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Prediction of Second-Moment Properties in Spectral Form for Non-Divergent Barotropic Flow

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The ability of the 'natural modes' in a system of non-linear differential equations to predict first- and second-order statistical moments has been investigated for a problem arising in the study of long waves in the atmosphere by means of a one-level non-divergent barotropic model written in spectral form. The 'natural modes' have been compared to the solution of a non-linear moment equation truncated such that all cumulants from K_3 up are assumed to vanish at all times. In the latter model a two-days integration showed that the trace of the covariance matrix of the expansion coefficients increased almost linearly with time. The 'natural modes' may be useful to obtain first-order approximations to the time-dependent variance of the fitted field in physical space over a time space of perhaps one day.

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INTRODUCTION

The problem of predicting the future state of a dynamic system for incomplete initial data with an exact dynamic model has many aspects, two of which are especially important: the selection of statistics to describe the initial state, and a matched dynamical model to transform the statistics in time. Although problems of this nature have been most successfully tackled within the area of statistical turbulence theory the subject matter belongs properly within the realm of stochastic processes (see e.g. Moyal 1949). The problem has reappeared in recent papers in the atmospheric sciences where long waves have been studied using barotropic models. In this field the historical background of the problem appears to be relatively short. An excellent exposition of the difficulties in meteorological applications was given by Freiberger & Grenander (1965), who discussed the implications of the non-linear character of the governing equations. Epstein (1969) examined a problem of this type when starting with a barotropic flow on a flat earth. Using the same model Fleming (1970) has discussed the effect of various schemes to obtain completeness in the statistical variables.

The main theme of this paper is how a particular moment truncation method works with a non-divergent barotropic model written in spectral form using spherical harmonics as expansion functions. Since the dynamical equations are non-linear, it is of some interest to assess the goodness of various approximate solutions valid over a short time space. A 'natural mode' type solution is compared to the non-linear solution found by assuming cumulants from K_3 upwards, vanishing at all times.

A. Eliassen submitted this paper to the Norwegian Academy of Science and Letters in Oslo, 6th April 1973.

1. STATISTICAL DYNAMICAL PREDICTION

Consider a set of S dynamical variables y_i ($i = 1,2,\ldots,S$) whose initial state is uncertain but where individual transformations in time, viz.

$$\frac{d}{dt} y_i = \xi_i(y_1,y_2,\ldots,y_S;t) \qquad (1)$$

are determined by the differential equations of mechanics. Integrating such a statistical dynamical system implies finding ensemble averages of y_i at $t_1, t_2, \ldots$ and specifying the probability that the variables occupying a given region in the phase space would transform into another given region of the same space. A process where all uncertainty is limited to the initial value but where the laws of mechanics apply has been called a crypto-deterministic process (Moyal 1949).

Let the column vector containing the averages of the S components y_i be written as follows

$$\text{Average of } y_i = \langle y_i \rangle .$$

Similarly one can define the covariance of the components y_p and y_r which measures the degree of linear covariability between the two components of y. In statistical dynamical prediction the mean values, variances and covariances, and higher statistical moments will in general change in time depending on the form of (1). In practice, the equations comprising the statistical dynamical system will have to be integrated with the various statistical moments replaced by their *estimates*; the actual values of these estimates depend on the statistical method used in obtaining them. For example, if the method of least squares is used in estimating the y's in a regression model, a method of obtaining estimates of the covariance matrix of the y's can readily be found (see appendix). Prediction of, say, second moment properties can be used for constructing time-dependent confidence intervals for each of the y's, or for linear functions of them.

Let $\mathbf{V}$ be the matrix that contains the variances and covariances of the y's. The element v_{rp} in the rth row and the pth column is the covariance of the components y_r and y_p; clearly the matrix is symmetric. Starting from a single confidence interval for one variable, it is intuitively clear that the more narrow the confidence interval is the more determined this variable is. If (1) contains more than one variable an ellipsoid of concentration may be used to describe the joint uncertainty. For the case $S=2$ we may find two new random variables that are uniformly distributed within an ellipse and have means, variances, and covariances coinciding with those of the original variables. The *area* of this ellipse is then a joint measure of the uncertainty in these two variables. In the general case of a non-singular distribution one can define a new set of variables Z that are uniformly distributed within the ellipsoid

$$(\mathbf{Z}-\langle\mathbf{Z}\rangle)'\,\mathbf{V}^{-1}(\mathbf{Z}-\langle\mathbf{Z}\rangle) = s+2\,,$$

where the prime, here and in the rest of this paper, denotes the transposing of the vector or matrix. These variables will have mean values $\langle\mathbf{Z}\rangle$ and variances–covariances $\mathbf{V}$ (Cramer 1946, p. 300). The volume of this ellipsoid is given by

$$(s+2)^{s/2}\,\pi^{s/2}|\mathbf{V}|^{1/2}/\Gamma\left(\frac{s}{2}+1\right).$$

For a non-singular distribution the uncertainty is thus proportional to $|\mathbf{V}|$; this latter quantity is called the generalized variance (Cramer 1946, p. 301). It follows that a joint measure of the uncertainty in a statistical dynamical process can be found by studying the generalized variance.

If $\mathbf{V}$ is full rank and $\mathbf{y}$ is distributed according to the gaussian distribution, the quadratic form

$$\mathbf{Y}'\,\mathbf{V}^{-1}\,\mathbf{Y} \qquad (2)$$

will be distributed according to a chi-square distribution with S degrees of freedom. Whenever $\langle\mathbf{Y}\rangle = \mathbf{O}$ this distribution is the ordinary chi-square distribution; otherwise the form is distributed according to the *non-central* chi-square distribution with non-centrality parameter δ, where

$$\delta^2 = \langle\mathbf{Y}\rangle'\,\mathbf{V}^{-1}\langle\mathbf{Y}\rangle$$

(Scheffe 1959, p. 412). The connection between δ and the generalized variance may be shown by noting (Scheffe 1959, p. 417) that

$$1+\delta^2 = \frac{|\mathbf{V}+\langle\mathbf{Y}\rangle\,\langle\mathbf{Y}\rangle'|}{|\mathbf{V}|}$$

which shows that δ^2 increase when $|\mathbf{V}|$ decreases. If $|\mathbf{V}|$ decreases in such a way that all elements on the main diagonal approach zero simultaneously, the non-centrality parameter tends to infinity. A small value of $|\mathbf{V}|$ could be brought about by improved measuring techniques of the initial values. Roughly speaking then, the larger the δ (or smaller the $|\mathbf{V}|$) the more 'predictable' is the dynamic process. It will be shown below that δ can be used to characterize the degree of uncertainty in statistical dynamical processes with gaussian distributed variables. In fact, in linear systems, where the right-hand side of (1) is a linear function of the y's, δ is invariant.

