	7	25	−125
	9	138	212
	11	245	−55
3	4	−152	−160
	6	37	−236
	8	3	−20
	10	39	1
	12	−131	−18
4	5	10	94
	7	37	65
	9	−53	−98
	11	4	−93
	13	−11	63

metres after two days. Up to 36 hours the growth of the difference is approximately linear with time; after that time the increase is less. Without reference to other comparisons these results are difficult to interpret. A natural comparison can be made with the size of non-linear prediction minus initial state. The lower part of Table III

Table II. The contribution (at initial time, after 24 and 48 hours) for the tendency equation for A and B from $\omega\left(\frac{2m}{n(n+1)}+A_l^0 H_{lnn}^{0mm}\right)$ × expansion coefficient and from the remaining quadratic sums (all units 10^{-12} m² sec⁻¹)

		A_j^i		B_j^i	
i	*j*	*Linear*	*Quadratic*	*Linear*	*Quadratic*
0	3	0	−1350		
			−1540		
			599		
	5	0	617		
			948		
			314		
	7	0	4040		
			2800		
			525		
	9	0	2540		
			−34		
			−746		
1	2	−28,900	−1280	1780	−2230
		8290	1640	28,600	−428
		−19,500	172	−18,800	−1670
	4	−4630	−2240	−4370	2090
		−4540	−1430	−1350	3830
		−2730	516	768	3090
	6	−804	−2450	451	5870
		148	377	641	4340
		603	1460	373	−632
	8	−575	−2240	−760	1620
		−654	145	−696	−3730
		−913	−339	−662	−4190
	10	−69	30	−36	−6260
		−11	−2420	−47	−3970
		−11	−1430	−75	3000
2	3	28,700	2190	−6600	−2400
		−14,800	653	−25,400	−4490
		−25,100	867	27,800	−3770
	5	4800	3700	−10,100	10,700
		3260	2500	−14,000	6980
		−3070	854	−15,200	2470
	7	−2930	335	−573	7580
		−1790	135	−166	6740
		−441	408	−41	6040
	9	7	3140	−463	981
		54	575	−507	4430
		204	980	−503	5370

Table II (continued)

		A_j^i		B_j^i	
i	*j*	*Linear*	*Quadratic*	*Linear*	*Quadratic*
2	11	393	1740	174	1570
		280	1440	284	1260
		247	−688	342	226
3	4	−28,600	10,100	27,000	−10,900
		903	8850	33,100	−12,600
		12,900	4540	6330	−13,100
	6	−14,500	8640	−2300	−1990
		−15,000	5950	1320	2160
		−10,400	5370	5450	6420
	8	−342	6960	−44	−107
		−314	6030	−884	1880
		34	3650	−1600	5080
	10	−3	1560	153	−3970
		92	4270	246	−3160
		149	4230	400	65
	12	282	−898	−2080	−3240
		873	1760	−1960	−2550
		1360	3330	−1440	−900
4	5	13,000	−5610	−1440	−2360
		7240	−6840	−5640	436
		1970	−5560	−3820	−470
	7	3040	1450	−1720	2190
		3250	2730	−3690	4300
		3500	−895	−5590	6510
	9	−666	5070	362	−1610
		−585	5470	114	4030
		−196	3050	−116	8760
	11	1320	1520	50	−1150
		1200	1410	347	2400
		854	279	589	1350
	13	−1680	147	299	1590
		−1920	−359	679	1570
		−1900	−923	1270	−99

contains the RMS difference between the solution of (5) minus initial state at various epochs. These comparisons show that after 24 hours we should not expect to see much similarity between predictions from (5) and (7).

Fig. 1 shows the contour maps of geometric height for the 500 mb surface on 12 January 1963 at 12 GMT using the 45 fitted values in Table I. Fig. 2, 3, 4, and 5 show the two solutions after a 12, 24, 36, and 48 hours prediction using (5) and (7). From these maps it is concluded that solutions from (7) eventually move westward at a more rapid rate than the solutions from (5), but it becomes a matter of taste when (7) becomes useless. Within the period of one day (7) extrapolates (5) fairly well and may be used for predic-

Table III. Root-mean-square (rms) differences between (5) and (7) (in metres of geometric height) compared with rms differences between (5) and initial state for various epochs and degrees of latitude

	(5) Minus (7)							
	Degrees latitude							
Hours	*80*	*70*	*60*	*50*	*40*	*30*	*20*	*10*
3	7	14	15	11	14	9	5	5
6	14	34	31	21	26	21	10	11
9	23	45	43	31	40	34	16	21
12	30	61	60	38	53	42	23	24
15	36	76	74	44	66	53	31	30
18	43	92	94	49	77	59	38	36
24	58	120	111	57	97	82	55	47
36	75	170	147	75	160	107	96	64
48	92	206	170	103	170	109	136	75

	(5) Minus initial state			
Hours	*70*	*50*	*30*	*10*
0–24	51	106	117	90
0–36	55	134	139	119
0–48	58	140	136	132

tion; beyond 24 hours the quality of the extrapolation deteriorates rapidly.