To show this let (1) be of the form

$$\frac{d}{dt}\mathbf{Y} = \mathbf{A}\,\mathbf{Y} \tag{3}$$

where $\mathbf{Y}$ is a $S\times 1$ and $\mathbf{A}$ a real $S\times S$ matrix of constants. The mean of $\mathbf{Y}$ can be shown to follow the equation

$$\frac{d}{dt}\langle \mathbf{Y}\rangle = \mathbf{A}\langle \mathbf{Y}\rangle \tag{4}$$

Thus, for linear systems, ensemble means follow the same differential equation as individual solutions of (3). Furthermore, one can show that

$$\frac{d}{dt}\mathbf{V} = \mathbf{A}\,\mathbf{V}+\mathbf{V}\,\mathbf{A}' . \tag{5}$$

Note that (3) is linear in the variances and covariances, and in particular independent of other statistical moments. Similarly, the equations for higher moments may be found.

Since $\mathbf{V}\,\mathbf{V}^{-1} = \mathbf{I}$

$$\frac{d}{dt}\mathbf{V}\cdot\mathbf{V}^{-1}+\mathbf{V}\cdot\frac{d}{dt}\mathbf{V}^{-1} = \mathbf{O}$$

and because of (5)

$$\frac{d\mathbf{V}^{-1}}{dt} = -\mathbf{V}^{-1}\mathbf{A}-\mathbf{A}'\mathbf{V}^{-1} .$$

Taking the time derivative of δ^2 we have

$$\frac{d}{dt}(\langle\mathbf{Y}'\rangle\mathbf{V}^{-1}\langle\mathbf{Y}\rangle) = \frac{d}{dt}\langle\mathbf{Y}'\rangle\cdot\mathbf{V}^{-1}\langle\mathbf{Y}\rangle$$

$$+\langle\mathbf{Y}'\rangle\frac{d}{dt}\mathbf{V}^{-1}\cdot\langle\mathbf{Y}\rangle+\langle\mathbf{Y}'\rangle\mathbf{V}^{-1}\frac{d}{dt}\langle\mathbf{Y}\rangle$$

$$= \langle\mathbf{Y}'\rangle\mathbf{A}'\mathbf{V}^{-1}\langle\mathbf{Y}\rangle$$

$$+\langle\mathbf{Y}'\rangle(-\mathbf{V}^{-1}\mathbf{A}-\mathbf{A}'\mathbf{V}^{-1})\langle\mathbf{Y}\rangle$$

$$+\langle\mathbf{Y}\rangle'\mathbf{V}^{-1}\mathbf{A}\langle\mathbf{Y}\rangle = 0$$

which proves our assertion.

The sparsity of similar simple mathematical results on systems of differential equations where the right-hand side of (1) is a quadratic or higher order polynomial is regrettable, since many physical processes can be shown to be governed by such systems. Differential equations can always be found for statistical moments of any order, but it seems impossible to obtain a closed set of variables and equations unless special assumptions are made on the structure of the moments; this in turn restricts the computations of the probability distributions of (y_i). For example, if

$$\frac{d}{dt}y_i = \sum_{t,r} b_{irt}\, y_r y_t , \tag{6}$$

where b_{irt} $(i,r,t = 1,2,\ldots,S)$ do not depend on the y's,

$$\frac{d}{dt}\langle y_i\rangle = \sum_{r,t} b_{irt}(\langle y_r' y_t'\rangle+\langle y_r\rangle\langle y_t\rangle) \tag{7}$$

$$\frac{d}{dt}\langle y_r' y_i'\rangle$$

$$= \sum_{l,v} b_{ilv}\{\langle y_r' y_l' y_v'\rangle+\langle y_l' y_r'\rangle\langle y_v\rangle+\langle y_v' y_r'\rangle\langle y_l\rangle\}$$

$$+\sum_{l,v} b_{rlv}\{\langle y_i' y_l' y_v'\rangle+\langle y_l' y_i'\rangle\langle y_v\rangle+\langle y_v' y_i'\rangle\langle y_l\rangle\} \tag{8}$$

Here $y' = y-\langle y\rangle$, so that

$$\langle y_i' y_j'\rangle = \langle(y_i-\langle y_i\rangle)(y_j-\langle y_j\rangle)\rangle ,$$

and

$$\langle y_i' y_j' y_k'\rangle = \langle(y_i-\langle y_i\rangle)(y_j-\langle y_j\rangle)(y_k-\langle y_k\rangle)\rangle .$$

Similarly higher moment equations can be found. Evidently the time derivatives of any rth-order moment will depend on moments of order up to $r+1$. Rather than assuming that certain higher-order moments vanish at all times, closure can be obtained by assuming certain high-order cumulants (Kendall & Stuart, Vol. I, p. 70) vanishing. The connection between the moments and cumulants is as follows: Let K_r be the rth cumulant, μ_r the (central) moment, and write

$$\mu_1 = \langle y \rangle$$
$$\mu_2 = \langle (y')^2 \rangle = \langle (y - \langle y \rangle)^2 \rangle \,;$$

in general

$$\mu_r = \langle (y - \langle y \rangle)^r \rangle \,.$$

Then the connection between moments and cumulants is given by a series of relations of which we give the first five, viz.

$$\mu_2 = K_2$$
$$\mu_3 = K_3$$
$$\mu_4 = K_4 + 3K_2^2$$
$$\mu_5 = K_5 + 10K_3K_2$$
$$\mu_6 = K_6 + 15K_4K_2 + 10K_3^2 + 15K_2^3 \,.$$

The importance of the cumulants stems from the fact that they give a measure of the non-normality of the distribution. For the gaussian distribution one finds that $K_r = 0$ $(r \geq 3)$. Another reason for dealing with cumulants rather than moments is their convenient relation to convolutions: if a variable is written as the sum of independent terms the cumulant of any order is the sum of cumulants of that order for the independent terms.

The assumption of vanishing higher cumulants produces a new approach to truncation of moments. To what extent these *cumulant-discard schemes* (Kraichnan 1962) show any similarity with the original statistical dynamical process is undoubtedly the most crucial question. The simplest system is found by assuming that all third-order cumulants vanish, in which case only (7) and (8) are needed with a total of $S(S+3)/2$ variables; this model will be used in subsequent sections. If all fourth-order and higher cumulants vanish and the fourth moment is related to the second moment as in the multivariate gaussian distribution, a new closure scheme can be found ('quasinormal truncation scheme', Kraichnan, op. cit.). Although intuitively more attractive than arbitrarily assuming vanishing higher moments, the cumulant-discard scheme may nevertheless lead to solutions with moments that violate basic properties of moments in general. One of these properties is the positive definiteness of the covariance matrix; this aspect will be dealt with in the following sections.

The remarks above have been aimed at the problems of obtaining predictions of statistical quantities under the assumption that the physical process may be successfully modelled by (1). Many physical processes are, however, known to be computationally well approximated by linear and quadratic right-hand sides in (1) (Baer & King 1967).

2. INITIAL UNCERTAINTY AND ITS ESTIMATION

Of the methods presently available for obtaining estimates of the physical variables, regression methods have found their most widespread use in the atmospheric sciences (Panofsky 1949, Gilchrist & Cressman 1954). In regression a large number of observations with errors attached are supposed to have arisen from a model with a small number of constants to which has been added random errors. The problem is usually posed quite oppositely: given a model containing a small number of constants which are undetermined and a large number of observations which have errors attached to them, estimate the constants and their errors. In obtaining these estimates various principles of judging the fit may be employed; only least squares will be used here.

Regression methods are convenient to use when the data consist of observations at irregularly distributed points in the plane. Let $\varphi_j(\theta,\lambda)$ $(j = 1,2,\ldots)$ be a series of functions in the plane and let θ_r, λ_r $(r = 1,2,\ldots,m)$ be the coordinates to the *rth* point in the plane for which an observation $\psi(\theta_r,\lambda_r)$ exists. A function $Y(\theta,\lambda)$ can be fitted such that the residuals (i.e. the difference between observed and fitted) ε's are jointly minimized,

$$\sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} (\psi(\theta_i,\lambda_i) - Y(\theta_i,\lambda_i))^2 \qquad (9)$$

Here $Y(\theta,\lambda)$ is a linear function in constants a_i $(i = 1,\ldots,S)$, viz.

$$Y(\theta,\lambda) = \sum_{i=1}^{S} a_i\,\varphi_i(\theta,\lambda) \qquad (10)$$

and the functions $\varphi_i(\theta,\lambda)$ are linearly independent. In Appendix it is shown that the estimates of the

a's are given by

$$\mathbf{a} = (\boldsymbol{\varphi}'\boldsymbol{\varphi})^{-1}\,\boldsymbol{\varphi}'\boldsymbol{\psi} \tag{11}$$

where $\mathbf{a}$ is an s-dimensional vector containing the estimates, $\boldsymbol{\varphi}$ is an $m \times s$ matrix containing the column vectors $\boldsymbol{\varphi}_i$,

$$\boldsymbol{\varphi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_s] \tag{12}$$

and

$$\boldsymbol{\varphi}_i = [\varphi_i(\theta_1, \lambda_1), \varphi_i(\theta_2, \lambda_2), \ldots, \varphi_i(\theta_m, \lambda_m)]' \,;$$

here as usual the prime denotes the transposing of the vector or matrix. If the residuals are considered to be independently distributed with the same variance σ^2, the variance-covariance matrix of the estimated a's in (15) will be

$$\mathbf{V}_{\mathbf{a}} = \sigma^2(\boldsymbol{\varphi}'\boldsymbol{\varphi})^{-1}\,. \tag{13}$$

One of the benefits from using regression methods is that an estimate of the variance can be found at any of the m points θ_i, λ_i in the plane. In fact

$$\text{Variance of fitted } Y(\theta_r, \lambda_r) = \boldsymbol{\varphi}_p' \mathbf{V}_a \boldsymbol{\varphi}_p \tag{14}$$

where $\boldsymbol{\varphi}_p$ is an s-dimensional column vector that contains the values of the S regressors φ_i at the point θ_r, λ_r:

$$\boldsymbol{\varphi}_p = [\varphi_1(\theta_r, \lambda_r), \varphi_2(\theta_r, \lambda_r), \ldots, \varphi_S(\theta_r, \lambda_r)]'\,.$$

To use (13) and (14) in practice, an estimated value of σ^2 is needed; such an estimate can be found from (9) by

$$\text{Estimate of } \sigma^2 = \sum_{i=1}^{m} \varepsilon_i^2/(m-S)\,.$$

More complex methods than least squares are usually used in the atmospheric sciences and needless to say the assessment of second moments of the estimates tend to be difficult. In meteorology it is common to relate the state of one atmospheric field in terms of other observable atmospheric fields. These relationships may take the form of diagnostic relations which are second-order partial differential equations; in such cases it becomes extremely difficult to relate the statistical properties of the observed fields to those of the derived fields. For instance, if the balance equation (Bolin 1955) is used, how can the statistical properties of the stream field be expressed, given the various assumptions on the corresponding properties of the pressure field? The solution of these and similar theoretical problems are important for a better understanding of the effects of error propagation and the workings of statistical dynamics in meteorological prediction.

3. MOMENT INTEGRATION OF THE NON-DIVERGENT BAROTROPIC VORTICITY EQUATION IN SPECTRAL FORM

The combining of least square with the non-divergent barotropic vorticity equation in spectral form yields a series of simple results concerning the precision in the predictions of long waves in the atmosphere using a heavily truncated system. Let the two-dimensional stream function $\psi(\theta, \lambda)$ be represented by (10) such that

$$\varphi_r(\theta, \lambda) = \begin{Bmatrix} \cos \\ \text{or} \\ \sin \end{Bmatrix} m\lambda P_n^m(\theta) \tag{15}$$

where r $(r = 1, \ldots, S)$ determines m and n. The S equations in (1) become

$$\frac{d}{dt} a_i = \sum_j C_{ij} a_j + \sum_{j,k} d_{ijk} a_j a_k \tag{16}$$

(Silberman 1954). C_{ij} and d_{ijk} are theoretical constants; in the present case the d's determine which of the spherical harmonics (15) are allowed to interact to produce changes in the coefficients a_i. For the non-divergent barotropic vorticity equation C_{ij} reduces to a single term in each of the S equations in (16). In fact, if a_j is the coefficient of cos $m\lambda P_n^m(\theta)$ and a_{j+1} the coefficient of sin $m\lambda P_n^m(\theta)$ $(m > 0)$, then

$$C_{j,j+1} = \frac{2m\omega}{n(n+1)} = -\alpha = -C_{j+1,j}$$

Here ω is the rate of rotation of the earth. The amount of non-linear interaction of the various pairs of waves is determined by d_{ijk}; if d_{ijk} vanished or were small in comparison with C_{ij}, the solutions of (16) would behave much like trigonometric solutions of the form

$$a_j\ (t) = A\cos\alpha t + B\sin\alpha t$$
$$a_{j+1}(t) = B\cos\alpha t - A\sin\alpha t$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$a_k(t) = D\cos\beta t + E\sin\beta t$$

A discussion on the validity of this approximation has been reported elsewhere (Knudsen 1973). If a trigonometric solution applies, the statistical dynamic system would have solutions

$$\begin{aligned}
&\operatorname{cov}(a_j(t); a_k(t)) \\
&= \tfrac{1}{2}(\langle D'A'\rangle - \langle E'B'\rangle)\cos(\alpha+\beta)t \\
&+ \tfrac{1}{2}(\langle D'A'\rangle + \langle E'B'\rangle)\cos(\alpha-\beta)t \\
&+ \tfrac{1}{2}(\langle D'B'\rangle + \langle E'A'\rangle)\sin(\alpha+\beta)t \\
&+ \tfrac{1}{2}(\langle D'B'\rangle - \langle E'A'\rangle)\sin(\alpha-\beta)t\,. \qquad (17)
\end{aligned}$$

where, as before, $A' = A - \langle A\rangle$.