Second-order approximation

If expansion coefficients with constant magnitude as analytic expressions prove to be of insufficient accuracy in approximating (5) one is led to try second-order approximations. If the solution for each expansion coefficient of (2) is written

$$K_n^m(t) = \{K_n^m(0)+k_n^m(t)\}\exp(i\alpha_{m,n}t) \qquad (10)$$

where $K_n^m(0)$, the value of K_n^m at $t = 0$, is a constant term and $k_n^m(t)$ a variable term, the actual development of k_n^m can be studied by numerical integration. If (10) is inserted into (2) we get

$$\frac{d}{dt}k_n^m = \frac{i\omega}{2}\sum_{r,s,j,l}(K_s^r(0)+k_s^r(t))\,(K_l^j(0)+k_l^j(t)) \times \exp\{i(\alpha_{r,s}+\alpha_{j,l}-\alpha_{m,n})t\}H_{snl}^{rmj} \qquad (11)$$

where $\alpha_{r,s}$ is given by (8) and the zonal term K_1^0 is excluded from the sum. The initial conditions for the deviations are $k_n^m(0) = 0$ (all m,n). (11) may be linearized by dropping the product term $k_s^r k_l^j$. Such a procedure is legitimate if the product of deviations remains small throughout the integration. As a further point of attack we may separate (11) into its equivalent of (5) and study the tendency at initial time. Write the expansion coefficients of (5) in the form of a vector **Y**; then the linearized equation (11) may be written

$$\dot{\mathbf{Y}} = \mathbf{A}(t)\cdot\mathbf{Y}+\mathbf{c} \qquad (12)$$

Here **A**(t) is a matrix of time-dependent coeffi-

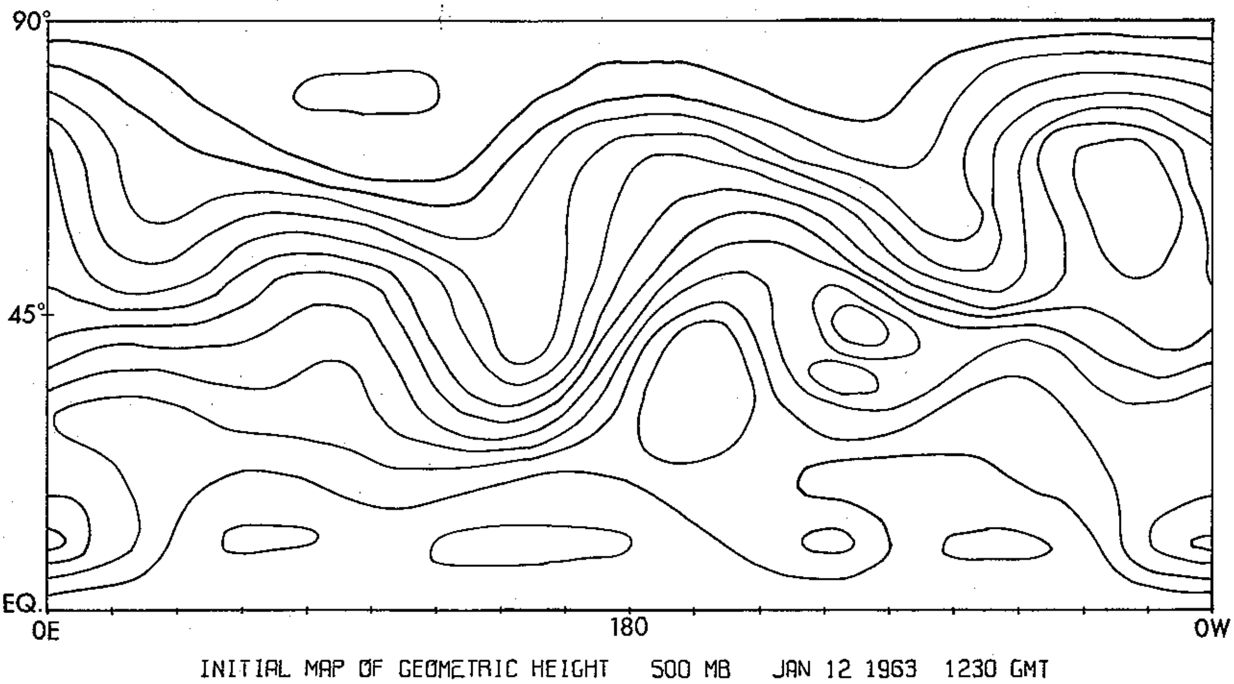

Fig. 1. The 12 January 1963 contour map of fitted geometric height field for the 500 mb surface using 45 variables. Only Northern Hemisphere depicted. Contour distance 80 metres.