We have chosen to investigate the short-term properties of (17) in comparison with the statistical dynamical solution of (16) when cumulants from the third order up vanish. Since in the statistical dynamical solution of (16) there will always be a component of type (17) due to the linear part of (16), one may interpret the difference between (17) and the statistical dynamical solution of (16) as being brought about by non-linear interaction between the first and second moments. If this interaction is negligible one may interprete (17) as the contribution to the development of the second moments in (4) and (8) that is independent of the initial **a**.

4. RESULTS

Data were obtained through the kind cooperation of Geophysical Fluid Dynamics Laboratories, NOAA, at Princeton University. They consisted of reported geometric heights of the 500 mb surface at the observing sites of around 400 stations on the northern hemisphere. An expansion containing 45 odd spherical harmonic functions was fitted and the coefficients rescaled to represent stream function (Knudsen 1973). The dates involved in this study were 12–16 January 1963. The geographical distribution of the observing stations in the sample was quite uneven; large areas over the oceans had no stations at all. The

Table I. The eigenvalues and the trace of the covariance matrix of the stream function at various epochs in a two-day prediction using equations for first and second moment (eigenvalues and trace in units of $10^{-10}\mathrm{m}^4\mathrm{sec}^{-2}$)

Rank of Eigen-value	Time (hours) 0	6	12	24	48
1	8400	7700	7300	7100	8900
2	2700	2700	2700	2900	3600
3	1500	1500	1700	2100	2700
4	950	1000	1100	1100	1600
5	810	780	760	950	1300
6	530	550	590	680	1100
7	460	480	520	610	910
8	380	370	410	500	690
9	320	350	390	470	600
10	310	330	350	410	570
11	280	290	310	390	470
12	250	250	270	300	380
13	210	200	200	300	330
14	190	190	190	240	290
15	170	180	170	210	270
16	140	130	140	180	220
17	120	130	130	140	220
18	110	120	120	140	180
19	100	110	110	130	170
20	94	110	97	120	160
21	92	96	91	110	130
22	80	81	86	110	110
23	77	74	75	94	99
24	73	71	75	88	89
25	66	68	63	80	82
26	59	61	57	74	71
27	58	58	53	65	58
28	58	56	52	60	50
29	52	53	48	51	48
30	50	51	45	44	42
31	48	49	43	40	34
32	45	46	41	39	30
33	43	43	38	33	25
34	41	40	36	31	21
35	40	38	32	27	19
36	37	36	28	27	16
37	37	33	26	22	16
38	35	30	26	21	14
39	33	28	22	18	12
40	29	20	20	15	9
41	29	21	17	13	7
42	24	18	15	12	7
43	18	17	11	9	6
44	11	13	11	7	6
45	9	3	5	−16	−26
Trace:	19393	18886	18791	20006	25794

45 basic expansion coefficients are still far short of the 200 to 300 variables needed to represent phenomena on a synoptic scale, but it was felt that the extra effort needed for handling 20,000 to 45,000 second-moment equations was not justified at present. Only one case was studied; this choice was dictated by the high cost of performing the numerical integration of the statistical dynamical system of differential equations. The numerical integration used centred differences. If $a(t)$ is a vector of variables, then

$$\frac{d}{dt}\mathbf{a} \cong \frac{\mathbf{a}(t+\Delta t)-\mathbf{a}(t-\Delta t)}{2\Delta t}$$

$$\mathbf{a}(t+\Delta t) \cong \mathbf{a}(t-\Delta t)+2\Delta t\frac{d}{dt}\mathbf{a}(t)\,.$$

On the first step, uncentred differences are used, viz.

$$\mathbf{a}(\Delta t) = \mathbf{a}(0)+\Delta t\frac{d}{dt}\mathbf{a}(0)\,.$$

With 45 expansion coefficients and equations for first and second moment the total number of equations is $45+45\times 46/2 = 1080$. The maximum stepsize was determined by experimenting on (16); when using heavily truncated expansions very large stepsizes (6 hours) can be used on this equation. For the statistical dynamical system $\Delta t = 3$ hours was chosen. The programs were written in Fortran for the Princeton University Computer Centre's IBM 360/91. The time used for running a single 48 hour statistical dynamical prediction was about seven and a half minutes.

Initially (13) is a positive definite quadratic form but no guarantee exists that it will remain so throughout the integration since the cumulants from K_3 up have been excluded. If the positive definiteness is lost at some point one could even expect that (18) would show negative variances in some geographical areas.

Table I contains the eigenvalues of the covariance matrix of the expansion coefficients **a** throughout the integration. Initially this covariance matrix is given by (13), at later times the time derivative of any of its elements is given by a combination of (5) and (8). Evidently the definiteness was lost after one day, since a single eigenvalue became slightly negative, but the loss does not seem to be too serious. Figs. 1, 2, 3, 4, and 5 show the contours in θ, λ – space for the solution for the standard deviation (that is, the square root of (14)) throughout the integrated 48 hours. In no case did negative variances appear. Table I shows that in the first few hours of the integration the largest eigenvalue actually shrank in size. From the sixth hour onwards, there is a tendency for the initially largest eigenvalues

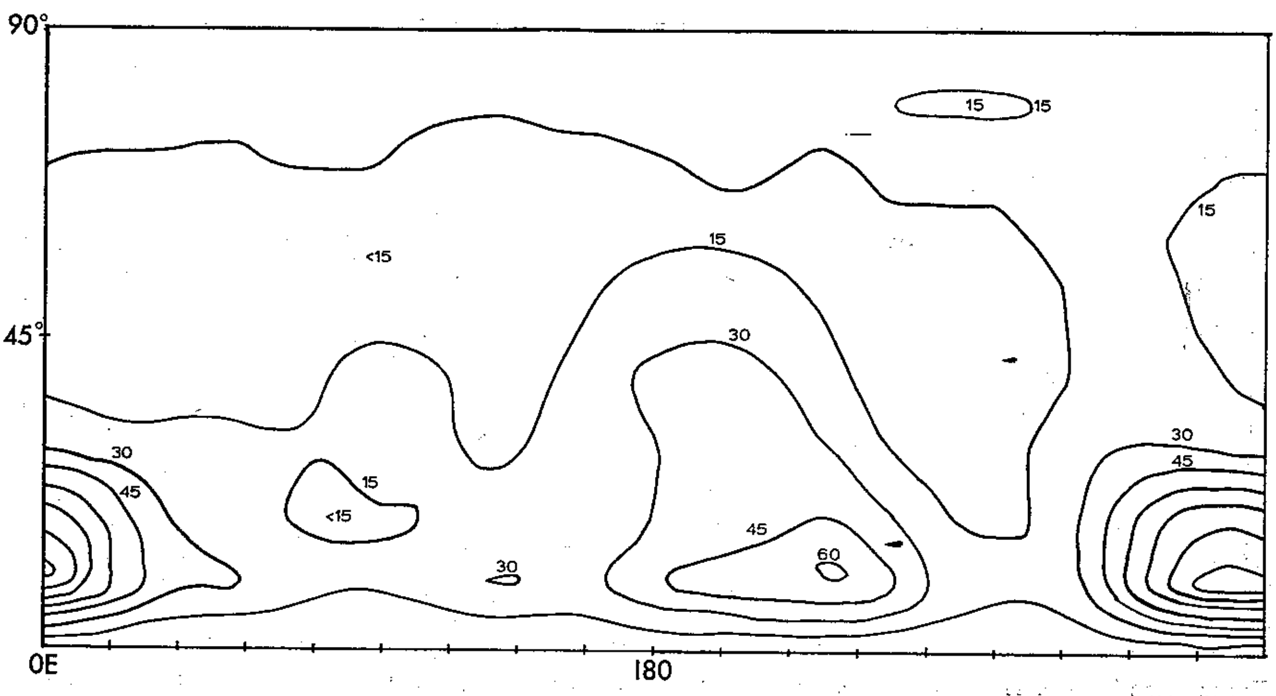

Fig. 1. Initial standard deviation of the geometric height field using (14) (Contour distance 15m.)

to grow larger and the smallest to become still smaller. This indicates a move towards a reduction of the rank of the matrix and, what amounts to the same thing, a greater linear dependence between the components of **a**.

A question of practical importance is whether the time development of (17) and the statistical dynamical solution is close enough so that (17) may be used over shorter time space rather than integrating the more complicated statistical dynamical solution. One way to investigate this is to compare maps of the geographical distribution of the variance of the fitted expansion. Fig. 1 shows the initial standard deviation (14); Figs. 2, 3, 4, and 5 show the time development of the statistical dynamical solution using (17). A com-

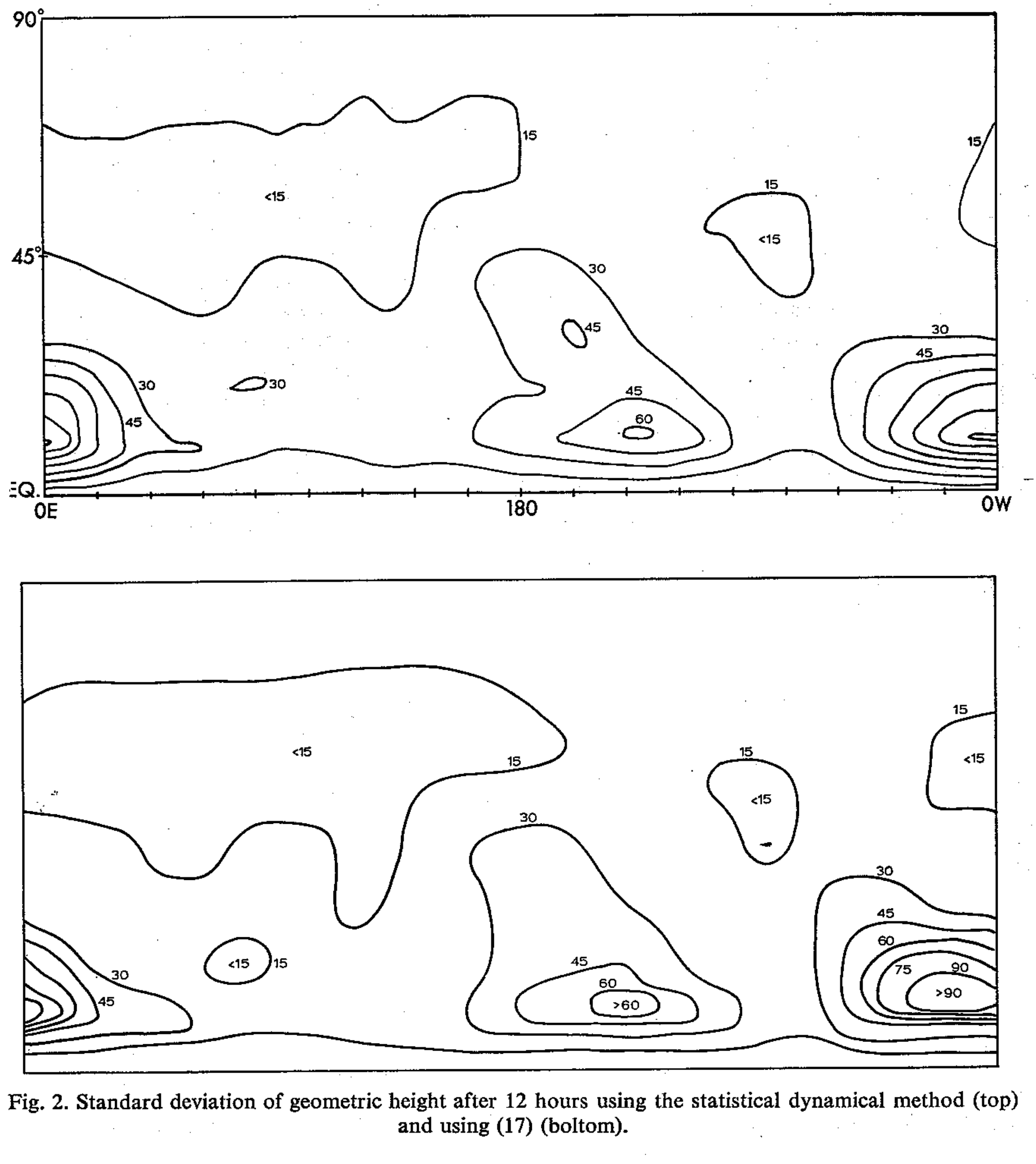

Fig. 2. Standard deviation of geometric height after 12 hours using the statistical dynamical method (top) and using (17) (boltom).

parison shows that (17) is adequate for some time but does not predict the non-linear build up of uncertainty at 120 degrees west which was started after 24 hours. The smoothing out of the North American–Siberian low uncertainty area is partly predicted by (17), but the overall increase in uncertainty as predicted by the statistical dynamical system is not present. From the figures one may conclude that (17) gives a reasonably true picture of the time development in the present case up to 24 hours.