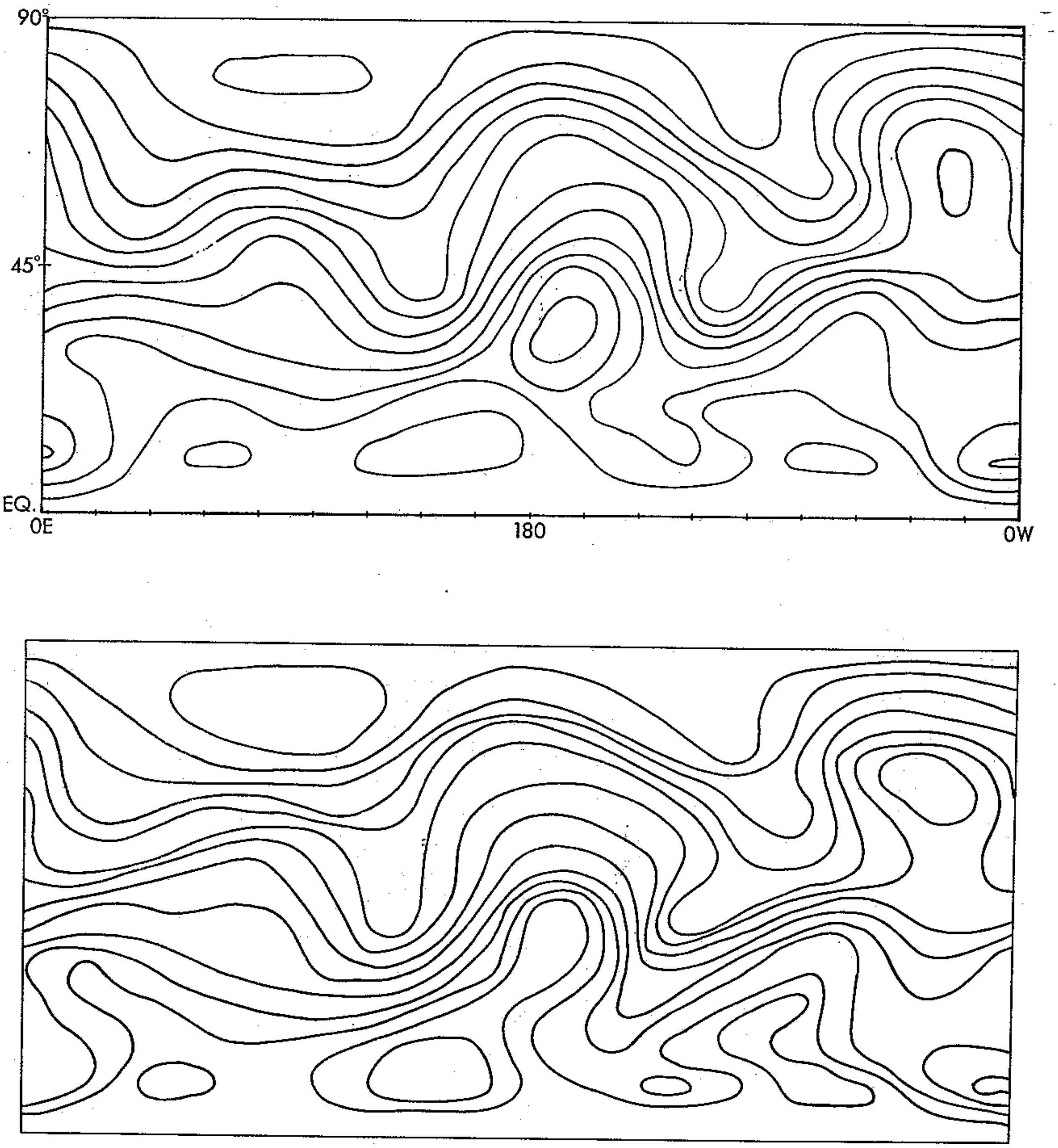

Fig. 2. The predicted height field after 12 hours using (5) (top) and (7) (bottom).