Another way of looking at the similarities is to compare the changes in the eigenvectors of the covariance matrix from both methods (Table II). If (17) has any merit the discrepancies between the eigenvectors as predicted dynamically and by

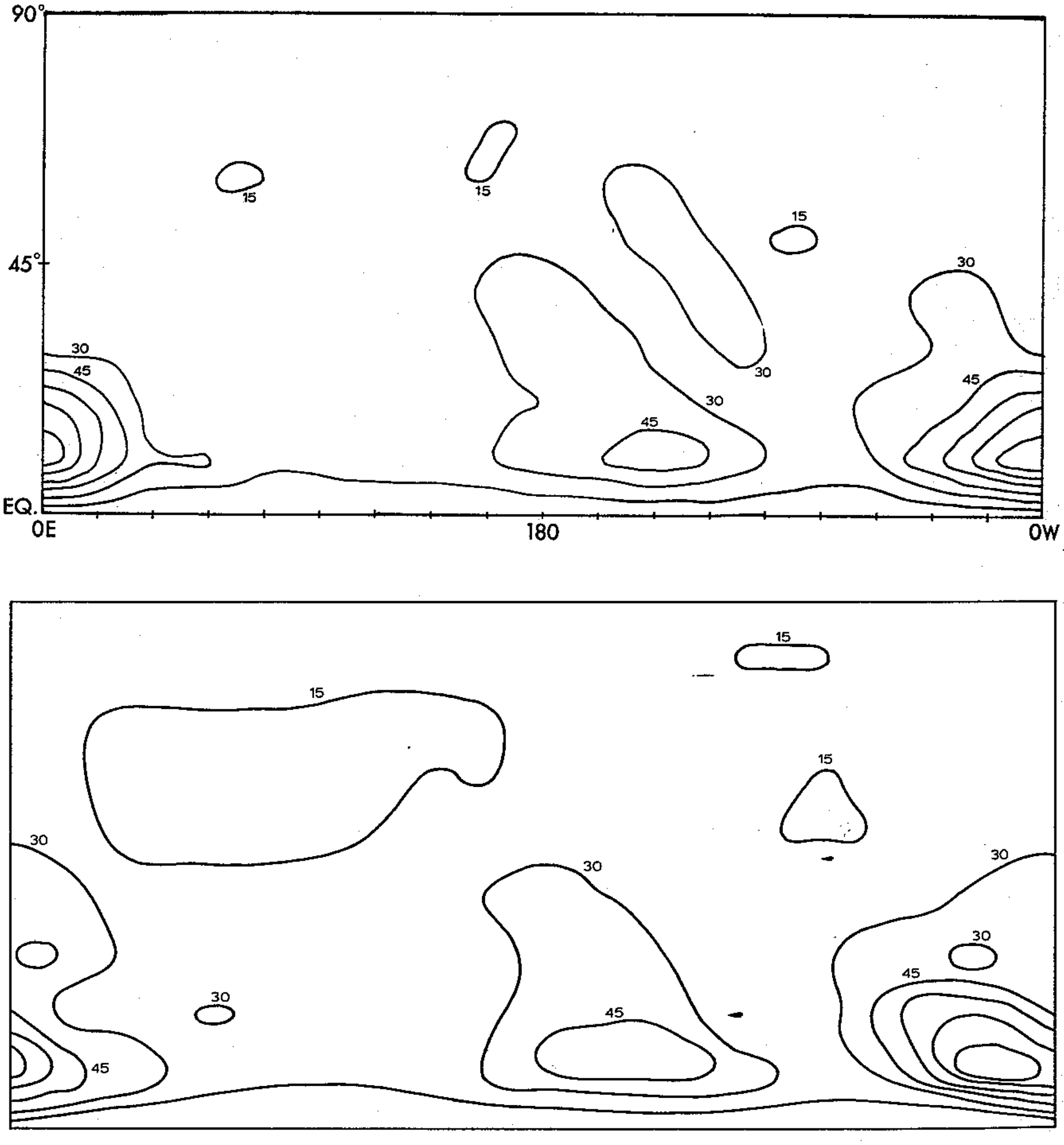

Fig. 3. Same comparison as in Fig. 2, but after 24 hours.

(17) should be small. Table II contains the eigenvectors belonging to the four largest eigenvalues, initially and as they appear after 48 hours.

Correlation coefficients have been computed between corresponding eigenvectors at a few epochs within the 48 hours of integration. In Table III the absolute value of the correlation coefficient is given between the four most important eigenvectors initially and at later epochs. An important finding here is that the persistence of the largest eigenvector is much greater than the three other vectors. After 24 hours, for instance, persistence seems to be more important the larger the eigenvalue is; this fact is not so clear after 48 hours. The lower part of the table shows the correlation between the corresponding pairs of

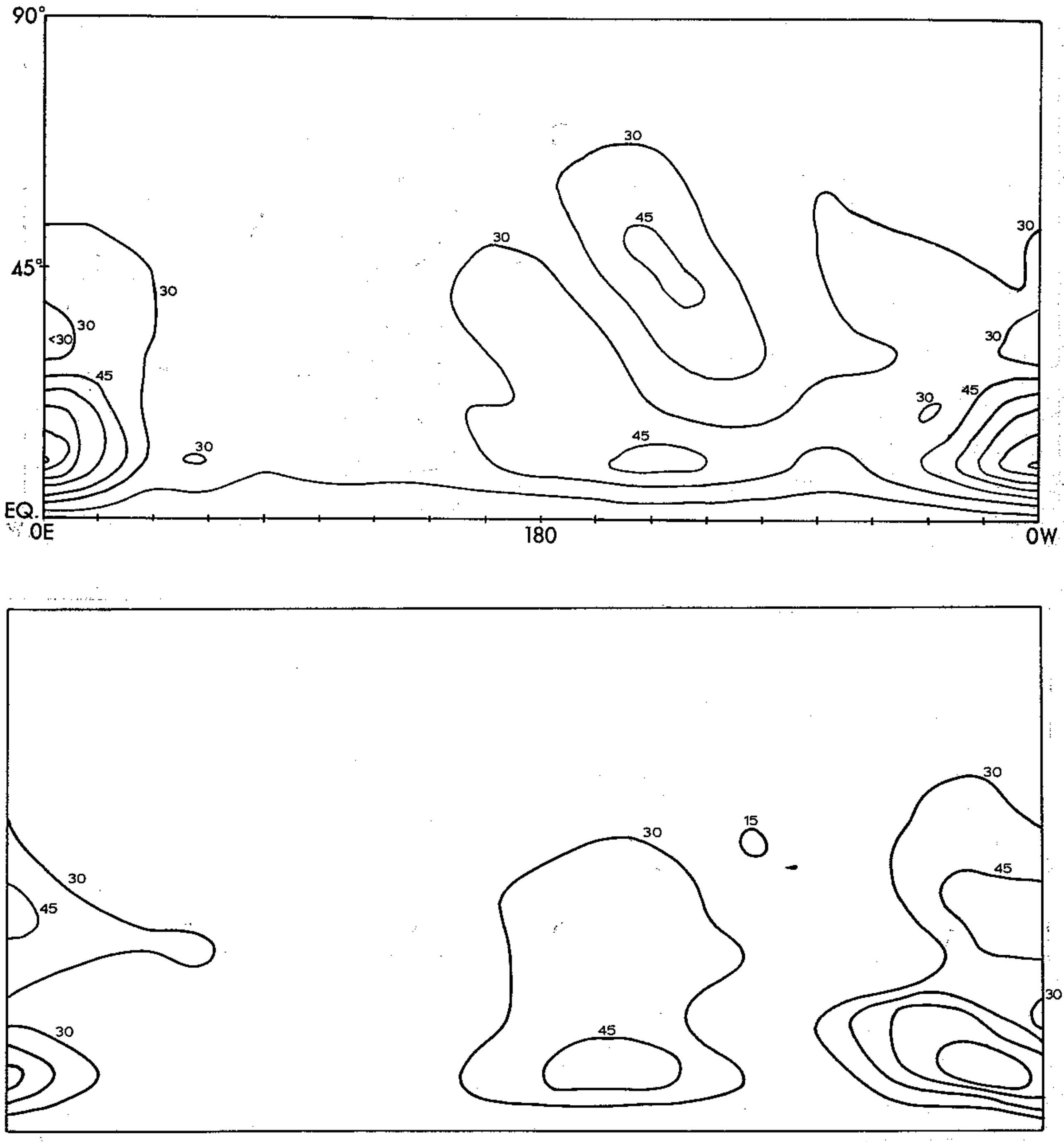

Fig. 4. Same comparison as in Fig. 2, but after 36 hours.

eigenvectors. Again the greater persistence of the most important eigenvector seems to be the reason for the decay of the solution from (17).