cients and **c** a column vector of constants. If $\mathbf{A}(t)$ is well approximated by $\mathbf{A}(0)$ for $t < t_d$, we may ask ourselves how much insight into (11) one can obtain by solving the following equations with constant coefficients, viz.

$$\dot{\mathbf{Y}} = \mathbf{A}(0)\mathbf{Y} + \mathbf{C} \qquad (13)$$

This system of linear equations has a solution of the form

$$\mathbf{Y} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \ \dots \ + c_s\mathbf{K}_s e^{\lambda_s t} + \mathbf{C}_q \qquad (14)$$

where the eigenvalues $\lambda_i (i = 1, 2, \dots, s)$ are found from the determinantal equation

$$|\mathbf{A}(0) - \lambda \mathbf{I}| = 0 \qquad (15)$$

where **I** is a unit matrix of dimension *sxs*. The

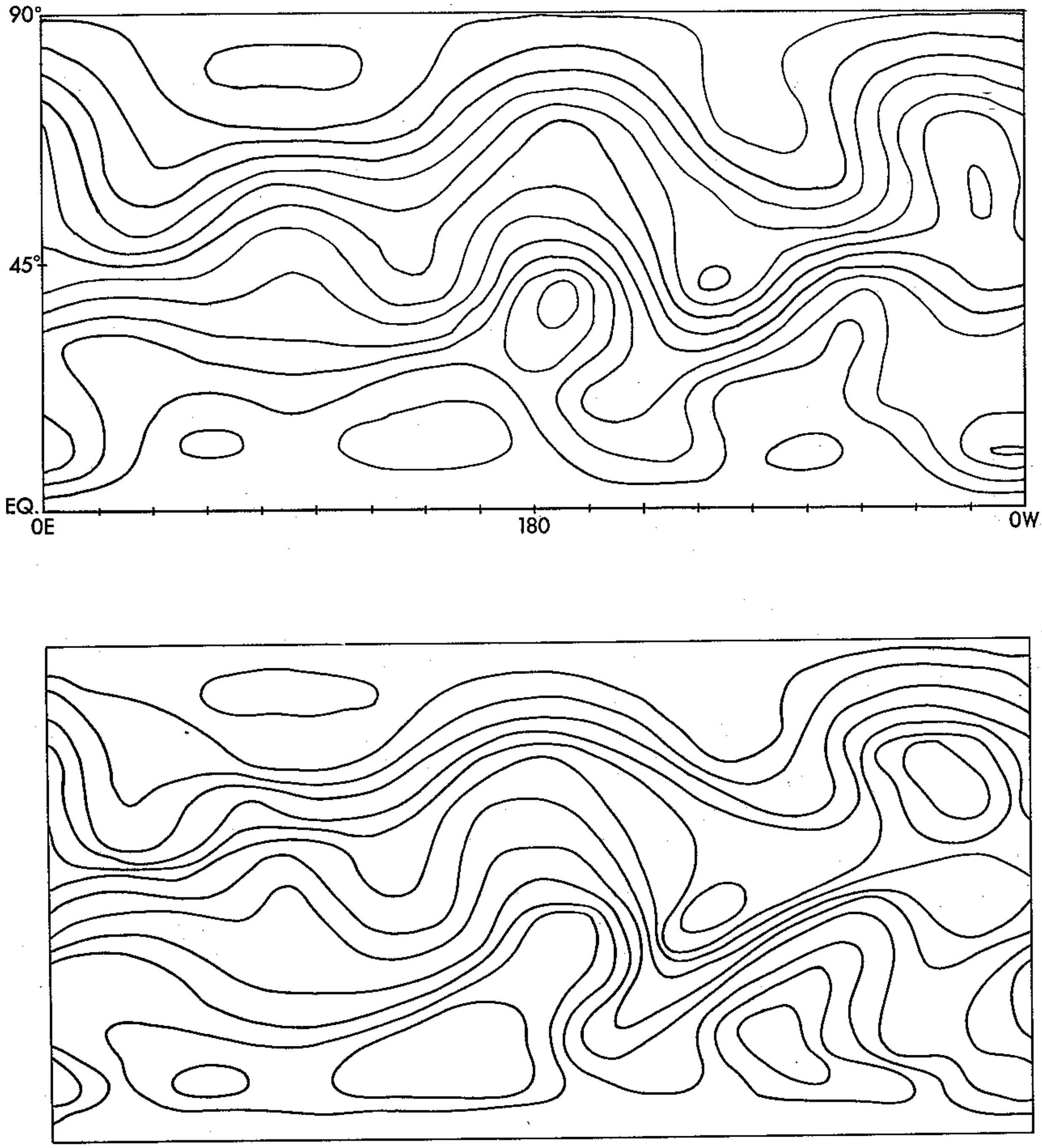

Fig. 3. The predicted height field after 24 hours using (5) (top) and (7) (bottom).

eigenvectors **K** are found by solving the homogeneous system of linear equations

$$\{\mathbf{A}(0)-\lambda\mathbf{I}\}\mathbf{K} = 0 \, .$$

We have investigated the similarity between the non-linear eqs. (11), (12) and (14) by comparing the results using the initial data in Table I. Since K_1^0 is invariant in a non-divergent barotropic model, there are 44 real variables ($s = 44$) in (13) and (14).