5. DISCUSSION

The inconvenience of integrating a large number of second-moment equations leads to a search for simpler ways of solving the non-divergent barotropic statistical dynamical system. If the similarity between solutions using (17) and the statistical dynamical solution is relatively high over a short period, (17) could be inserted into the statistical dynamical equations and equations on higher moments deleted altogether. One may generalize this idea further and obtain closure in the statisti-

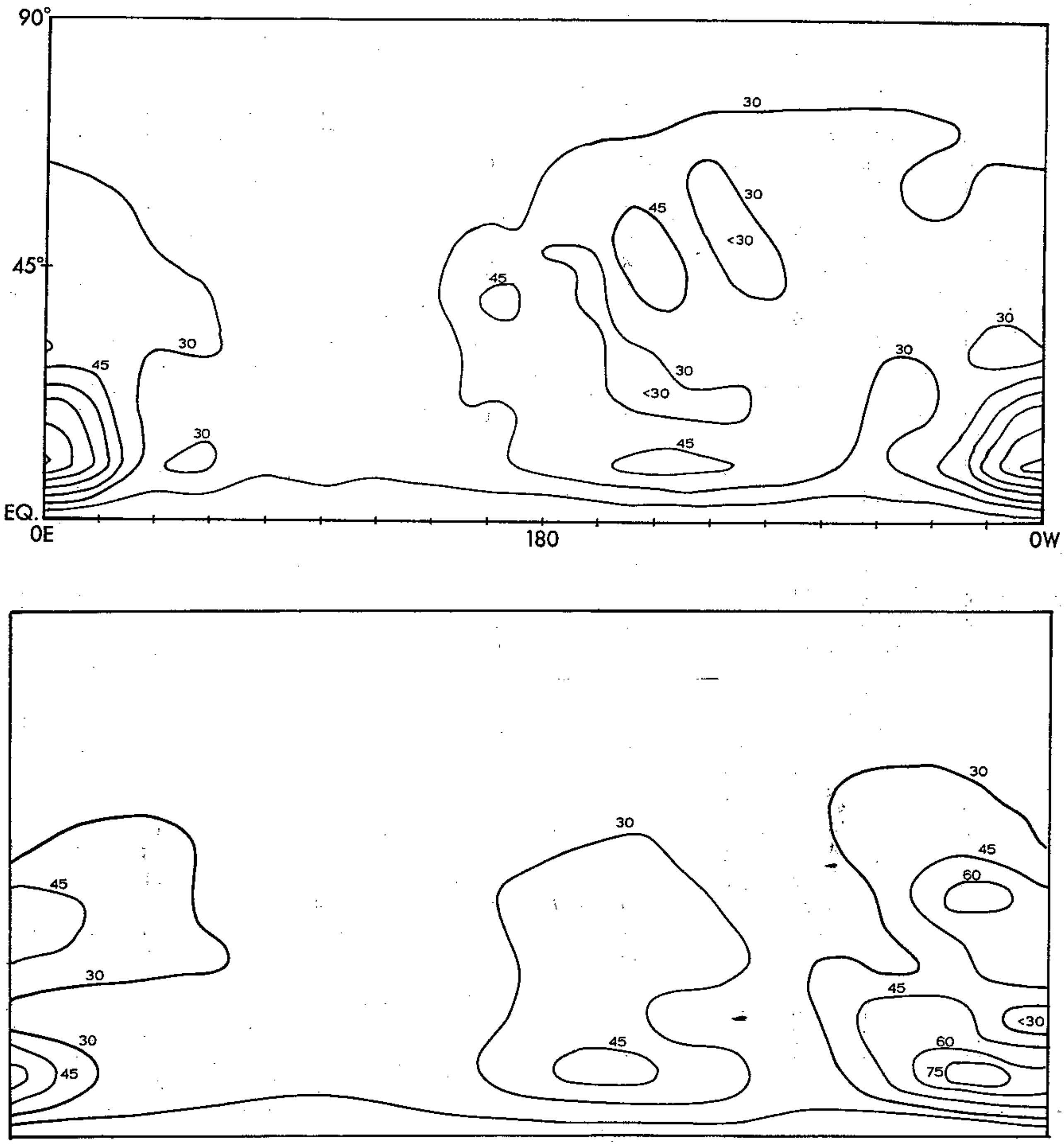

Fig. 5. Same comparison as in Fig. 2, but after 48 hours.

cal dynamical system for any chosen moment truncation by allowing the highest moment to be transformed by a periodic function like (17). In assessing any harmful effect one is forced to use experimental sampling with a completely deterministic model; very large samples may then be

Table II. Eigenvectors (in terms of spherical harmonic coefficients) for the four largest eigenvalues (A) initially, (B) after 48 hours using the moment equations and (C) after 48 hours using (17)

*	**	Largest			2nd Largest			3rd Largest			4th Largest		
		A	*B*	*C*	*A*	*B*	*C*	*A*	*B*	*C*	*A*	*B*	*C*
0	1	9	9	9	−4	−6	−4	3	0	3	18	10	18
	3	−17	−17	−17	8	5	8	3	16	3	−22	4	−22
	5	19	17	19	−10	−8	−10	−18	−17	−19	11	4	11
	7	−13	−12	−13	8	−1	8	23	18	23	0	−7	0
	9	7	7	7	−4	5	−4	−14	−13	−14	−2	0	−2
1	2	24	16	−11	5	−21	−17	−6	−4	11	−29	35	30
	2	−6	1	22	18	2	−8	0	−8	−13	−16	−22	−12
	4	−36	−24	−12	−9	−25	−27	−3	−19	−7	15	19	20
	4	8	19	35	−25	−23	−6	−7	−19	−1	14	−23	−5
	6	38	32	33	10	−10	18	29	29	32	8	−12	6
	6	−8	−3	−20	24	38	20	15	23	4	−4	−15	−7
	8	−26	−27	−26	−5	18	−7	−29	−22	−11	−31	6	−12
	8	6	7	8	−16	−7	−16	−6	−30	−13	−3	−6	−2
	10	13	17	13	1	3	1	12	−2	13	0	−14	0
	10	−1	−4	−1	7	−9	7	9	18	9	6	6	6
2	3	23	33	−14	14	0	−14	−2	−13	3	13	−20	−35
	3	−9	8	21	4	6	5	−2	−16	0	41	6	−26
	5	−28	−12	1	−16	−12	−10	21	−8	27	2	−21	−33
	5	11	8	30	−4	12	14	21	−14	−12	−36	−10	−14
	7	24	15	17	15	22	16	−28	−13	−39	−9	13	1
	7	−11	−5	−20	4	−3	−3	−32	12	−18	22	−10	24
	9	−14	−13	−14	−12	12	−13	16	3	18	2	−17	1
	9	8	10	9	−3	−6	−3	25	16	24	−13	0	−14
	11	6	7	6	8	3	8	−3	8	−1	3	6	2
	11	−1	−5	−1	2	10	3	12	−13	−13	10	11	10
3	4	18	33	−19	−10	−9	11	−10	−13	10	5	13	−6
	4	−10	−6	9	22	−33	−22	−3	5	3	−13	−9	13
	6	−21	−13	−3	12	−5	−17	−9	−14	−15	−20	−22	−19
	6	9	−12	23	−25	−22	−23	−17	8	−7	−10	12	13
	8	17	11	15	−12	−10	−6	3	9	10	19	13	26
	8	−5	−3	−10	19	4	22	23	20	21	23	20	16
	10	−10	−16	−10	10	4	11	3	12	4	−9	4	−8
	10	1	4	0	−8	4	−8	−13	−17	−13	−19	−15	−19
	12	6	3	4	−6	1	−6	−4	−7	−5	2	−4	0
	12	2	2	3	1	−3	0	−3	3	1	10	0	10
4	5	11	34	−14	32	−23	−7	−1	22	1	−8	2	21
	5	−8	2	−2	26	−25	−41	0	−18	0	21	14	−11
	7	−12	−23	−1	−29	−37	−39	4	4	18	4	−7	−1
	7	10	−10	16	−27	5	3	21	−27	11	−5	26	−6
	9	10	8	8	19	11	21	1	8	−2	9	32	9
	9	−8	−2	−9	22	20	20	−23	6	−23	0	−11	−1
	11	−7	−8	−8	−8	2	−4	−4	−4	−7	−16	−23	−15
	11	4	7	2	−4	−13	−16	12	−3	11	−3	−15	−7
	13	4	3	4	1	−14	−2	27	1	4	11	6	8
	13	0	2	1	7	0	7	−29	1	−1	40	11	8

* Order of spherical harmonic.

** Degree of spherical harmonic; cosines precede sines.

necessary to obtain precise moment estimates.

The time-dependent covariance matrix of the expansion coefficients can be used to display the loss in prediction in time of the fitted regression model using (14). Since it relates to the precision (or lack thereof) in the numerical prognosis it should be useful in giving weights to newly entered data in continuous time prediction systems. In a similar vein information on the climatology could be entered with weights related to the sparsity of data in the area as suggested by Bergthorsson & Döös (1955) or alternatively by using various bayesian procedures.

Table IIIa. Difference in direction as measured by the correlation coefficient between the eigenvectors of the covariance matrix of the four largest eigenvalues *initially* compared with *later* epochs

Epoch		Rank of initial eigenvalue			
(Hours)	*	1	2	3	4
	1	100	0	0	0
0	2	0	100	0	0
	3	0	0	100	0
	4	0	0	0	100
	1	95	2	9	0
	2	6	97	6	6
6	3	0	7	86	11
	4	4	4	10	87
	1	80	5	15	5
	2	9	90	8	11
12	3	1	16	77	17
	4	3	7	32	59
	1	92	20	4	8
24	2	8	62	14	15
	3	1	38	51	9
	4	1	14	42	49
	1	83	21	2	13
	2	1	24	7	10
48	3	6	42	32	3
	4	20	4	7	21

* Rank of eigenvalue at Epoch.

Table IIIb. Difference in direction (at the *same* epochs) measured by the correlation coefficient (× 100) between the eigenvectors of the covariance matrix of the four largest eigenvalues from a prediction with the moment equations and with (17).

Epoch	Rank of eigenvalue			
(Hours)	1	2	3	4
6	93	98	97	96
12	75	94	89	65
24	67	86	73	74
48	32	72	49	51

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APPENDIX

Let Y be a random variable whose mean (expected value) is completely described by a linear function in a set of k regressors $\mathbf{X}_i$ ($i = 1,2,\ldots,k$) such that

$$E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta} \qquad \text{(A1)}$$

where $\mathbf{X}$ is a $n \times k$ matrix consisting of column vectors of $\mathbf{X}_i$, each containing the value of the ith regressor at each of the n points in the sample. A sample of size n for Y may be considered to be brought about by having a random variable ε added to each of the Y's in the sample so that

$$\mathbf{Y} = E(\mathbf{Y}) + \boldsymbol{\varepsilon}$$

or

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}. \qquad \text{(A2)}$$

The random variables ε may have any distribution; in particular the covariance between any pair (i,j) of random variables may be given by a covariance matrix $\sigma^2\mathbf{V}$. The Gauss–Markoff theorem states that of those estimators with the property that $E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$ the ones with the (jointly) smallest variances are given by

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y} \qquad \text{(A3)}$$

and the covariance between any pair of the estimators $\hat{\boldsymbol{\beta}}$ is given by the elements in the covariance matrix

$$E\{(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\} = \sigma^2(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} \qquad \text{(A4)}$$

(for proof see Kendall & Stuart, Vol. 2, 1961, p. 87 ff). In some cases $\boldsymbol{\beta}$ is found by minimizing the sum of the squares of the residuals e_i where

$$\mathbf{e} = \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}.$$

This solution, the least squares solution, is found to be

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}. \qquad (A5)$$

It is seen that one could have arrived at the same result by assuming $\mathbf{V} = \sigma^2\mathbf{I}$ where $\mathbf{I}$ is the $n \times n$ identity matrix. This assumes that the errors $\boldsymbol{\varepsilon}$ are uncorrelated. With this latter assumption the covariance matrix between the least squares estimates becomes

$$E\{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\} = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}. \qquad (A6)$$

In the present paper we shall consider the case where a functional expansion has been truncated to a finite number of terms. Such expansion may have had mathematical advantages if continuous data were available, since the expansion coefficients may have been obtained by definite integrals. When data are given only in discrete points one may approximate the integrals by various quadrature formulae; alternatively one may use the method of least squares. In the latter case the expansion coefficients will be given by (A5). The spherical (surface) harmonics are such an expansion. The regressors $\mathbf{X}_i$ are given by the expression

$$\cos m\lambda P_n^m(\theta)$$

or

$$\sin m\lambda P_n^m(\theta) \qquad (A7)$$

where λ is longitude, θ is colatitude, and $P_n^m(\theta)$ are associated Legendre polynomials of degree n and order m. Although the functions are orthogonal when multiplied together and integrated over the surface of the sphere, this aspect is of little use when the data are given in discrete points.

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