The numerical solutions of the non-linear eqs. (11) and (12) were obtained using centred differences. The computation of the complex eigenvalues were performed using a computer program published in the literature (Grad & Brebner 1968); this program computes all the eigenvalues and

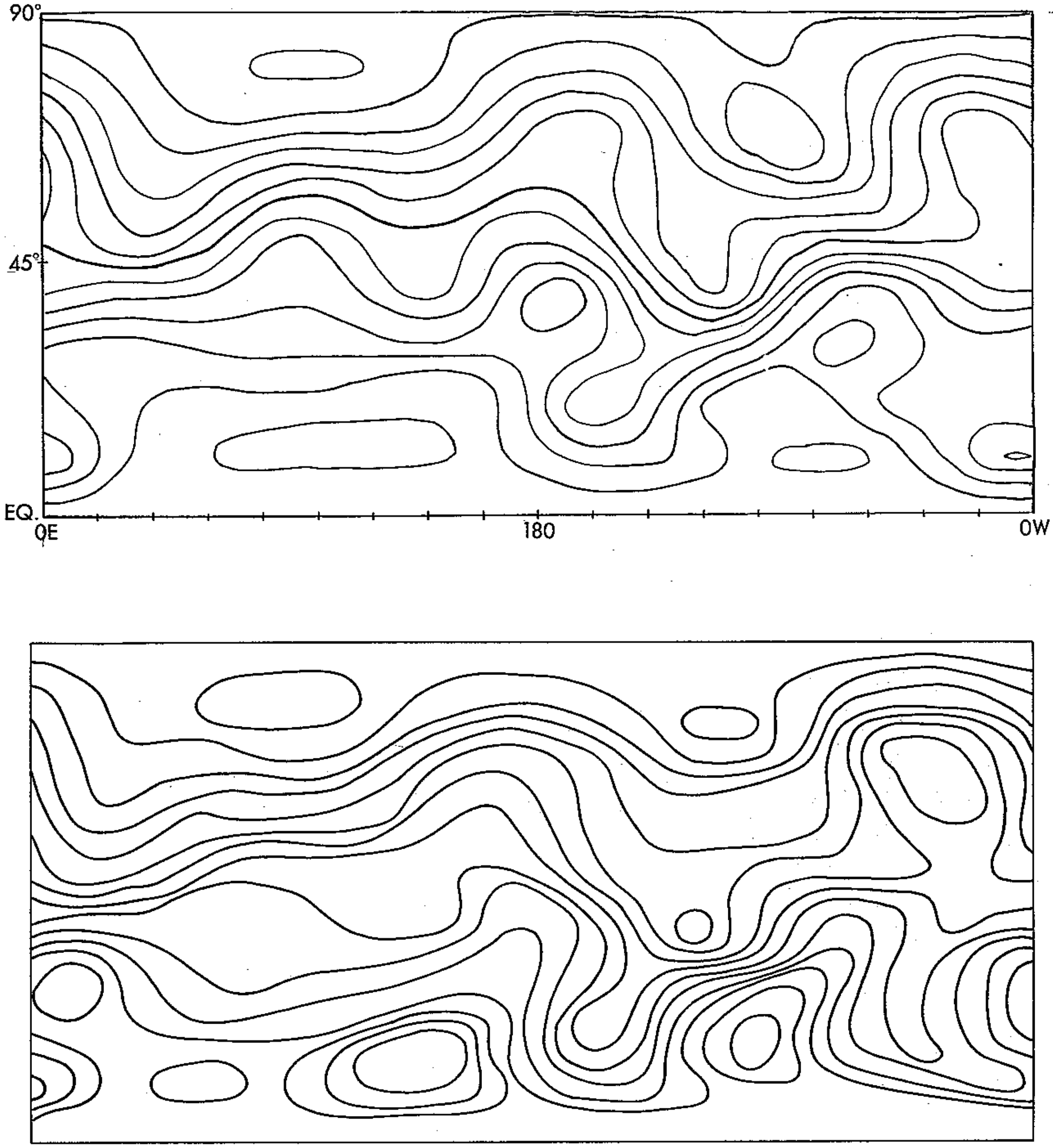

Fig. 4. The predicted height field after 36 hours using (5) (top) and (7) (bottom).

eigenvectors of a real (not necessarily symmetric) matrix. The eigenvalues were computed by the *QR* double step method and the eigenvectors by inverse iteration.

Table IV contains the real and imaginary part of the eigenvalues measured in day^{-1}. There were 4 real positive and 20 pairs of complex eigenvalues; 9 out of the 20 pairs had a positive real part. This means that the approximation provided by (14) contains both exponentially increasing as well as decreasing and oscillating components. The exponentials with positive arguments will eventually dominate the solution, and since the solution of (11) has bounded variation it suggests that the algebraic solution (14) will approximate $k_n^m(t)$ only for a short time. From Table IV it ap-

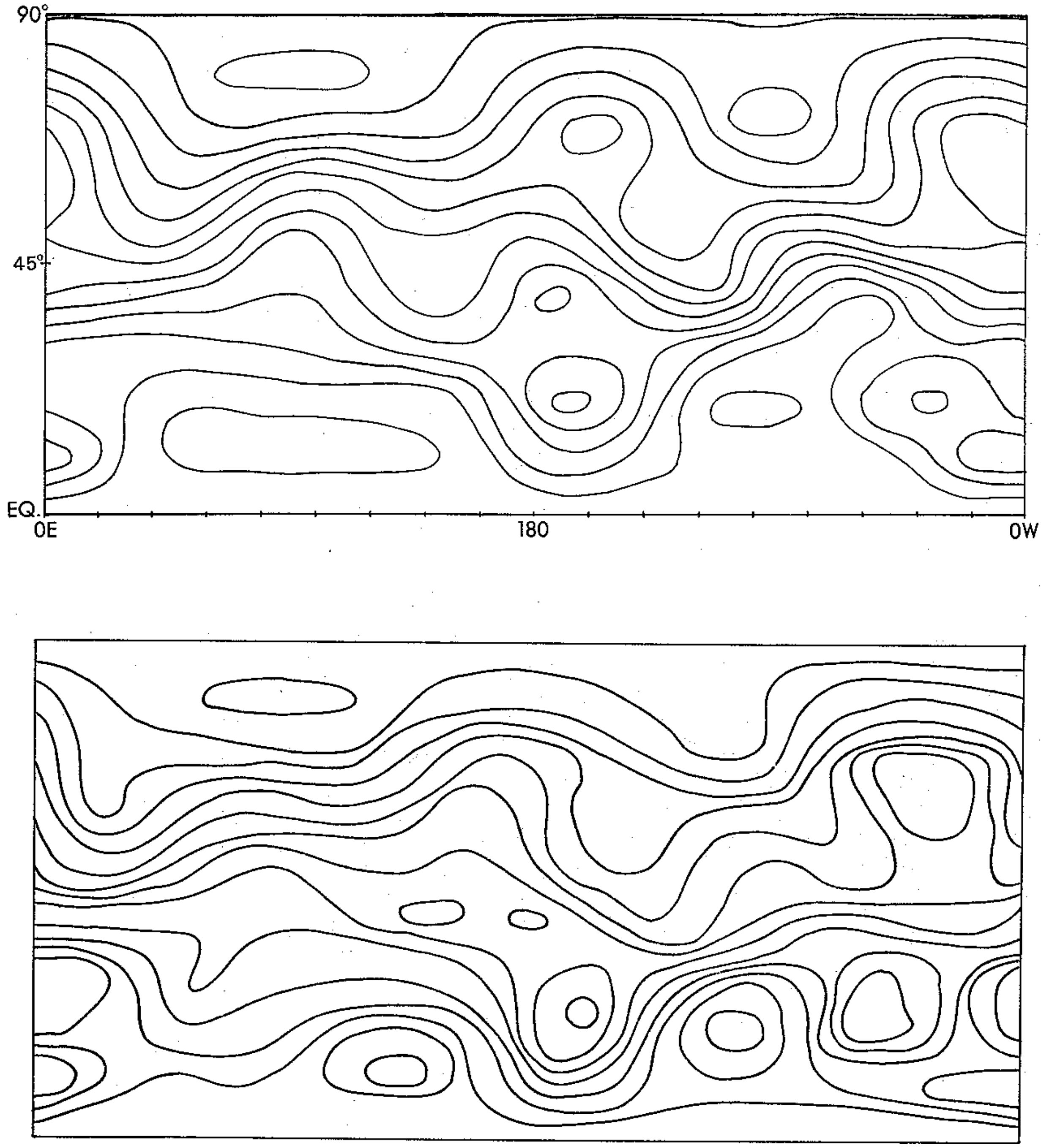

Fig. 5. The predicted height field after 48 hours using (5) (top) and (7) (bottom).

pears that the fastest growing exponential uses 17 days to grow *e*-fold; hence over the first few days one should expect that (14) behaves much like an ordinary periodic function.

An inspection of the graphs of the solutions of the non-linear, linearized and algebraic solution shows that the latter solution of course always represents the initial tendency correctly. Within the first day the non-linear and algebraic solution agree fairly well and the difference between them is always less than the magnitude of the non-linear solution. This suggests that (14) offers a short term improvement compared to (7) alone. After 24 hours the solutions start to move apart. In cases where the non-linear solution reverses its sign within the first two days, the algebraic solution has great difficulties following. Fig. 6 shows the development of the non-linear, the linearized and the algebraic solution for the variable b_5^2. Further approximations could be found by repeating the process and finding the equations for the deviations of the already determined approximation. These new functions will all be of type (14). Such further approximations have at present not been attempted.

Strength of interaction

In (11) it is seen that the importance of the interaction between (r,s) and (j,l) also depends on the term

$$\alpha_{r,s}+\alpha_{j,l}-\alpha_{m,n}$$

which we may call the *interaction frequency*. If this frequency is large it is indicative of an interaction which will change direction often and thus will not have a long term effect on $k_n^m(t)$; the opposite is true for a small value. We have computed the interaction frequency for selected wave vectors in order to establish if there are more permanent long-range couplings between particular wave vectors.

Table IV. The eigenvalues of (15) in day^{-1} and the corresponding periods in days for the functions $e^{-\lambda_d t}\cos\omega_d t$ to change by a factor of e

Periodic part		Exponential part		
ω_d(day^{-1})	T_ω(days	λ_d(day^{-1})	T_λ(days) (*)	(**)
1.382	4.5	−0,0213		294
0.8985	7	−0.2827		22
0.9503	7	−0.0226		279
0.8500	7	0.0600	105	
0.7266	9	0.0420	150	
0.5495	11	−0.0671		94
0.3983	16	0.2618	24	
0.5419	12	0.0711	88	
0	∞	0.3629	17	
0.4128	15	0.0968	65	
0.4130	15	−0.0621		101
0.3871	16	−0.0105		598
0.2972	21	0.0086	734	
0.2730	23	0.0290	216	
0.0349	180	−0.2169		29
0.1927	33	−0.0873		72
0.1503	42	−0.1261		50
0.1858	34	−0.0135		466
0.0586	107	−0.1305		48
0	∞	0.2212	28	
0.0301	209	0.1650	38	
0.0477	132	0.0591	106	
0	∞	0.0847	74	
0	∞	0.0104	605	

(*) Days to increase by a factor of e.
(**) Days to decrease by a factor of e.

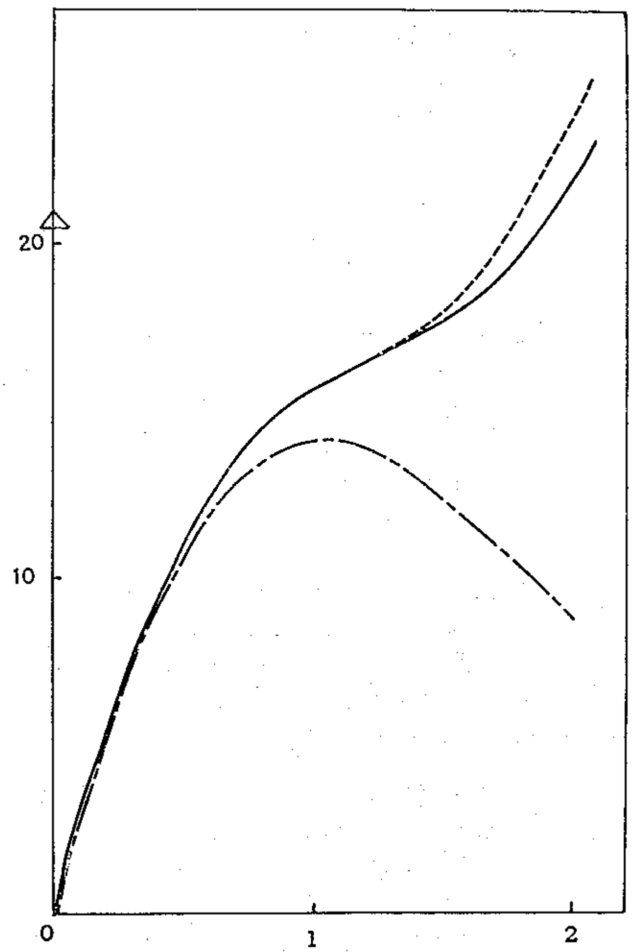

Fig. 6. A comparison between predictions using (11) (fully drawn), (12) (dashed and (14) (dashed and dotted), all for the component b_5^2. Arrow shows magnitude of $B_5^2(0)$.

In a comparison between the wave vectors (0,3) and (2,7) with an even-odd expansion with 10 complex terms per (longitudinal) wave number ((m,n) = (0,3) and (m,n) = (2,7)), we have computed the period T, where

$$T=\frac{2\pi}{(\alpha_{r,s}+\alpha_{j,l}-\alpha_{m,n})}$$

expressed in days, the (r,j) and (j,l) being subjected to the usual selection rules. For (0,3) a median period of 10 days was found and 20 days for

(2,7). A vanishing interaction frequency can occur if $\alpha_{r,s}$ appears twice, once as a contributing wave vector and once as the vector to which it contributes. Thus we have that

$$\alpha_{m,n}+\alpha_{o,s}-\alpha_{m,n} = 0$$

which would constitute a constant coupling if H_{snn}^{omm} is non-vanishing. By the selection rules we must have $m = 0$ since a zonal component cannot get contributions from other zonal terms. The net result from this observation is that (2,7) has constant (steady) contributions from zonal terms whereas (0,3) can only have periodic contributions. For (2,7) a few periods were quite long; the combination $(-6, 15)$ and (8, 11) gave rise to a period of 4521 days and cases with periods larger than one year were also observed. (0,3) on the other hand had only one vector which gave rise to periods larger than 100 days, namely, the combination $(-1,10)$ and (1,8). Other observations, some obvious, were the following:

(a) The influence of the zonal term A_1^0 on the interaction frequency is negligible; we may simplify computations by leaving it in the quadratic term in (11).
(b) Usually long periods are found when an interacting pair consists of one Legendre polynomial with a large number of zero intercepts (i.e. $k-|j|$ is large) and one with few zero intercepts (i.e. $s-|r|$ small). For (2,7) the pair (7,10) and $(-5,14)$ had a period of 120 days, whereas (7,10) and $(-5,6)$ had a period of 5 days.
(c) The magnitude of the interaction coefficient does not seem to be related to the interaction period, except for the zonal term of lowest degree. For zonal components the longer periods are found when the degree of the interacting polynomials increases.

4. DISCUSSION

The spectral form of the non-divergent barotropic vorticity equation lends itself better to studies of the propagation of uncertainty in the initial data than does the same equation in physical space. This follows since a dynamical study of the uncertainty in physical space would require the knowledge of the covariance function between any pair of points in physical space. The spectral approach would only require the knowledge of the covariance between the expansion coefficients and in studies of long waves in the atmosphere a small number (≤ 100) could be used. It is of great importance, however, to find out to what extent (2) or (5) behaves like a linear system, since statistical dynamical predictions are greatly simplified in linear systems of differential equations. The fact that the quadratic term in (2) or (5) proves to be important means that dynamical predictions using inexact initial states must necessarily use open-ended systems of differential equations on the higher order statistical moments.

To what extent analytical approximations at the origin used above can be of help in gaining understanding of the time development of spectral solutions in general is still unresolved. It was shown above, however, that the analytical solution gave real improvement over a simple trigonometric solution over a period up to one day. For expansions with 45 terms or less we should expect to be able to obtain analytical solutions valid for up to 24 hours. Ours is a rather highly truncated expansion and it remains to be seen to what extent expansions with more terms have greater temporal validity. The results on the strength of the interactions indicate that the contributions from some pairs of wave vectors are almost random in nature. With larger systems one would expect to see a larger number of such unimportant interactions. It may be profitable at a future time to explore the effect of systematically deleting the rapidly changing contributions and compare the results with predictions which retain them. Alternatively one may perhaps replace some of the interactions with contributions from a stochastic process.

ACKNOWLEDGEMENTS

The author would like to thank his advisor, Dr. John W. Tukey, for the valuable assistance received during the execution of this research project. The many fruitful discussions with Dr. K. Miyakoda and other staff members of Geophysical Fluid Dynamics Laboratories, NOAA, Princeton, N.J., helped the project to a measurable degree. While at Princeton the author had support through a research assistantship sponsored by the Office of Army Research through contract number DA-31-124-ARO-(D)-215.

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